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STABILITY AND SQUARE INTEGRABILITY OF SOLUTIONS TO THIRD ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, sufficient conditions to guarantee the square integrability of all solutions and the asymptotic stability of the zero solution of a non-autonomous third-order neutral delay differential equation are established. An example is given to illustrate the main results.

1. Introduction

In this paper, we discuss three classic questions on the behavior of solutions of differential equations, namely, their boundedness, stability, and square integrability. In particular, we examine the uniform asymptotic stability of solutions of the third order nonlinear neutral delay differential equation

$$\begin{aligned}\big[x(t)+\beta x(t-\tau)\big]'''+a(t)\Big(Q\big(x(t)\big)x'(t)\Big)'+\\ b(t)\Big(R\big(x(t)\big)x'(t)\Big)+c(t)f\big(x(t-\tau)\big)=0,\end{aligned} \tag{1.1}$$

as well as the boundedness and square integrability of solutions of the corresponding forced equation

$$\begin{aligned}\big[x(t)+\beta x(t-\tau)\big]'''+a(t)\Big(Q\big(x(t)\big)x'(t)\Big)'+\\ b(t)\Big(R\big(x(t)\big)x'(t)\Big)+c(t)f\big(x(t-\tau)\big)=h(t).\end{aligned} \tag{1.2}$$

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Here β and τ are constants with $0 \le \beta \le 1$ and $\tau \ge 0$, the functions a, b, $c\colon [0,\infty) \to [0,\infty)$, Q, $R\colon \mathbb{R} \to [0,\infty)$, $h\colon [0,\infty) \to \mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$ are continuous, and $xf(x) > 0$ for $x \neq 0$.

For second order equations, determining the asymptotic stability and square integrability of solutions has been a very active area of research over the years; see, for example, the monographs [3] and [4]. These properties have received far less attention for third order equations; some early well-known results on special cases of equation (1.1) can be found in Ezeilo [12], Hara [19], and the classic work of Reissig, Sansone, and Conti [25]. More recent results have appeared in the monograph of Padhi and Pati [23] and the papers Ademola and Arawomo [1], Baculíková and Džurina [2], Bartušek and Graef [5], [6], Došlá [9], Graef et al. [13]–[17], Mihalíková and Kostiková [20], Omeike [21], Oudjedi [22], Qian [24], Remili et al. [26]–[38], Tian et al. [39], Tunç [40]–[45], and Zhang and Yu [46].

By a *solution* of (1.1) or (1.2) we mean a continuous function $x\colon [t_x,\infty) \to \mathbb{R}$ such that $x(t) + \beta x(t-\tau) \in C^3\big([t_x,\infty),\mathbb{R}\big)$ and which satisfies the equation on $[t_x,\infty)$.

2. Asymptotic stability

We shall make use of the following assumptions on the functions appearing in the equations. Assume that there are positive constants a_0, a_1, c_0, b_1, q_0, q_1, r_0, r_1, L, δ, d, M, η, and J such that the following conditions are satisfied:

(H_1) $0 < a_0 \le a(t) \le a_1$ and $0 < c_0 \le c(t) \le b(t) \le b_1$;

(H_2) $0 < q_0 \le Q(x) \le q_1$ and $1 \le r_0 \le R(x) \le r_1$;

(H_3) $\delta\big(1 + \frac{\beta}{2}\big) < d < a_0 q_0$ and $-L \le b'(t) \le c'(t) \le 0$;

(H_4) $f(0) = 0$, $\frac{f(x)}{x} \ge M > 0$ for $x \neq 0$, f' is continuous and $f'(x) \le \delta$ for all x;

(H_5) $\frac{1}{2} d a'(t) Q(x) - c_0\big(d - \big(1 + \frac{\beta}{2}\big)\delta\big) + \frac{b_1\beta}{2}(r_1 + r_1\beta + \delta) \le -\eta < 0$;

(H_6) $\beta(a_1 q_1 - d) + b_1 \beta r_1 (1+\beta) - (2-\beta)(a_0 q_0 - d) < 0$;

(H_7) $\int_{-\infty}^{+\infty} \big(|Q'(u)| + |R'(u)|\big)\,du \le J < +\infty$.

Our main result on the asymptotic stability of the zero solution of equation (1.1) is contained in the following theorem.

Theorem 2.1. *Assume that conditions (H_1)–(H_7) hold. Then, the zero solution of equation* (1.1) *is uniformly asymptotically stable if*

$$\tau < \min\left\{ \frac{2\eta}{b_1\delta(1+\beta+2d)}, \frac{(2-\beta)(a_0q_0 - d) - \beta(a_1q_1 - d) - b_1\beta r_1(1+\beta)}{b_1\delta(1+\beta)} \right\}.$$

Proof. For convenience, we introduce the notation

$$\theta_1(t) = \Big(Q\big(x(t)\big)\Big)' = Q'\big(x(t)\big)x'(t), \quad \theta_2(t) = \Big(R\big(x(t)\big)\Big)' = R'\big(x(t)\big)x'(t),$$

and

$$X(t) = x(t) + \beta x(t-\tau).$$

Then,

$$X'(t) = Y(t) = y(t) + \beta y(t-\tau) \quad \text{and} \quad X''(t) = Z(t) = z(t) + \beta z(t-\tau).$$

We will write equation (1.1) as the equivalent system:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \\ Z'(t) &= -a(t)Q(x)z(t) - a(t)Q'(x)y^2(t) - b(t)R(x)y(t) \\ &\qquad - c(t)f\big(x(t)\big) + c(t)\int\limits_{t-\tau}^{t} f'\big(x(s)\big)y(s)\,\mathrm{d}s. \end{aligned} \tag{2.1}$$

Define a Lyapunov functional $U(t,x,y,Z)$ such that $U(t,0) = 0$ and

$$U = \exp\left(-\frac{1}{\kappa}\int\limits_{t_1}^{t}\big(|\theta_1(s)| + |\theta_2(s)|\big)\,\mathrm{d}s\right)V, \tag{2.2}$$

where

$$V = V_0 + V_1 + \mu\int\limits_{t-\tau}^{t} z^2(s)\,\mathrm{d}s + \sigma\int\limits_{t-\tau}^{t} y^2(s)\,\mathrm{d}s + \lambda\int\limits_{-\tau}^{0}\int\limits_{t+s}^{t} y^2(u)\,\mathrm{d}u\,\mathrm{d}s, \tag{2.3}$$

and

$$\begin{aligned} V_0 &= dc(t)F(x) + c(t)Y(t)f(x) + \frac{b(t)R(x)}{2}Y^2(t), \\ V_1 &= \frac{1}{2}\;Z^2(t) + dyZ(t) + \frac{1}{2}da(t)Q(x)y^2, \\ F(x) &= \int\limits_{0}^{x} f(u)\,\mathrm{d}u, \end{aligned}$$

and κ, μ, σ and λ are constants to be suitably selected below.

First, we shall show that $V(t)$ defined by (2.3) is positive definite. From (H_1) and (H_3), we have

$$\begin{aligned} V_1 &= \frac{1}{2}\left(Z^2 + 2dyZ + da(t)Q(x)y^2\right) \\ &= \frac{1}{2}\Big((Z+dy)^2 + dy^2\big(a(t)Q(x) - d\big)\Big) = V_{11}. \end{aligned}$$

In the same way, it follows that

$$V_1 = \frac{da(t)Q(x)}{2}\left(y + \frac{1}{a(t)Q(x)}Z\right)^2 + \frac{1}{2}\left(\frac{a(t)Q(x)-d}{a(t)Q(x)}\right)Z^2 = V_{12}.$$

Then

$$\begin{aligned}
V_1 &= \frac{1}{2}V_{11} + \frac{1}{2}V_{12} \\
&= \frac{1}{4}(Z+dy)^2 + \frac{1}{4}da(t)Q(x)\left(y + \frac{1}{a(t)Q(x)}Z\right)^2 \\
&\quad + \frac{1}{4}d\big(a(t)Q(x)-d\big)y^2 + \frac{1}{4a(t)Q(x)}\big(a(t)Q(x)-d\big)Z^2 \\
&\geq \frac{d(a_0q_0-d)}{4}y^2 + \frac{(a_0q_0-d)}{4a_1q_1}Z^2.
\end{aligned}$$

From this inequality we see that there is a positive constant k_0 such that

$$V_1 \geq k_0(y^2+Z^2),$$

where $k_0 = \min\left\{\frac{d}{4}(a_0q_0-d),\ \frac{1}{4a_1q_1}(a_0q_0-d)\right\}$. From (H_1) and (H_3), we obtain

$$\begin{aligned}
V_0 &= dc(t)F(x) + \frac{b(t)R(x)}{2}Y^2 + \frac{c(t)}{2}\big(Y+f(x)\big)^2 - \frac{c(t)}{2}Y^2 - \frac{c(t)}{2}f^2(x) \\
&\geq dc(t)\int_0^x\left(1-\frac{f'(u)}{d}\right)f(u)\,\mathrm{d}u + \frac{b(t)}{2}\left(R(x)-\frac{c(t)}{b(t)}\right)Y^2 \\
&\geq dc(t)\int_0^x\left(1-\frac{\delta}{d}\right)f(u)\,\mathrm{d}u + \frac{c_0}{2}(r_0-1)Y^2 \\
&\geq \delta_1F(x) + \frac{c_0}{2}(r_0-1)Y^2,
\end{aligned}$$

where $\delta_1 = dc_0\big(1-\frac{\delta}{d}\big)$. Observe that by (H_4), we have

$$\frac{f^2(x)}{x^2} \geq M^2,$$

which implies that

$$F(x) \geq \frac{1}{2\delta}f^2(x) \geq \frac{M^2}{2\delta}x^2(t).$$

Since

$$\sigma\int_{t-\tau}^t y^2(s)\,\mathrm{d}s + \mu\int_{t-\tau}^t z^2(s)\,\mathrm{d}s + \lambda\int_{-\tau}^0\int_{t+s}^t y^2(u)\,\mathrm{d}u\,\mathrm{d}s > 0,$$

it follows that

$$V \geq k_1(Z^2+y^2+x^2+Y^2), \tag{2.4}$$

where $k_1 = \min\{k_0, \frac{M^2\delta_1}{2\delta}, \frac{c_0}{2}(r_0-1)\}$. By ($H_7$), we have

$$U \geq k_2(Z^2 + y^2 + x^2 + Y^2) \tag{2.5}$$

for some constant $k_2 > 0$. It is not difficult to see that

$$W(x, y, Z) = k_2(Z^2 + y^2 + x^2 + Y^2) = 0 \quad \text{if and only if} \quad x = y = Z = 0, \tag{2.6}$$

and

$$U \geq k_2(Z^2 + y^2 + x^2 + Y^2) = W(x, y, Z) > 0 \quad \text{if} \quad (x, y, Z) \neq 0. \tag{2.7}$$

The derivative of the functional V along the trajectories of the system (2.1) is given by

$$\begin{aligned}
V' = {} & H(t, x, y) + \frac{1}{2}da'(t)Q(x)y^2 + \beta c(t)yy(t-\tau)f'(x) + b(t)\beta R(x)y(t-\tau)z \\
& + b(t)\beta^2 R(x)y(t-\tau)z(t-\tau) - \sigma y^2(t-\tau) \\
& - db(t)R(x)y^2 + c(t)y^2 f'(x) + \sigma y^2 + \lambda\tau y^2 \\
& + \big(d - a(t)Q(x)\big)z^2(t) + \mu z^2(t) + \beta\big(d - a(t)Q(x)\big)zz(t-\tau) - \mu z^2(t-\tau) \\
& - \lambda\int_{t-\tau}^{t} y^2(s)\,\mathrm{d}s + c(t)\big(z + \beta z(t-\tau) + dy\big)\int_{t-\tau}^{t} f'\big(x(s)\big)y(s)\,\mathrm{d}s + \psi(Y, Z),
\end{aligned}$$

where

$$\psi(Y, Z) = \frac{b(t)}{2}\theta_2(t)Y^2 - \frac{1}{2}da(t)\theta_1(t)y^2 - a(t)\theta_1(t)yZ$$

and

$$H(t, x, y) = dc'(t)F(x) + c'(t)Yf(x) + \frac{b'(t)R(x)}{2}Y^2.$$

Notice that

$$\begin{aligned}
\psi(Y, Z) &\leq \frac{b_1}{2}|\theta_2(t)|Y^2 + \frac{a_1}{2}|\theta_1(t)|(1+d)(y^2 + Z^2) \\
&\leq \omega\big(|\theta_1(t)| + |\theta_2(t)|\big)(y^2 + Y^2 + Z^2),
\end{aligned}$$

with $\omega = \frac{1}{2}\max\{b_1, a_1(1+d)\}$.

If $c'(t) = 0$, then $H(t, x, y) = \frac{b'(t)R(x)}{2}Y^2 \leq 0$. If $c'(t) < 0$, then $H(t, x, y)$ can be written as

$$H(t, x, y) = dc'(t)H_1(t, x, y),$$

where

$$H_1(t, x, y) = \left[F(x) + \frac{b'(t)R(x)}{2dc'(t)}\left\{Y + \frac{c'(t)}{b'(t)R(x)}f(x)\right\}^2 - \frac{c'(t)}{2db'(t)R(x)}f^2(x)\right].$$

From (H_3), we have $0 < \frac{c'(t)}{b'(t)} \le 1$, so

$$H_1(t,x,y) \ge F(x) - \frac{1}{2d} f^2(x) \ge \int_0^x \left(1 - \frac{\delta}{d}\right) f(u)\,\mathrm{d}u \ge \frac{\delta_1}{dc_0} \int_0^x f(u)\,\mathrm{d}u \ge 0.$$

It follows immediately that

$$H(t,x,y) = dc'(t) H_1(t,x,y) \le 0.$$

Hence, on combining the two cases for $c'(t)$, we have $H(t,x,y) \le 0$ for all $t \ge 0$, x, and y.

From condition (H_4) and applying the fact that $2uv \le u^2 + v^2$, we obtain

$$z \int_{t-\tau}^{t} f'\big(x(s)\big) y(s)\,\mathrm{d}s \le \frac{\delta\tau}{2} z^2 + \frac{\delta}{2} \int_{t-\tau}^{t} y^2(s)\,\mathrm{d}s, \tag{2.8}$$

$$\beta z(t-\tau) \int_{t-\tau}^{t} f'\big(x(s)\big) y(s)\,\mathrm{d}s \le \frac{\beta\delta\tau}{2} z^2(t-\tau) + \frac{\delta\beta}{2} \int_{t-\tau}^{t} y^2(s)\,\mathrm{d}s, \tag{2.9}$$

and

$$dy \int_{t-\tau}^{t} f'\big(x(s)\big) y(s)\,\mathrm{d}s \le \frac{\delta\tau}{2} dy^2 + \frac{\delta d}{2} \int_{t-\tau}^{t} y^2(s)\,\mathrm{d}s. \tag{2.10}$$

Applying conditions (H_1) and (H_3) and using (2.8)–(2.10), we have

$$\begin{aligned}
V' \le{} & \left(\frac{1}{2} da'(t) Q(x) - b(t)\left(dR(x) - \delta\left(1 + \frac{\beta}{2}\right)\frac{c(t)}{b(t)}\right) + \sigma + \frac{d\delta\tau}{2} b_1 + \lambda\tau\right) y^2(t) \\
& + \left(\mu - \frac{(2-\beta)(a_0 q_0 - d) - \beta b_1 r_1}{2} + \frac{\delta\tau}{2} b_1\right) z^2(t) \\
& + \left(\frac{b_1 \beta r_1}{2}(1+\beta) + \frac{\delta\beta b_1}{2} - \sigma\right) y^2(t-\tau) \\
& + \left(\frac{\beta(a_1 q_1 - d) + b_1 \beta^2 r_1}{2} - \mu + \beta\frac{\delta\tau}{2} b_1\right) z^2(t-\tau) \\
& + \left(\frac{\delta}{2} b_1 + \beta\frac{\delta}{2} b_1 + \frac{d\delta}{2} b_1 - \lambda\right) \int_{t-\tau}^{t} y^2(s)\,\mathrm{d}s \\
& + \omega\big(|\theta_1(t)| + |\theta_2(t)|\big)\big(y^2 + Y^2 + Z^2\big).
\end{aligned}$$

Let
$$\mu = \frac{\beta(a_1q_1 - d) + b_1r_1\beta^2 + \beta\delta\tau b_1}{2}, \qquad \lambda = \frac{\delta b_1}{2}(1+\beta+d)$$
and
$$\sigma = \frac{b_1\beta}{2}(r_1 + \beta r_1 + \delta).$$
Then,
$$\begin{aligned}
V' \le & \left(\frac{1}{2}da'(t)Q(x) - c_0\left(d - \left(1+\frac{\beta}{2}\right)\delta\right)\right.\\
&\left. + \frac{b_1\beta}{2}(r_1+\beta r_1+\delta) + \frac{b_1\delta\tau}{2}(1+\beta+2d)\right)y^2(t)\\
&+\frac{1}{2}\Big(\beta(a_1q_1-d) + b_1\beta r_1(1+\beta) - (2-\beta)(a_0q_0-d) + b_1\delta\tau(1+\beta)\Big)z^2(t)\\
&+\omega\Big(|\theta_1(t)| + |\theta_2(t)|\Big)\left(y^2+Y^2+Z^2\right)\\
\le & \frac{1}{2}\Big(\beta(a_1q_1-d) + b_1\beta r_1(1+\beta) - (2-\beta)(a_0q_0-d) + b_1\delta\tau(1+\beta)\Big)z^2(t)\\
&+\left(-\eta + \frac{b_1\delta\tau}{2}(1+\beta+2d)\right)y^2(t) + \omega\Big(|\theta_1(t)| + |\theta_2(t)|\Big)\left(y^2+Y^2+Z^2\right).
\end{aligned}$$
From (2.4), (2.2), and taking $\frac{1}{\kappa} = \frac{\omega}{k_1}$, we see that
$$\begin{aligned}
\frac{d}{dt}U &= \exp\left(-\frac{\omega}{k_1}\int_{t_1}^{t}(|\theta_1(s)| + |\theta_2(s)|)\,\mathrm{d}s\right)\left(\frac{d}{dt}V - \frac{\omega(|\theta_1(t)|+|\theta_2(t)|)}{k_1}V\right)\\
&\le \frac{1}{2}\Big(\beta(a_1q_1-d) + b_1\beta r_1(1+\beta) - (2-\beta)(a_0q_0-d) + b_1\delta\tau(1+\beta)\Big)z^2(t)\\
&\quad+\left(-\eta + \frac{b_1\delta\tau}{2}(1+\beta+2d)\right)y^2(t).
\end{aligned}$$
Therefore, from (H_5) and (H_6) there exists a positive constant N such that
$$U' \le -N\left(y^2(t) + z^2(t)\right), \tag{2.11}$$
provided that
$$\tau < \min\left\{\frac{2\eta}{b_1\delta(1+\beta+2d)},\ \frac{(2-\beta)(a_0q_0-d) - \beta(a_1q_1-d) - b_1\beta r_1(1+\beta)}{b_1\delta(1+\beta)}\right\}.$$
Finally, it follows that
$$U'(x,y,z) = 0 \quad \text{if and only if} \quad x=y=z=0, \tag{2.12}$$
and
$$U'(x,y,z) < 0 \qquad \text{for} \qquad (x,y,z) \ne 0. \tag{2.13}$$

From the properties of the Lyapunov function U, namely (2.5), (2.6), (2.7), (2.12), and (2.13), we see that the zero solution of the system (2.1) is uniformly asymptotically stable (see [7], [18]), and this completes the proof of the theorem. $\square$

3. Boundedness of solutions of (1.2)

To study the boundedness of solutions of the forced equation (1.2), we will write it as the system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \\ Z'(t) &= -a(t)Q(x)z(t) - a(t)Q'(x)y^2(t) - b(t)R(x)y(t) \\ &\quad - c(t)f\big(x(t)\big) + h(t) + c(t)\int_{t-\tau}^{t} f'\big(x(s)\big)y(s)\,\mathrm{d}s. \end{aligned} \tag{3.1}$$

Our main theorem in this section is as follows.

Theorem 3.1. *Assume that the conditions of Theorem 2.1 are satisfied and there is a positive constant D_1 such that*

(I$_1$) $\displaystyle\int_{t_1}^{t} |h(s)|\,\mathrm{d}s < D_1.$

Then there exists a positive constant D such that any solution $x(t)$ of (3.1) satisfies

$$|x(t)| \le D, \quad |y(t)| \le D, \quad \text{and} \quad |Z(t)| \le D. \tag{3.2}$$

Proof. Differentiating (2.3) along the solutions of system (3.1), we obtain

$$V'_{(3.1)} = V'_{(2.1)} + h(t)(dy + Z).$$

Since $V'_{(2.1)} \le 0$, it follows that

$$V'_{(3.1)} \le K_2|h(t)|(|y| + |Z|),$$

where $K_2 = \max\{d, 1\}$. Now the inequality (2.4) and the fact that $|p| \le p^2 + 1$ imply

$$\begin{aligned} V'_{(3.1)} &\le K_2\,|h(t)|\,(y^2 + Z^2 + 2) \\ &\le K_2\,|h(t)|\ V(t) + 2K_2\,|h(t)|\,. \end{aligned} \tag{3.3}$$

Integrating from $t_1 = t_0 + \tau$ to t, we obtain

$$V(t) - V(t_1) \le 2K_2\int_{t_1}^{t} |h(s)|\ \mathrm{d}s + K_2\int_{t_1}^{t} V(s)\,|h(s)|\ \mathrm{d}s,$$

or

$$V(t) \leq V(t_1) + 2K_2 D_1 + K_2 \int_{t_1}^{t} V(s)\,|h(s)|\,\mathrm{d}s.$$

Applying Gronwall's inequality, it follows that

$$V(t) \leq \big(V(t_1) + 2K_2 D_1\big) \exp\left(K_2 \int_{t_1}^{t} |h(s)|\,\mathrm{d}s \right) \leq D_2, \tag{3.4}$$

i.e., $V(t)$ is bounded. In view of (2.4), this implies the conclusion of the theorem holds. □

Corollary 3.2. *Under the conditions of Theorem 2.1, the zero solution of equation* (1.1) *is globally uniformly asymptotically stable.*

P r o o f. By Theorem 2.1, the zero solution of equation (1.1) is uniformly asymptotically stable, and by Theorem 3.1, all solutions are bounded. The conclusion then follows by the well-known LaSalle's invariance principle. □

4. Square integrability of solutions

In this section, we are concerned with the square integrability of solutions of equation (1.2). Our main result in this direction is contained in the following theorem.

Theorem 4.1. *In addition to the assumptions of Theorem 3.1, assume that*

(I_2) $c_0 M - \dfrac{b_1 r_1}{2} > 0$;

(I_3) $\displaystyle\int_{t_1}^{+\infty} |a'(s)|\,\mathrm{d}s < A$.

Then all solutions of equation (1.2) *and their derivatives belong to* $L^2[t_1, +\infty)$.

P r o o f. Define W(t) by

$$W(t) = U(t) + \varepsilon \int_{t_1}^{t} \big(z^2(s) + y^2(s)\big)\,\mathrm{d}s, \tag{4.1}$$

where $\varepsilon > 0$ is a constant to be specified later. By differentiating W(t) and using (2.11), (3.3), and the fact that

$$\exp\left(-\frac{1}{\kappa} \int_{t_1}^{t} (|\theta_1(s)| + |\theta_2(s)|)\,\mathrm{d}s \right) \leq 1,$$

we obtain

$$W'(t) \le (\varepsilon - N)\big(z^2(t) + y^2(t)\big) + \big(K_2 V(t) + 2K_2\big)|h(t)|.$$

Choosing $\varepsilon < N$, from (3.4) we obtain

$$W'(t) \le K_4\,|h(t)|\,, \tag{4.2}$$

where $K_4 = K_2 D_2 + 2K_2$. Integrating (4.2) from $t_1 = t_0 + \tau$ to t and using condition (I_1) of Theorem 3.1, we have

$$W(t) - W(t_1) = \int_{t_1}^{t} W'(s)\,\mathrm{d}s \le K_4 D_1.$$

Now,

$$U(t_1) = W(t_1),$$

so

$$W(t) \le K_4 D_1 + U(t_1).$$

Hence, by (4.1),

$$\int_{t_1}^{t} \big(y^2(s) + z^2(s)\big)\,\mathrm{d}s < \frac{K_4 D_1 + U(t_1)}{\varepsilon},$$

which implies the existence of positive constants σ_1 and σ_2 such that

$$\int_{t_1}^{t} x''^2(s)\,\mathrm{d}s = \int_{t_1}^{t} z^2(s)\,\mathrm{d}s \le \sigma_2$$

and

$$\int_{t_1}^{t} x'^2(s)\,\mathrm{d}s = \int_{t_1}^{t} y^2(s)\,\mathrm{d}s \le \sigma_1.$$

To prove that $\int_{t_1}^{t} x^2(s)\,\mathrm{d}s < \infty$, multiply (1.2) by $x(t-\tau)$ to obtain

$$\begin{aligned} x(t-\tau)x'''(t) &+ \beta x(t-\tau)x'''(t-\tau) \\ &+ a(t)Q(x)x(t-\tau)\,x''(t) + a(t)Q'(x)x(t-\tau)x'^2(t) \\ &+ b(t)R(x)x(t-\tau)\,x'(t) + c(t)x(t-\tau)f\big(x(t-\tau)\big) \\ &\qquad\qquad\qquad\qquad = x(t-\tau)h(t). \end{aligned} \tag{4.3}$$

Integrating (4.3) from t_1 to t, we have

$$\int_{t_1}^{t} c(s)x(s-\tau)f\big(x(s-\tau)\big)\,\mathrm{d}s = L_1(t) + L_2(t) + L_3(t), \tag{4.4}$$

where

$$L_1(t) = -\int_{t_1}^{t}\big(x(s-\tau)x'''(s) + \beta x(s-\tau)x'''(s-\tau)\big)\,\mathrm{d}s,$$

$$L_2(t) = -\int_{t_1}^{t}\Big(a(s)Q\big(x(s)\big)x(s-\tau)x''(s) + a(s)Q'\big(x(s)\big)x'^2(s)x(s-\tau)$$

$$+b(s)R(x(s))x(s-\tau)x'(s)\Big)ds,$$

$$L_3(t) = \int_{t_1}^{t} h(s)x(s-\tau)\,\mathrm{d}s.$$

Integrating by parts

$$\begin{aligned} L_1(t) &= M_1(t) - M_1(t_1) + \int_{t_1}^{t} x'(s-\tau)x''(s)\,\mathrm{d}s \\ &\le |M_1(t) - M_1(t_1)| + \int_{t_1}^{t} \frac{1}{2}\big(x'^2(s-\tau) + x''^2(s)\big)\,\mathrm{d}s, \end{aligned}$$

where

$$M_1(t) = -x(t-\tau)X''(t) + \frac{\beta}{2}x'^2(t-\tau).$$

Now,

$$\int_{t_1}^{t} x'^2(s-\tau)\,\mathrm{d}s = \int_{t_0}^{t-\tau} x'^2(u)\,\mathrm{d}u \le \int_{t_0}^{t_1} x'^2(u)\,\mathrm{d}u + \sigma_1 \le \sigma_3 + \sigma_1 \quad \text{for some } \sigma_3 > 0.$$

In view of (3.2), we see that

$$|M_1(t) - M_1(t_1)| \le D^2\left(\frac{3\beta}{2} + 1\right) + |M_1(t_1)| \quad \text{for all } t \ge t_1.$$

Thus,

$$L_1(t) \le D^2\left(\frac{3\beta}{2} + 1\right) + |M_1(t_1)| + \frac{1}{2}(n + \sigma_1 + \sigma_2) = l_1.$$

Similarly we have

$$
\begin{aligned}
L_2(t) = -\int_{t_1}^{t} \Big(& a(s)Q\big(x(s)\big)x(s-\tau)x''(s) + a(s)Q'\big(x(s)\big)x'^2(s)x(s-\tau) \\
& + b(s)R\big(x(s)\big)x(s-\tau)x'(s)\Big)\,\mathrm{d}s \\
= & -a(t)Q\big(x(t)\big)x(t-\tau)x'(t) + a(t)Q\big(x(t)\big)\int_{t_1}^{t} x'(s)x'(s-\tau)\,\mathrm{d}s \\
& + \int_{t_1}^{t} a'(s)Q\big(x(s)\big)x(s-\tau)x'(s)\,\mathrm{d}s \\
& - \int_{t_1}^{t} a'(s)Q\big(x(s)\big)\left[\int_{t_1}^{s} x'(u)x'(u-\tau)\,\mathrm{d}u\right]\mathrm{d}s \\
& + \int_{t_1}^{t} a(s)Q'\big(x(s)\big)x(s-\tau)x'^2(s)\,\mathrm{d}s \\
& - \int_{t_1}^{t} a(s)Q'\big(x(s)\big)x'(s)\left[\int_{t_1}^{s} x'(u)x'(u-\tau)\,du\right]\mathrm{d}s \\
& - \int_{t_1}^{t} b(s)R\big(x(s)\big)x(s-\tau)x'(s)\,\mathrm{d}s + M_2(t_1),
\end{aligned}
$$

where $M_2(t_1) = a(t_1)Q\big(x(t_1)\big)x(t_1-\tau)x'(t_1)$. Then

$$
\begin{aligned}
L_2(t) \le q_1\int_{t_1}^{t} & \left(|a'(s)||x'(s)||x(s-\tau)| + |a'(s)|\left[\int_{t_1}^{s} x'(u)x'(u-\tau)\,\mathrm{d}u\right]\right)\mathrm{d}s \\
& + a_1\int_{t_1}^{t} \Big(|Q'\big(x(s)\big)x'(s)||x'(s)||x(s-\tau)| \\
& + |Q'\big(x(s)\big)x'(s)|\left[\int_{t_1}^{s} x'(u)x'(u-\tau)\,\mathrm{d}u\right]\Big)\,\mathrm{d}s + \frac{b_1r_1}{2}\int_{t_1}^{t} x^2(s-\tau)\,\mathrm{d}s \\
& + \frac{b_1r_1}{2}\int_{t_1}^{t} x'^2(s)\,\mathrm{d}s + |M_2(t_1)| + a_1q_1\left(D^2 + \sigma_1 + \frac{n}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le a_1 q_1\left(D^2+\sigma_1+\frac{n}{2}\right)+|M_2(t_1)|+q_1\left(D^2+\sigma_1+\frac{n}{2}\right)\int_{t_1}^{t}|a'(s)|\,\mathrm{d}s\\
&\quad + a_1\left(D^2+\sigma_1+\frac{n}{2}\right)\int_{x(t_1)}^{x(t)}|Q'(u)|\,\mathrm{d}u+\frac{b_1 r_1}{2}\int_{t_1}^{t}x^2(s-\tau)\,\mathrm{d}s+\frac{b_1 r_1}{2}\sigma_1\\
&\le a_1 q_1\left(D^2+\sigma_1+\frac{n}{2}\right)+|M_2(t_1)|+q_1\left(D^2+\sigma_1+\frac{n}{2}\right)A\\
&\quad + a_1\left(D^2+\sigma_1+\frac{n}{2}\right)J+\frac{b_1 r_1}{2}\sigma_1+\frac{b_1 r_1}{2}\int_{t_1}^{t}x^2(s-\tau)\,\mathrm{d}s.
\end{aligned}
$$

Also,

$$
L_3(t)\le\int_{t_1}^{t}|x(s-\tau)|\,|h(s)|\,\mathrm{d}s\le D\int_{t_1}^{t}h(s)\,\mathrm{d}s\le D_1 D.
$$

From (4.4) and condition (I_2), we obtain

$$
\begin{aligned}
c_0 M\int_{t_1}^{t}x^2(s-\tau)\,\mathrm{d}s &\le \int_{t_1}^{t}c(s)x(s-\tau)f\big(x(s-\tau)\big)\,\mathrm{d}s\\
&\le K+\frac{b_1 r_1}{2}\int_{t_1}^{t}x^2(s-\tau)\,\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
K = l_1 &+ \left(D^2+\sigma_1+\frac{n}{2}\right)(a_1 q_1+q_1 A+a_1 J)\\
&+|M_2(t_1)|+\frac{b_1 r_1}{2}\sigma_1+D_1 D.
\end{aligned}
$$

Observe that

$$
\left(c_0 M-\frac{b_1 r_1}{2}\right)\int_{t_1}^{t}x^2(s-\tau)\,\mathrm{d}s\le K,
$$

from which it follows that

$$
\int_{t_1}^{t}x^2(s-\tau)\,\mathrm{d}s<\infty \qquad \text{for all} \quad t\ge t_1.
$$

Hence, $\int_{t_1}^{+\infty}x^2(s)\,\mathrm{d}s<\infty$, and this completes the proof of theorem. $\square$

5. Example

As an example of our results, consider the forced third order non-autonomous delay neutral differential equation

$$\begin{aligned}\left[x(t)+\frac{1}{10}x(t-\tau)\right]''' &+ \left(\frac{1}{\pi}\arctan t+\frac{13}{2}\right)\left(\left(2-\frac{1}{1+x^2}\right)x'\right)' \\ &+ \left(\frac{1}{2+t^2}+1\right)\left(\left(1+\frac{1}{3+x^2}\right)x'\right) \\ &+ \left(\frac{1}{4+t^2}+1\right)\left(\frac{3}{2}x(t-\tau)+\frac{x(t-\tau)}{1+x^2(t-\tau)}\right)=\frac{\sin t}{1+t^2}.\end{aligned} \tag{5.1}$$

It is easy to see that for all $t \geq t_1$:

$$6=a_0\leq a(t)=\frac{1}{\pi}\arctan t+\frac{13}{2}\leq 7=a_1,$$

$$a'(t)=\frac{1}{\pi}\frac{1}{1+t^2}\leq\frac{1}{\pi},$$

$$1=c_0\leq c(t)=\frac{1}{4+t^2}+1\leq b(t)=\frac{1}{2+t^2}+1\leq\frac{3}{2}=b_1,$$

$$1=q_0\leq Q(x)=2-\frac{1}{1+x^2}\leq 2=q_1,$$

$$1\leq R(x)=1+\frac{1}{3+x^2}\leq\frac{4}{3}=r_1,$$

$$\frac{3}{2}=M\leq\frac{f(x)}{x}=\frac{3}{2}+\frac{1}{1+x^2}\quad\text{with}\quad x\neq 0\quad\text{and}\quad |f'(x)|\leq\frac{5}{2}=\delta,$$

$$\delta\left(1+\frac{\beta}{2}\right)=\frac{105}{40}<d<6=a_0q_0\quad\text{for}\quad\beta=\frac{1}{10}\quad\text{and}\quad d=5,$$

$$c_0M-\frac{b_1r_1}{2}=\frac{3}{2}-1>0,$$

$$\frac{1}{2}da'(t)Q(x)-c_0(d-\left(1+\frac{\beta}{2}\right)\delta)+\frac{b_1\beta}{2}(r_1+r_1\beta+\delta)\leq-\frac{48}{100}<0\quad\text{for}\quad d=5,$$

$$\beta(a_1q_1-d)+b_1\beta r_1(1+\beta)-(2-\beta)(a_0q_0-d)\leq-\frac{39}{50}<0\quad\text{for}\quad d=5,$$

$$\int_{t_1}^{+\infty}|a'(s)|\,\mathrm{d}s=\frac{1}{\pi}\int_{t_1}^{+\infty}\frac{1}{1+s^2}<+\infty,$$

$$\int_{t_1}^{+\infty}|h(s)|\,\mathrm{d}s\leq\int_{t_1}^{+\infty}\frac{1}{1+s^2}<+\infty,$$

and

$$\int_{-\infty}^{+\infty} (|Q'(u)| + |R'(u)|)\, \mathrm{d}u \le J < +\infty.$$

All the conditions of Theorem 4.1 are satisfied, so every solution of (5.1) and their derivatives are bounded and belong to $L^2[t_1, +\infty)$. In addition, if $h(t) \equiv 0$, then the zero solution of (5.1) is uniformly asymptotically stable.

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