sciendo

TATRA MOUNTAINS
Mathematical Publications

DOI: 10.2478/tmmp-2018-0012
Tatra Mt. Math. Publ. **71** (2018), 139–148

NEUTRAL DIFFERENCE SYSTEM AND ITS NONOSCILLATORY SOLUTIONS

Jana Pasáčková

University of Economics, Prague, CZECH REPUBLIC

ABSTRACT. The paper deals with a system of four nonlinear difference equations where the first equation is of a neutral type. We study nonoscillatory solutions of the system and we present sufficient conditions for the system to have weak property B.

1. Introduction

In this paper, we study a four-dimensional system of this form

$$\begin{aligned} \Delta\left(x_n + p_n x_{n-\sigma}\right) &= A_n\, f_1\left(y_n\right), \\ \Delta y_n &= B_n\, f_2\left(z_n\right), \\ \Delta z_n &= C_n\, f_3\left(w_n\right), \\ \Delta w_n &= D_n\, f_4\left(x_{\gamma_n}\right), \end{aligned} \tag{S}$$

where $n \in \mathbb{N}_0 = \{n_0, n_0+1, \dots\}$, n_0 is a positive integer, σ is a nonnegative integer, $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ are positive real sequences defined for $n \in \mathbb{N}_0$. Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$.

The sequence $\gamma\colon \mathbb{N} \to \mathbb{N}$ satisfies

$$\lim_{n\to\infty} \gamma_n = \infty. \tag{H1}$$

The sequence $\{p_n\}$ is a sequence of the real numbers and it satisfies

$$\lim_{n\to\infty} p_n = P, \qquad \text{where} \quad |P| < 1. \tag{H2}$$

2010 Mathematics Subject Classification: 39A10.
Keywords: difference system, nonoscillatory solutions, neutral term, property B, weak property B.
This paper was processed with contribution of long term institutional support of research activities by Faculty of Informatics and Statistics, University of Economics, Prague.

Functions $f_i \colon \mathbb{R} \to \mathbb{R}$ for $i = 1, \dots, 4$ satisfy

$$\frac{f_i(u)}{u} \geq M, \qquad u \in \mathbb{R}\backslash\{0\}, \quad M \in \mathbb{R} \quad \text{and} \quad M > 0. \tag{H3}$$

By a solution of the system (S) we mean a vector sequence (x, y, z, w) which satisfies the system (S) for $n \in \mathbb{N}_0$. We study nonoscillatory solutions. Therefore, the first important terms are oscillatory and nonoscillatory solutions. The component x is said to be **nonoscillatory** if there exists $n_1 \geq n_0$ such that $x_n \geq 0$ (respectively $x_n \leq 0$) for all $n \geq n_1$. The component x is said to be **oscillatory** if for any $n_1 \geq n_0$ there exists $n \geq n_1$ such that $x_{n+1}x_n < 0$. A solution of system (S) is said to be nonoscillatory (respectively oscillatory) if all of its components x, y, z, w are nonoscillatory (respectively oscillatory).

Definition 1. The system (S) has **weak property B** if every nonoscillatory solution of (S) satisfies

$$x_n z_n > 0 \quad \text{and} \quad y_n w_n > 0 \qquad \text{for large } n. \tag{1}$$

Definition 2. The system (S) has **property B** if any of its solutions is either oscillatory or either satisfies

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |y_n| = \lim_{n\to\infty} |z_n| = \lim_{n\to\infty} |w_n| = \infty, \tag{2}$$

or

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = 0. \tag{3}$$

Strongly monotone solutions are solutions satisfying (1) and $x_n y_n > 0$. While solutions satisfying (1) and $x_n y_n < 0$ are called *Kneser solutions.* Property B is defined in accordance with those for the higher-order differential equations or for the system of differential equations, see [9] and references therein.

2. Summary of previous results

This paper continues in our previous research, see [7], [8]. We studied the weak property B and the property B in both papers.

In [7], we investigate the system (S), where $0 \leq p_n < 1$ and

$$\sum_{n=n_0}^{\infty} A_n = \infty, \quad \sum_{n=n_0}^{\infty} B_n = \infty, \quad \sum_{n=n_0}^{\infty} C_n = \infty. \tag{CF}$$

We say that the system (S) is in the *canonical form* when conditions (CF) are satisfied. In [7], we established sufficient conditions for the system in the canonical form to have property B. The main theorem from this paper is the following.

Theorem 1 ([7, Theorem 3]). *Assume* $\lim p_n = P$, $0 < P < 1$, (CF) *and*

$$\sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{\gamma_i-\sigma-1} A_j \left(\sum_{k=n_0}^{j-1} B_k \left(\sum_{l=n_0}^{k-1} C_l \right)\right)\right) = \infty. \tag{4}$$

In addition, if

$$\sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{i-1} C_j \right) = \infty \tag{5}$$

holds, then the system (S) *has property B.*

The conditions (CF) and (5) ensure that (S) has weak property B and the condition (4) helps to ensure that all nonoscillatory solutions satisfy the asymptotic properties.

In [8], we investigated oscillatory and nonoscillatory solutions of (S) and we established sufficient conditions for the system to have strongly monotone solutions or Kneser solutions. We found sufficient conditions for the system to have property B as well. Unlike [7], in [8] we studied (S) without any conditions for sequences $\{A_n\}, \{B_n\}$ and $\{C_n\}$ which led to the large number of conditions for (S) to have weak property B or property B. The main theorems are the following.

Theorem 2 ([8, Theorem 8]). *Let*

$$\sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{\gamma_i-\sigma-1} A_j \right) = \infty, \tag{6}$$

and

$$\sum_{i=n_0}^{\infty} B_i \left(\sum_{j=n_0}^{i-1} C_j \right) = \infty \tag{7}$$

hold. In addition, if

$$\sum_{i=n_0}^{\infty} B_i \left(\sum_{j=i}^{\infty} C_j \right) = \infty \tag{8}$$

and

$$\sum_{i=n_0}^{\infty} C_i \left(\sum_{j=n_0}^{i-1} D_j \right) = \infty \tag{9}$$

hold, then the system (S) *has weak property B.*

Theorem 3 ([8, Theorem 9]). *Let* (6)–(8) *and* (9) *hold. In addition, if*

$$\sum_{i=n_0}^{\infty} A_i \left(\sum_{j=n_0}^{i-1} B_j \right) = \infty \tag{10}$$

and

$$\sum_{i=n_0}^{\infty} A_i \left(\sum_{j=i}^{\infty} B_j \right) = \infty \tag{11}$$

hold, then the system (S) *has property B.*

Since we have found it very complicated, we try to find some simplification of these conditions. For example, if only $\sum A_n = \infty$. Thus, the aim of this paper is to extend results from these papers.

The first motivation for our previous papers was our paper "Nonoscillatory solutions of the four-dimensional difference system", see [2]. In this paper we investigated asymptotic properties of nonoscillatory solutions of the system (S*). It is the system (S) without a neutral term (i.e., $p_n = 0$), with power functions instead of functions f_i and with $\gamma_n = n + \tau$, which is the most common form of the sequence γ

$$\begin{aligned} \Delta x_n &= A_n\, y_n^{\frac{1}{\alpha}}, \\ \Delta y_n &= B_n\, z_n^{\frac{1}{\beta}}, \\ \Delta z_n &= C_n\, w_n^{\frac{1}{\gamma}}, \\ \Delta w_n &= D_n\, x_{n+\tau}^{\delta}. \end{aligned} \tag{S*}$$

By using the notation

$$A_n = a_n^{-\frac{1}{\alpha}} \quad B_n = b_n^{-\frac{1}{\beta}} \quad C_n = c_n^{-\frac{1}{\gamma}} \quad D_n = d_n,$$

the system (S*) can be written as a fourth-order nonlinear difference equation of the form

$$\Delta\left(c_n \left(\Delta\left(b_n \left(\Delta\big(a_n (\Delta x_n)^{\alpha} \big) \right)^{\beta} \right) \right)^{\gamma} \right) - d_n x_{n+\tau}^{\delta} = 0. \tag{E}$$

Thus, if functions f_i are invertible, then the system (S) can be easily rewritten as a fourth-order nonlinear neutral difference equation, similarly as (E). Equations with quasi-differences have been widely studied in the literature; for example see [3]–[6], [11], [12].

In [6], they studied property A which means that they investigated oscillatory properties of solutions of the fourth-order difference equations. Their approach is based on studying the four-dimensional difference system, where $\{D_n\}$ is a negative real sequence, instead of the considered equation.

Difference equations with a neutral term have been studied, for example, in [10], [13]. In [10], they established conditions under which for every real constant there exists a solution of considered equation convergent to this constant. The investigated equation is of the m-order, where $m \geq 2$.

In [13], they studied nonoscillatory solutions of a fourth-order nonlinear neutral difference equations of the form

$$\Delta\Big(a_n\Delta\Big(b_n\Delta\big(c_n\Delta(x_n - p_n x_{n-\tau})\big)\Big)\Big) + f(n, x_{n-\sigma}) = 0. \tag{12}$$

They defined an asymptotically zero solution as a minimal solution and asymptotically $Q_{n,N}$ solution as a maximal solution, where

$$Q_{n,N} = \sum_{k=N}^{n-1} \frac{1}{c_k} \sum_{j=N}^{k-1} \frac{1}{b_j} \sum_{i=N}^{j-1} \frac{1}{a_i}.$$

Then they have found the necessary and sufficient conditions for the equation to have a minimal and a maximal solution.

Theorem 4 ([13, Theorem 1]). *Assume that the operator of the difference equation* (12) *is in the canonical form and conditions*

$$xf(n,x) > 0 \quad \text{for all} \quad x \neq 0, \quad n \in N \quad \text{and} \quad \sum_{i=1}^{\infty} p_i < \infty \tag{13}$$

hold. Let

$$n \sum_{i=n}^{\infty} \frac{1}{c_i} \sum_{j=1}^{i-1} \frac{1}{b_j} \sum_{k=1}^{j-1} \frac{1}{a_k} \sum_{s=1}^{k-1} f\left(s, \frac{1}{s-\sigma}\right) < \infty$$

for $n > 3$. *Then* (12) *has an eventually positive solution* x_n *which converges to zero.*

Theorem 5 ([13, Theorem 3]). *Assume that the operator of the difference equation* (12) *is in the canonical form, conditions* (13) *hold and* f *is a nondecreasing function in the second argument. Then a necessary and sufficient condition for* (12) *to have solution* x_n *satisfying*

$$\lim \frac{x_n}{Q_{n,N}} = \beta \neq 0$$

is that

$$\sum_{n=1}^{\infty} |f(n, CQ_{n,N})| < \infty$$

for some integer $N \geq 1$ *and some nonzero constant* C.

Another direction which can be investigated are the bounded and unbounded solutions. A solution of a system is said to be bounded if all its components are bounded, otherwise, it is called unbounded. The bounded and unbounded solutions of a four-dimensional system with a neutral term have been studied in [1]. They investigated system (S) with $\gamma_n = n - \tau$. They presented sufficient conditions for solutions of the system to be bounded or unbounded. The conditions depend on the type of nonoscillatory solutions.

In proofs of our theorems, we use the change of summation which is described by the following remark.

Remark 1 (**Change of summation**)**.** Let $\{X_n\}$ and $\{Y_n\}$ be positive real sequences defined for $n \in \mathbb{N}_0$. Then

$$\sum_{i=n_0}^{\infty} X_i \left(\sum_{j=n_0}^{i} Y_j \right) = \infty \quad \text{if and only if} \quad \sum_{i=n_0}^{\infty} Y_i \left(\sum_{j=i}^{\infty} X_j \right) = \infty.$$

3. Nonoscillatory solutions and their asymptotic properties

The system (S) has property B if any of its nonoscillatory solutions satisfy special asymptotic properties. Therefore, we start with the classification of all possible types of nonoscillatory solutions. Throughout the paper, we can focus on solutions whose first component is eventually positive for large n. Since the system (S) has a solution (x, y, z, w), then it has the solution $(-x, -y, -z, -w)$ as well.

We use the notation

$$s_n = x_n + p_n x_{n-\sigma}, \tag{14}$$

where $n \in \mathbb{N}_0$.

The following lemma establishes the relation between the sequences $\{s_n\}$ and $\{x_n\}$. By (H2) and (14), the boundedness of x implies the boundedness of s. The opposite implication was proved in [8, Lemma 2] for $|P| < 1$. Therefore we present the following lemma without the proof.

Lemma 1. *Let $\{x_n\}$ be eventually positive sequence and $\{p_n\}$ satisfies* (H2)*, $n \in \mathbb{N}_0$. Let $\{s_n\}$ be the sequence defined by* (14)*. Then $\{x_n\}$ is bounded if and only if $\{s_n\}$ is bounded.*

If we assume nonoscillatory solutions with eventually positive x, then any nonoscillatory solution (x, y, z, w) of (S) is one of the following types:

$$\begin{array}{llllll}
\text{type (a)} & x_n > 0 & y_n > 0 & z_n > 0 & w_n > 0 & \text{for large } n, \\
\text{type (b)} & x_n > 0 & y_n > 0 & z_n > 0 & w_n < 0 & \text{for large } n, \\
\text{type (c)} & x_n > 0 & y_n < 0 & z_n > 0 & w_n < 0 & \text{for large } n, \\
\text{type (d)} & x_n > 0 & y_n < 0 & z_n < 0 & w_n < 0 & \text{for large } n, \\
\text{type (e)} & x_n > 0 & y_n > 0 & z_n < 0 & w_n < 0 & \text{for large } n, \\
\text{type (f)} & x_n > 0 & y_n > 0 & z_n < 0 & w_n > 0 & \text{for large } n, \\
\text{type (g)} & x_n > 0 & y_n < 0 & z_n < 0 & w_n > 0 & \text{for large } n, \\
\text{type (h)} & x_n > 0 & y_n < 0 & z_n > 0 & w_n > 0 & \text{for large } n.
\end{array}$$

If we assume some conditions that hold for sequences $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ and $\{D_n\}$, we find that some of the solutions cannot exist.

In the following, we assume that

$$\sum_{i=n_0}^{\infty} B_i < \infty \quad \text{and} \quad \sum_{i=n_0}^{\infty} D_i < \infty.$$

Lemma 2. *Assume*

$$\sum_{i=n_0}^{\infty} A_i = \infty. \tag{15}$$

Then any nonoscillatory solution (x, y, z, w) of the system (S) *with eventually positive x cannot be of type (d) or (g).*

P r o o f. Assume that (x, y, z, w) is a solution of type (d) or (g). Since $y_n < 0$ and $z_n < 0$ there exist $n_1 \in \mathbb{N}_0$ such that $y_n \le k < 0$ for $n \ge n_1 \ge n_0$. Using the summation of the first equation of (S) we get

$$s_n - s_{n_0} = \sum_{i=n_0}^{n-1} A_i f_1(y_i) \le M \sum_{i=n_0}^{n-1} A_i y_i \le Mk \sum_{i=n_0}^{n-1} A_i.$$

Passing $n \to \infty$ we get that $s_n \to -\infty$. Thus, s_n is unbounded, by Lemma 1, x_n is unbounded too. Since $x_n > 0$ and $y_n < 0$, x is positive and decreasing which gives a contradiction with the unboundedness of x. Therefore, the solution cannot be of type (d) or (g). □

Lemma 3. *Assume*

$$\sum_{i=n_0}^{\infty} B_i \left(\sum_{j=n_0}^{i-1} C_j \left(\sum_{k=j}^{\infty} D_k \right) \right) = \infty. \tag{16}$$

Then any nonoscillatory solution (x, y, z, w) of the system (S) *with eventually positive x cannot be of type (b), (e), (f), (g), (h).*

P r o o f. Since $\sum B$ and $\sum D$ are convergent, the condition (16) implies that

$$\sum_{j=n_0}^{\infty} C_j \left(\sum_{k=j}^{\infty} D_k \right) = \infty \quad \text{and} \quad \sum_{i=n_0}^{\infty} C_i = \infty.$$

Assume that there exist $n_1 \in \mathbb{N}_0$ and a solution such that $z_n < 0$, $w_n > 0$ for $n \geq n_1 \geq n_0$. From the fourth equation of (S) we have $\Delta w_n > 0$ and this implies that there exists $k > 0$ such that $w_n \geq k$ for large n. Using (H3) we have $f_3(w_n) \geq w_n \geq Mk$. By the summation of the third equation of (S) we have

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} C_i f_3 (w_i) \geq Mk \sum_{i=n_0}^{n-1} C_i. \tag{17}$$

Passing $n \to \infty$, we get a contradiction with the fact that $z_n < 0$. This excludes solutions of types (f) and (g).

Now, assume that (x, y, z, w) is a solution of type (h). Using the same argument and substituting (17) into the summation of the second equation we obtain

$$y_n - y_{n_0} = \sum_{i=n_0}^{n-1} B_i f_2 (z_i) \geq M \sum_{i=n_0}^{n-1} B_i z_i \geq M^2 k \sum_{i=n_0}^{n-1} B_i \left(\sum_{j=n_0}^{i-1} C_j \right).$$

Since $\sum C$ is divergent we get using the change of summation and by passing $n \to \infty$ a contradiction with a boundedness of y.

Assume that (x, y, z, w) is a solution of type (b). Since x is positive and increasing, there exists $n_2 \in \mathbb{N}_0$ such that $x_n \geq l > 0$ for $n \geq n_2 \geq n_0$. Using the summation of the fourth equation of (S) we get

$$w_\infty - w_n = \sum_{i=n}^{\infty} D_i f_4 (x_{\gamma_i}) \geq M \sum_{i=n}^{\infty} D_i x_{\gamma_i} \geq Ml \sum_{i=n}^{\infty} D_i.$$

Thus,

$$w_n \leq -Ml \sum_{i=n}^{\infty} D_i.$$

Substituting this into the summation of the third equation of (S) we obtain

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} C_i f_3 (w_i) \leq M \sum_{i=n_0}^{n-1} C_i w_i \leq -M^2 l \sum_{i=n_0}^{n-1} C_i \left(\sum_{j=i}^{\infty} D_j \right). \tag{18}$$

Passing $n \to \infty$ we get the contradiction with the boundedness of z.

Finally, assume that (x, y, z, w) is a solution of type (e). Using the substitution of (18) into the summation of the second equation we obtain

$$y_n - y_{n_0} = \sum_{i=n_0}^{n-1} B_i f_2(z_i) \leq M \sum_{i=n_0}^{n-1} B_i z_i \leq -M^3 l \sum_{i=n_0}^{n-1} B_i \left(\sum_{j=n_0}^{i-1} C_j \left(\sum_{k=j}^{\infty} D_k \right) \right).$$

This gives the contradiction with the boundedness of y. □

4. Weak property B

From the previous section, we get the following theorem.

Theorem 6. *Let the following conditions be satisfied:*

$$\sum_{i=n_0}^{\infty} B_i < \infty \quad \text{and} \quad \sum_{i=n_0}^{\infty} D_i < \infty,$$

and (15), (16) *hold, then the system* (S) *has weak property B.*

Proof. The conditions imply that solutions of type (b), (d), (e), (f), (g), (h) do not exist. □

Example 1. Assume the difference system

$$\begin{aligned} \Delta\left(x_n - \frac{1}{2}x_{n-1}\right) &= \frac{3}{4}n \cdot y_n\,, \\ \Delta y_n &= \frac{1}{n(n+1)} z_n\,, \\ \Delta z_n &= n(n+1) w_n\,, \\ \Delta w_n &= \frac{1}{n(n+1)} x_n \left(\log_2 x_n - 1\right). \end{aligned} \tag{E}$$

We have

$$A_n = \frac{3}{4}n, \qquad B_n = \frac{1}{n(n+1)},$$
$$C_n = n(n+1), \qquad D_n = \frac{1}{n(n+1)}.$$

Thus, $\sum B_n, \sum D_n$ are convergent. We can easily check that conditions (15), (16) are satisfied. Therefore, the system (E) has a weak property B. In fact, the solution is

$$(x, y, z, w) = \left(2^n, \frac{2^n}{n}, 2^n(n-1), \frac{2^n}{n}\right)$$

and it is the solution of type (a).

REFERENCES

[1] DIBLÍK, J.—ŁUPIŃSKA, B.—RŮŽIČKOVÁ, M.—ZONENBERG, J.: *Bounded and unbounded non-oscillatory solutions of a four-dimensional nonlinear neutral difference system,* Adv. Difference Equ. **319** (2015), 1–11.

[2] DOŠLÁ, Z.—KREJČOVÁ, J.: *Nonoscillatory solutions of the four-dimensional difference system,* Electron. J. Qual. Theory Differ. Equ. **4** (2011), 1–11.

[3] ______ *Oscillation of a class of the fourth-order nonlinear difference equations,* Adv. Difference Equ. **99** (2012), 1–14.

[4] ______ *Asymptotic and oscillatory properties of the fourth-order nonlinear difference equations,* Appl. Math. Comput. **249** (2014), 164–173.

[5] DOŠLÁ, Z.—KREJČOVÁ, J.—MARINI, M.: *Minimal and maximal solutions of the fourth-order nonlinear difference equations,* Difference Equ. Appl. **20** (2014), 1–14.

[6] JANKOWSKI, R.—SCHMEIDEL, E.—ZONENBERG, J.: *Oscillatory properties of solutions of the fourth order difference equations with quasidifferences,* Opuscula Math. **34** (2014), 789–797.

[7] KREJČOVÁ, J.: *Nonoscillatory solutions of the four-dimensional neutral difference system,* in: Differ. and Difference Equ. with Appl.—ICDDEA '15 (S. Pinelas et al., eds.), Amadora, Portugal, 2015, Math. Statist., Vol. 164, Springer, Chan, 2017, pp. 215–224.

[8] ______ *Property B of the four-dimensional neutral difference system,* in: Proc. of the 22nd Internat. Conference on Difference Equations and Applications—ICDEA '16 (S. Elaydi et al., eds.), Osaka, Japan, 2016, Math. Statist., Vol. 212, Springer, Singapore, 2016, pp. 147–163.

[9] KIGURADZE, I. T.—CHANTURIA, A.: *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* In: Math. Appl., Vol. 89, Kluwer Acad. Publ., Dordrecht, 1993.

[10] MIGDA, M.—SCHMEIDEL, E.: *Convergence of solutions of higher order neutral difference equations with quasi-differences,* Tatra Mt. Math. Publ. **63** (2015), 205–213.

[11] ______ *Asymptotic properties of fourth-order nonlinear difference equations,* Math. Comput. Modelling, **39** (2004), 1203–1211.

[12] SCHMEIDEL, E.: *Oscillation and nonoscillation theorems for fourth order difference equations,* Rocky Mountain J. Math. **33** (2003), 1083–1094.

[13] ______ *Asymptotic trichotomy for positive solutions of a class of fourth-order nonlinear neutral difference equations with quasidifferences,* Nonlinear Anal. **63** (2005), 899–907.

Received May 30, 2017

University of Economics
Faculty of Informatics and Statistics
W. Churchill Sq. 4
CZ–130-67 Prague
CZECH REPUBLIC
E-mail: jana.pasackova@vse.cz