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APPROXIMATION OF SOLUTIONS TO NONAUTONOMOUS DIFFERENCE EQUATIONS

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ABSTRACT. We study the asymptotic properties of solutions to nonautonomous difference equations of the form

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n, \quad f\colon \mathbb{N}\times\mathbb{R}\to\mathbb{R}, \quad \sigma\colon \mathbb{N}\to\mathbb{N}.$$

Using the iterated remainder operator and asymptotic difference pairs we establish some results concerning approximative solutions and approximations of solutions. Our approach allows us to control the degree of approximation.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and real numbers, respectively. Let $m \in \mathbb{N}$. We consider the nonautonomous difference equations of the form

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n. \tag{E}$$

$$a_n, b_n \in \mathbb{R}, \quad f\colon \mathbb{N}\times\mathbb{R}\to\mathbb{R}, \quad \sigma\colon \mathbb{N}\to\mathbb{N}, \quad \sigma(n)\to\infty.$$

By a *solution* of (E) we mean a sequence $x\colon \mathbb{N}\to\mathbb{R}$ satisfying (E) for all large n. We say that x is a *full solution* of (E) if (E) is satisfied for all n. Moreover, if $p \in \mathbb{N}$ and (E) is satisfied for all $n \ge p$, then we say that x is a *p-solution*.

In recent years, the equation (E) and similar equations have been studied in many papers, see for example, [1], [4], [5], [9], [12], [15], [17], [18], [21]. Some classical results on asymptotic behavior of solutions can be found in [2], [3], [6]–[8], [10]–[14], [22]–[24].

In this paper we establish some results concerning approximative solutions of the equation (E). In particular, we present sufficient conditions under which for a given solution y of equation $\Delta^m y = b$ there exists a solution x of (E) such that x and y are asymptotically equivalent. We give also some results concerning

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approximations of solutions. The results obtained in this paper generalize the main results from [15] and [18].

2. Notation and terminology

We use the symbols
$$\mathrm{Sol(E)}, \quad \mathrm{Sol_p(E)}$$
to denote the set of all solutions of (E), and the set of all p-solutions of (E), respectively. The space of all sequences $x\colon \mathbb{N} \to \mathbb{R}$ we denote by SQ. Moreover
$$\mathrm{SQ}^* = \{\mathrm{x} \in \mathrm{SQ}\colon \mathrm{x_n} \neq 0 \ \text{ for any } n\}.$$
For integers p, q such that $0 \le p \le q$, we define
$$\mathbb{N}(p) = \{p, p+1, p+2, \dots\}, \quad \mathbb{N}(p,q) = \{p, p+1, \dots, q\}.$$
If x, y in SQ, then
$$xy \quad \text{and} \quad |x|$$
denote the sequences defined by $xy(n) = x_n y_n$ and $|x|(n) = |x_n|$, respectively. Let $a, b \in \mathrm{SQ}$, $t \in [1, \infty)$. We will use the following notations
$$\mathrm{Fin} = \{\mathrm{x} \in \mathrm{SQ}\colon \mathrm{x_n} = 0 \text{ for all large n}\}, \quad \mathrm{A(t)} := \left\{\mathrm{a} \in \mathrm{SQ}\colon \sum_{\mathrm{n}=1}^{\infty} \mathrm{n^{t-1}} |\mathrm{a_n}| < \infty\right\},$$
$$\begin{aligned}
&\mathrm{o(1)} = \{x \in \mathrm{SQ}\colon \text{x is convergent to zero}\}, && \mathrm{O(1)} = \{\mathrm{x} \in \mathrm{SQ}\colon \text{x is bounded}\},\\
&\mathrm{o(a)} = \{ax\colon x \in \mathrm{o(1)}\} + \mathrm{Fin}, && \mathrm{O(a)} = \{\mathrm{ax}\colon \mathrm{x} \in \mathrm{O(1)}\} + \mathrm{Fin},\\
&\Delta^{-m} b = \{y \in \mathrm{SQ}\colon \Delta^{\mathrm{m}} \mathrm{y} = \mathrm{b}\}, && \mathrm{Pol(m-1)} = \mathrm{Ker}\Delta^{\mathrm{m}} = \Delta^{-\mathrm{m}} 0.
\end{aligned}$$
Note that $\mathrm{Pol(m-1)}$ is the space of all polynomial sequences of degree less than m. Moreover for any $y \in \Delta^{-m} b$ we have
$$\Delta^{-m} b = y + \mathrm{Pol(m-1)}.$$
For a subset A of a metric space X and $\varepsilon > 0$ we define an *ε-framed interior* of A by
$$\mathrm{Int(A}, \varepsilon) = \left\{\mathrm{x} \in \mathrm{X}\colon \overline{\mathrm{B}}(\mathrm{x}, \varepsilon) \subset \mathrm{A}\right\},$$
where $\overline{\mathrm{B}}(x, \varepsilon)$ denotes a closed ball of radius ε about x. We say that a subset U of X is a *uniform neighborhood* of a subset Z of X, if there exists a positive number ε such that $Z \subset \mathrm{Int(U}, \varepsilon)$. Let $g\colon [0, \infty) \to [0, \infty)$ and $w \in \mathrm{SQ}^*$, we say that f is (g, w)-*dominated* if
$$|f(n,t)| \le g(|t w_n^{-1}|) \qquad \text{for} \quad (n,t) \in \mathbb{N} \times \mathbb{R}. \tag{1}$$
We say that a sequence $x \in \mathrm{SQ}$ is (f, σ)-*bounded* if the sequence $\big(f(n, x_{\sigma(n)})\big)$ is bounded. We say that f is *locally equibounded* if for any $t \in \mathbb{R}$ there exists a neighborhood U of t such that f is bounded on $\mathbb{N} \times U$.

Lemma 2.1. *If f is locally equibounded, then any bounded sequence x is (f,σ)--bounded.*

Proof. Choose $a, b \in \mathbb{R}$ such that $x(\mathbb{N}) \subset [a, b]$. For any $t \in [a, b]$ there exist an open subset U_t of $\mathbb{R}$ and a positive constant M_t such that

$$|f(n, s)| \le M_t$$

for any $s \in U_t$ and any $n \in \mathbb{N}$. There exists a finite subset $\{t_1, \dots t_n\}$ of $[a, b]$ such that

$$[a, b] \subset U_{t_1} \cup \dots \cup U_{t_n}.$$

If $M = \max(M_{t_1}, \dots M_{t_n})$, then $|f(n, x_{\sigma(n)})| \le M$ for any n. □

Lemma 2.2. *Assume X is a closed subset of $\mathbb{R}$, $g \colon X \to \mathbb{R}$ is locally bounded and Y is a bounded subset of X. Then the set $g(Y)$ is bounded.*

Proof. Choose a closed interval $[a, b]$ such that $Y \subset [a, b]$, and let $Z = X \cap [a, b]$. For any $t \in Z$ there exist a neighborhood U_t of t and a positive constant Q_t such that $|g(s)| \le Q_t$ for any $s \in U_t \cap X$. By compactness of Z we can choose $t_1, t_2, \dots, t_n \in Z$ such that

$$Z \subset U_{t_1} \cup U_{t_2} \cup \dots \cup U_{t_n}.$$

Then $Y \subset Z$ and for any $s \in Z$ we have

$$g(s) \le \max\{Q_{t_1}, \dots, Q_{t_n}\}.$$

□

In what follows, we present the main tools used in our paper, i.e., the remainder operator, the asymptotic difference pairs and the fixed point lemma based on the Schauder fixed point theorem.

2.1. The remainder operator

Let

$$\mathrm{S(m)} = \left\{ \mathrm{a} \in \mathrm{SQ} \colon \text{ the series } \sum_{\mathrm{i}_1=1}^{\infty} \sum_{\mathrm{i}_2=\mathrm{i}_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} a_{i_m} \text{ is convergent} \right\}.$$

For any $a \in \mathrm{S(m)}$ we define the sequence $r^m(a)$ by

$$r^m(a)(n) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} a_{i_m}.$$

Then $\mathrm{S(m)}$ is a linear subspace of $\mathrm{o}(1)$, $r^m(a) \in \mathrm{o}(1)$ for any $a \in \mathrm{S(m)}$ and

$$r^m \colon \mathrm{S(m)} \to \mathrm{o}(1)$$

is a linear operator which we call the *remainder operator of order* m. The value $r^m(a)(n)$ we denote also by $r^m_n(a)$ or simply $r^m_n a$. If $a \in \mathrm{A(m)}$, then $a \in \mathrm{S(m)}$ and

$$r^m(a)(n) = \sum_{j=n}^{\infty} \binom{m-1+j-n}{m-1} a_j \quad \text{for any} \;\; n \in \mathbb{N}. \tag{2}$$

The following lemma is a consequence of [16, Lemma 3.1, Lemma 4.2, and Lemma 4.8].

Lemma 2.3. *Assume* $a \in \mathrm{A(m)}$, $u \in \mathrm{O(1)}$, $k \in \{0, 1, \dots, m\}$, *and* $p \in \mathbb{N}$. *Then*

(*a*) $\mathrm{O(a)} \subset \mathrm{A(m)} \subset \mathrm{o(n^{1-m})}$, $|r^m(ua)| \le \|u\| r^m|a|$, $\Delta r^m|a| \le 0$,

(*b*) $|r^m_p a| \le r^m_p |a| \le \sum_{n=p}^{\infty} n^{m-1}|a_n|$, $r^k a \in \mathrm{A(m-k)}$,

(*c*) $\Delta^m r^m a = (-1)^m a$, $r^m \mathrm{Fin(p)} = \mathrm{Fin(p)} = \Delta^{\mathrm{m}} \mathrm{Fin(p)}$.

For more information about the remainder operator see [16].

2.2. Asymptotic difference pairs

We say that a pair (A, Z) of linear subspaces of SQ is an *asymptotic difference pair* of order m or, simply, *m-pair* if

$$\mathrm{Fin} + \mathrm{Z} \subset \mathrm{Z}, \quad \mathrm{O(1)A} \subset \mathrm{A}, \quad \mathrm{A} \subset \Delta^{\mathrm{m}}\mathrm{Z}.$$

We say that an m-pair (A, Z) is *evanescent* if $Z \subset \mathrm{o(1)}$. If $A \subset \mathrm{SQ}$ and (A, A) is an m-pair, then we say that A is an m-space. We will use the following lemma.

Lemma 2.4. *Assume* (A, Z) *is an* m*-pair, and* $a, b, x \in \mathrm{SQ}$. *Then*

(*a*) *if* $b - a \in A$, *then* $\Delta^{-m} b + Z = \Delta^{-m} a + Z$,

(*b*) *if* $b \in A$, *then* $\Delta^{-m} b + Z = \mathrm{Pol(m-1)} + \mathrm{Z}$,

(*c*) *if* $a \in A$ *and* $\Delta^m x \in \mathrm{O(a)} + \mathrm{b}$, *then* $x \in \Delta^{-m} b + Z$.

P r o o f. See [18, Lemma 3.5. Lemma 3.6, and Lemma 3.7]. □

Lemma 2.5. *Assume* (A, Z) *is an evanescent* m*-pair,* $a \in A$, $u \in \mathrm{O(1)}$, $\lambda \in \mathbb{R}$, *and* $x, y \in \mathrm{SQ}$. *Then*

(*a*) $\mathrm{O(a)} \subset \mathrm{A} \subset \mathrm{A(m)}$,

(*b*) $r^m A \subset Z$,

(*c*) *if* $x_n = y_n + \lambda r^m_n(au)$ *for large* n, *then* $y \in x + Z$.

P r o o f. By [18, Remark 3.4] we have (a) and (b). Assume $x_n = y_n + \lambda r^m_n(au)$ for large n. Then

$$y - x + r^m(\lambda a u) \in \mathrm{Fin},$$

and, by (b), we get $y - x \in r^m A + \mathrm{Fin} \subset \mathrm{Z} + \mathrm{Fin} \subset \mathrm{Z}$. Therefore $y \in x + Z$. □

Example 1. Assume $s \in \mathbb{R}$, $(s+1)(s+2)\dots(s+m) \neq 0$, and $t \in (-\infty, m-1]$. Then

$$\big(\mathrm{o(n^s)},\, \mathrm{o(n^{s+m})}\big), \quad \big(\mathrm{O(n^s)},\, \mathrm{O(n^{s+m})}\big), \quad \big(\mathrm{A(m-t)},\, \mathrm{o(n^t)}\big)$$

are m-pairs.

Example 2. Assume $s \in (-\infty, -m)$, $t \in (-\infty, 0]$, and $u \in [1, \infty)$. Then

$$\big(\mathrm{o(n^s)},\, \mathrm{o(n^{s+m})}\big), \quad \big(\mathrm{O(n^s)},\, \mathrm{O(n^{s+m})}\big), \quad \big(\mathrm{A(m-t)},\, \mathrm{o(n^t)}\big), \quad \big(\mathrm{A(m+u)},\, \mathrm{A(u)}\big)$$

are evanescent m-pairs.

We say that a subset A of SQ is an *m-space* if (A, A) is an m-pair.

Example 3. If $\lambda \in (0,1) \cup (1, \infty)$, then $\mathrm{o(\lambda^n)}$ and $\mathrm{O(\lambda^n)}$ are m-spaces.

2.3. The fixed point lemma

Lemma 2.6. *Assume $y \in \mathrm{SQ}$, $\rho \in \mathrm{o}(1)$, and*

$$S = \{x \in \mathrm{SQ} \colon |\mathrm{x} - \mathrm{y}| \leq |\rho|\}.$$

Then the formula

$$d(x,y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$

defines a metric on S such that any continuous map $H \colon S \to \mathrm{S}$ has a fixed point.

Proof. The assertion is a consequence of [17, Theorem 3.3 and Theorem 3.1]. □

3. Approximative solutions

This section is devoted to approximative solutions. By an approximative solution we mean a sequence y which is asymptotically equivalent to some solution. For a sequence $x \in \mathrm{SQ}$ we define the sequence $G(x)$ by

$$G(x)(n) = a_n f(n, x_{\sigma(n)}). \tag{3}$$

Theorem 3.1. *Assume (A, Z) is an evanescent m-pair, $a \in A$, $p \in \mathbb{N}$, $y \in \Delta^{-m} b$*

$$U \subset \mathbb{R}, \;\; M > 0, \;\; y(\mathbb{N}) \subset \mathrm{Int(U, Mr^m_p|a|)}, \;\; \mathrm{|f(n,t)| \leq M} \qquad \textit{for} \quad \mathrm{(n,t) \in \mathbb{N} \times U},$$

and f is continuous on $\mathbb{N} \times \mathbb{R}$. Then $y \in \mathrm{Sol_p(E) + Z}$.

Proof. Define $\rho \in \mathrm{SQ}$ and $S \subset \mathrm{SQ}$ by

$$\rho_n = \begin{cases} M r^m_n |a| & \text{for } n \geq p \\ 0 & \text{for } n < p \end{cases}, \qquad S = \{x \in \mathrm{SQ} : |x - y| \leq \rho\}. \tag{4}$$

Since the sequence $r^m|a|$ is nonincreasing, we have $\rho_n \le \rho_p$ for any n. Assume $x \in S$. If $k \in \mathbb{N}$, then $|x_{\sigma(k)} - y_{\sigma(k)}| \le \rho_{\sigma(k)} \le \rho_p$ and we obtain

$$x_{\sigma(k)} \in \overline{\mathrm{B}}(y_{\sigma(k)}, \rho_p) \subset U.$$

Hence $|f(k, x_{\sigma(k)})| \le M$. Thus, for any $x \in S$, we have

$$Gx \in \mathrm{O(a)} \subset \mathrm{A} \subset \mathrm{A(m)}.$$

Let

$$H\colon S \to \mathrm{SQ}, \qquad H(x)(n) = \begin{cases} y_n & \text{for} \quad n < p, \\ y_n + (-1)^m r_n^m Gx & \text{for} \quad n \ge p. \end{cases} \tag{5}$$

If $x \in S$ and $n \ge p$, then

$$|H(x)(n) - y_n| = |r_n^m Gx| \le r_n^m |Gx| \le M r_n^m |a| = \rho_n.$$

Hence $HS \subset S$. Let $\varepsilon > 0$. Choose $q \in \mathbb{N}$ and $\beta > 0$ such that

$$M \sum_{n=q}^{\infty} n^{m-1} |a_n| < \varepsilon \quad \text{and} \quad \beta \sum_{n=p}^{q} n^{m-1} |a_n| < \varepsilon. \tag{6}$$

Let

$$D = \big\{(n,t) \in \mathbb{N} \times \mathbb{R} : n \in \mathbb{N}(p,q) \quad \text{and} \quad |t - y_{\sigma(n)}| \le \rho_n\big\}.$$

Then D is a compact subset of $\mathbb{R}^2$. Hence f is uniformly continuous on D and there exists $\delta > 0$ such that if $(n,s), (n,t) \in D$ and $|s-t| < \delta$, then

$$|f(n,s) - f(n,t)| < \beta.$$

Let $x, y \in S$, $\|x - y\| < \delta$. Using Lemma 2.3 we obtain

$$\begin{aligned}
\|Hx - Hz\| &= \|r^m(Gx - Gz)\| = \sup_{n \ge p} |r_n^m (Gx - Gz)| \le \sup_{n \ge p} r_n^m |Gx - Gz| \\
&= r_p^m |Gx - Gz| \le \sum_{n=p}^{\infty} n^{m-1} |G(x)(n) - G(z)(n)| \\
&\le \sum_{n=p}^{q} n^{m-1} |G(x)(n) - G(z)(n)| + \sum_{n=q}^{\infty} n^{m-1} |G(x)(n) - G(z)(n)| \\
&\le \beta \sum_{n=p}^{q} n^{m-1} a_n + \sum_{n=q}^{\infty} n^{m-1} |G(x)(n)| + \sum_{n=q}^{\infty} n^{m-1} |G(z)(n)| \\
&\le \varepsilon + M \sum_{n=q}^{\infty} n^{m-1} |a_n| + M \sum_{n=q}^{\infty} n^{m-1} |a_n| \le 3\varepsilon.
\end{aligned}$$

Hence the map $H\colon S \to S$ is continuous. By Lemma 2.6, there exists an $x \in S$ such that $Hx = x$. Then, for $n \ge p$, we get

$$x_n = y_n + (-1)^m r_n^m Gx. \tag{7}$$

Hence for $n \geq p$, we have

$$\Delta^m x_n = \Delta^m y_n + \Delta^m (-1)^m r_n^m G(x) = b_n + G(x)(n).$$

Therefore $x \in \mathrm{Sol_p(E)}$. Using (7) and Lemma 2.5 we get $y \in x + Z$. □

Corollary 3.1. *Assume (A, Z) is an evanescent m-pair, $a \in A$, $y \in \Delta^{-m} b$, and there exists a uniform neighborhood U of the set $y(\mathbb{N})$ such that the restriction $f|\mathbb{N} \times U$ is continuous and bounded. Then $y \in \mathrm{Sol(E)} + \mathrm{Z}$.*

P r o o f. Choose a constant M such $|f(n,t)| \leq M$ for any $(n,t) \in \mathbb{N} \times U$. Moreover, choose a positive ε such that $y(\mathbb{N}) \subset \mathrm{Int(U}, \varepsilon)$. Since $r_n^m |a| = \mathrm{o}(1)$, there exists an index p such that $M r_p^m |a| < \varepsilon$. Then

$$y(\mathbb{N}) \subset \mathrm{Int(U, Mr_p^m|a|)}$$

and, by Theorem 3.1, we get

$$y \in \mathrm{Sol_p(E) + Z \subset Sol(E) + Z}.$$

□

Theorem 3.2. *Assume (A, Z) is an evanescent m-pair, $a \in A$, $L, M > 0$, $w \in \mathrm{SQ}^*$,*

$$g\colon [0,\infty) \to [0,\infty),\ g[0,L] \subset [0,M],\ |f(n,t)| \leq g(|t w_n^{-1}|) \quad \text{for} \quad (n,t) \in \mathbb{N} \times \mathbb{R},$$

$p \in \mathbb{N}$, f is continuous, $y \in \Delta^{-m} b$, and $|y \circ \sigma| \leq L|w| - M r_p^m |a|$. Then

$$y \in \mathrm{Sol_p(E) + Z}.$$

P r o o f. Define ρ and S by (4). Let $x \in S$. Using the inequality

$$|y \circ \sigma| \leq L|w| - M r_p^m |a|,$$

we get

$$\begin{aligned}
\left| \frac{x_{\sigma(n)}}{w_n} \right| &= \left| \frac{x_{\sigma(n)} - y_{\sigma(n)} + y_{\sigma(n)}}{|w_n|} \right| \\
&\leq \frac{|x_{\sigma(n)} - y_{\sigma(n)}| + |y_{\sigma(n)}|}{|w_n|} \\
&\leq \frac{M r_p^m |a| + |y_{\sigma(n)}|}{w_n} \leq L \quad \text{for any } n.
\end{aligned}$$

Using the inequality

$$|f(n,t)| \leq g(|t w_n^{-1}|) \quad \text{and inclusion} \quad g[0,L] \subset [0,M],$$

we have

$$|f(n, x_{\sigma(n)})| \leq g\left(\frac{|x_{\sigma(n)}|}{w_n} \right) \leq M \quad \text{for any } n.$$

Therefore,

$$|G(x)(n)| \le Ma_n.$$

Now, repeating the second part of the proof of Theorem 3.1, we obtain

$$y \in \mathrm{Sol_p(E)} + \mathrm{Z}. \qquad \square$$

Corollary 3.2. *Assume (A, Z) is an evanescent m-pair, $a \in A$, $w \in \mathrm{SQ}$, $|w| \ge \lambda > 0$, $y \in \Delta^{-m}b$, $y \circ \sigma \in \mathrm{O(w)}$, g is locally bounded, f is is continuous and (g, w)-dominated. Then*

$$y \in \mathrm{Sol(E)} + \mathrm{Z}.$$

Proof. Choose a positive constant P such that $|y \circ \sigma| \le P|w|$. Let

$$L = P + 1 \quad \text{and} \quad \alpha = \inf\{L|w_n| - |y_{\sigma(n)}| \colon n \in \mathbb{N}\}.$$

Then

$$L|w_n| - |y_{\sigma(n)}| = P|w_n| - |y_{\sigma(n)}| + |w_n| \ge P|w_n| - |y_{\sigma(n)}| + \lambda \ge \lambda \quad \text{for any } n.$$

Hence $\alpha \ge \lambda > 0$. By Lemma 2.2, there exists a positive constant M such that $g[0, L] \subset [0, M]$. Since $\lim_{n\to\infty} r_n^m |a| = 0$, there exists an index p such that

$$M r_p^m |a| \le \alpha.$$

Then

$$M r_p^m |a| \le L w_n - |y_{\sigma(n)}| \quad \text{for any } n.$$

Hence, by Theorem 3.2, $y \in \mathrm{Sol_p(E)} + \mathrm{Z} \subset \mathrm{Sol(E)} + \mathrm{Z}$. $\square$

4. Approximations of solutions

In this section, we present results concerning the approximations of solutions. In what follows, we assume that

$$g \colon [0, \infty) \to [0, \infty).$$

We say that g is of Bihari type if $g(t) > 0$ for $t > 0$ and for any $c > 0$ we have

$$\int_c^\infty \frac{dt}{g(t)} = \infty.$$

Theorem 4.1. *If (A, Z) is an m-pair, $a \in A$, and x is an (f, σ)-bounded solution of (E), then $x \in \Delta^{-m}b + Z$.*

Proof. Since x is a solution of (E), we have

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n \quad \text{for all large } n.$$

Hence, there exists a bounded sequence u such that $\Delta^m x = au + b$. Therefore $\Delta^m x \in \mathrm{O(a)} + \mathrm{b}$ and, by Lemma 2.4, we get $x \in \Delta^{-m}b + Z$. $\square$

Corollary 4.1. *If (A, Z) is an m-pair, $a, b \in A$, and x is an (f, σ)-bounded solution of* (E), *then* $x \in \mathrm{Pol(m-1)} + \mathrm{Z}$.

P r o o f. By Lemma 2.4, we have $\mathrm{Pol(m-1)+Z} = \Delta^{-\mathrm{m}}\mathrm{b+Z}$. Hence the assertion is a consequence of Theorem 4.1. □

Corollary 4.2. *Assume (A, Z) is an m-pair, $a \in A$, $Z \subset \mathrm{O}(1)$, $u \in \mathrm{O}(1)$, $b = \Delta^m u$, and f is locally equibounded. Then for any bounded solution x of* (E) *there exists a constant $c \in \mathbb{R}$ such that*

$$x \in u + c + Z. \tag{8}$$

P r o o f. By Lemma 2.1 x is (f, σ)-bounded. By Theorem 4.1, $x \in \Delta^{-m}b + Z$. Note that

$$\Delta^{-m}b = u + \mathrm{Pol(m-1)}.$$

Choose $z \in Z$ and $\varphi \in \mathrm{Pol(m-1)}$ such that

$$x = u + \varphi + z.$$

Then $\varphi = x - u - z$ is bounded. Hence φ is constant and we get (8). □

Theorem 4.2. *Assume that (A, Z) is an m-pair, $a \in A$, $w \in \mathrm{SQ}^*$, $\mathrm{O(w_{n+1}) = O(w_n)}$, the sequence $\sigma(n) - n$ is bounded, g is locally bounded, and f is (g, w)-dominated. Then*

$$\mathrm{O(w) \cap Sol(E) \subset \Delta^{-m}b + Z}.$$

P r o o f. Choose $k \in \mathbb{N}$ such that

$$|\sigma(n) - n| \le k \quad \text{for any } n.$$

Since $w_{n+1} = \mathrm{O(w_n)}$, there exists a constant $M > 1$ such that $|w_{n+1}| \le M|w_n|$ for large n. Then

$$|w_{n+2}| \le M|w_{n+1}| \le M^2|w_n|, \dots, |w_{n+k}| \le M^k|w_n|.$$

Hence, for any $p \in \mathbb{N}(0, k)$, we have

$$|w_{n+p}| \le M^k|w_n| \quad \text{for large } n.$$

Analogously, since $w_n = \mathrm{O(w_{n+1})}$, there exists a constant $Q > 1$ such that for any $p \in \mathbb{N}(0, k)$, we have

$$|w_{n-p}| \le Q^k|w_n| \quad \text{for large } n.$$

Hence, there exists a constant $L \ge \max(M^k, Q^k)$ such that

$$\big|w\big(\sigma(n)\big)\big| \le L|w_n| \quad \text{for any } n.$$

Let

$$x \in \mathrm{O(w) \cap Sol(E)}.$$

Choose a positive constant P_1 such that $|x_n| \le P_1|w_n|$ for any n. Let $P = LP_1$. Then

$$|x_{\sigma(n)}| \le P_1|w_{\sigma(n)}| \le P|w_n| \quad \text{for any } n. \tag{9}$$

By Lemma 2.2 there exists a positive constant Q_1 such that

$$g(s) \le Q_1 \quad \text{for any} \;\; s \in [0, P]. \tag{10}$$

Using (1), (9), and (10) we get

$$|f(n, x_{\sigma(n)})| \le g\left(\frac{|x_{\sigma(n)}|}{|w_n|}\right) \le Q_1.$$

Hence $x \in \mathrm{Sol(E)}$ and the sequence $(f(n, x_{\sigma(n)})$ is bounded. Therefore, using Lemma 2.4, we obtain $x \in \Delta^{-m}b + Z$. □

Corollary 4.3. *Assume the assumptions of Theorem 4.2 are satisfied and* $b \in A$. *Then*

$$\mathrm{O(w) \cap Sol(E) \subset Pol(m-1) + Z.}$$

P r o o f. The assertion is a consequence of Theorem 4.2 and Lemma 2.4. □

In the proof of the next theorem we will use the following two lemmas.

Lemma 4.1. *Assume* $u \in \mathrm{SQ}$, $u \ge 0$, $a \in \mathrm{A}(1)$, $g\colon [0, \infty) \to [0, \infty)$,

$$M, c \in (0, \infty), \quad p \in \mathbb{N}, \quad u_n \le c + M \sum_{j=p}^{n-1} |a_j| g(u_j) \qquad \textit{for} \quad n \ge p,$$

g is nondecreasing and of Bihari type. Then the sequence u is bounded.

P r o o f. The assertion is a consequence of [19, Lemma 4.1]. □

Lemma 4.2. [15, Lemma 7.3] *If x is a sequence of real numbers, $m \in \mathbb{N}$ and $p \in \mathbb{N}(m)$ then there exists a positive constant $L = L(x, p, m)$ such that*

$$|x_n| \le n^{m-1}\left(L + \sum_{i=p}^{n-1} |\Delta^m x_i|\right) \qquad \textit{for} \quad n \ge p.$$

Theorem 4.3. *Assume (A, Z) is an m-pair, $\sigma(n) \le n$ for large n,*

$$a \in A \cap \mathrm{A}(1), \quad \mathrm{b \in A(1), \quad w \in SQ^*, \quad w^{-1} \in O(n^{-m+1}),}$$

g is nondecreasing and of Bihari type, and f is (g, w)-dominated. Then

$$\mathrm{Sol(E) \subset \Delta^{-m}b + Z.}$$

Moreover, if $b \in \mathrm{A}$, *then*

$$\mathrm{Sol(E) \subset Pol(m-1) + Z.}$$

P r o o f. Assume x is a solution of (E). Choose an index p such that

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n \quad \text{for any} \quad n \ge p.$$

Choose $M > 0$ such that

$$|w_n^{-1}| \le M n^{1-m}.$$

For $n \in \mathbb{N}$ let

$$u_n = \left| x_{\sigma(n)} w_n^{-1} \right|.$$

By Lemma 4.2, there exists a positive constant L such that

$$|x_{\sigma(n)}| \le \sigma(n)^{m-1} \left(L + \sum_{i=p}^{\sigma(n)-1} |\Delta^m x_i| \right) \le n^{m-1} \left(L + \sum_{i=p}^{n-1} |\Delta^m x_i| \right).$$

Let $c = ML + M \sum_{i=1}^{\infty} |b_i|$. Then

$$u_n = \left| x_{\sigma(n)} w_n^{-1} \right| \le ML + M \sum_{i=p}^{n-1} |\Delta^m x_i| = ML + M \sum_{i=p}^{n-1} \left| b_i + a_i f\left(j, x_{\sigma(j)}\right) \right|$$

$$\le ML + M \sum_{i=1}^{\infty} |b_i| + M \sum_{i=p}^{n-1} |a_i| g(u_i) = c + M \sum_{i=p}^{n-1} |a_i| g(u_i).$$

Hence, by Lemma 4.1, the sequence u is bounded. Therefore, there exists a constant $Q > 1$ such that $g(u_n) \le Q$ for any n and we get

$$\left| f(n, x_{\sigma(n)}) \right| \le g\left(\left| x_{\sigma(n)} w_n^{-1} \right| \right) = g(u_n) \le Q$$

for any n. Hence x is an (f, σ)-bounded solution and, by Theorem 4.1, we get

$$x \in \Delta^{-m} + Z.$$

If $b \in A$, then, using Lemma 2.4 (b), we obtain

$$x \in \mathrm{Pol(m-1) + Z}.$$

□

Corollary 4.4. *Assume $s \in (-\infty, 0]$, $\sigma(n) \le n$ for large n,*

$$a \in \mathrm{A(m-s)}, \quad \mathrm{b \in A(1)}, \quad \mathrm{|f(n,t)| \le g(n^{1-m}|t|)} \qquad \textit{for any} \quad \mathrm{(n,t) \in \mathbb{N} \times \mathbb{R}},$$

and g is nondecreasing and of Bihari type. Then

$$\mathrm{Sol(E) \subset \Delta^{-m} b + o(n^s)}.$$

Moreover, if $b \in \mathrm{A(m-s)}$, then

$$\mathrm{Sol(E) \subset Pol(m-1) + o(n^s)}.$$

P r o o f. By Example 2, $\big(\mathrm{A(m-s), o(n^s)}\big)$ is an m-pair. Hence the assertion is a consequence of Theorem 4.3. □

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