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CONTROL OF CONSERVATION LAWS – – AN APPLICATION

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ABSTRACT. We present here three types of controlled boundary value problems for conservation laws arising from energy co-generation, hydraulic flows and water hammer for hydroelectric power plants and control of the open channel flows (shallow water). The novelty of these models, from the mathematical point of view, is that they are described by nonlinear hyperbolic partial differential equations of the conservation laws with (possibly) nonlinear boundary conditions. At their turn these boundary conditions are controlled by some systems of ordinary differential equations. The engineering requirements for such systems are asymptotic stability and disturbance rejection: these properties have to be achieved by feedback control. In our setting the main tool for tackling these problems is a suitable Lyapunov functional arising from the energy identity. The hints for "guessing" this functional are to be found in the linearized version of the aforementioned mathematical objects.

1. On conservation laws

1.1. Short overview

The conservation laws are nonlinear partial differential equations which arise from continuum physics, being discovered since the works of the *natural philosophers* of the XVIII century to name but Leonhard Euler. As pointed out in [7], the constitutive equations that encode material properties of the medium in continuum mechanics, thermo-mechanics, fluid mechanics, electrodynamics and others, being coupled with the field equations will generate closed systems of partial differential equations—the conservation laws—from which the "trajectories" of the continuum medium are to be determined. It is stated there [7] that "historically, the vast majority of noteworthy partial differential equations were

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generated through that process". To be more specific, nowadays it is understood by a system of conservation laws a structure written below

$$\partial_t u_k + \partial_x f_k(u_1, u_2, \ldots, u_n) = 0 \qquad k = 1, \ldots, n, \tag{1}$$

where ∂_t and ∂_x denote partial derivatives with respect to the variables denoted by the subscripts. Continuum physics is thus an invaluable source of partial differential equations even in our days.

Next, the conservation laws being highly nonlinear partial differential equations with a definite structure, display some specific properties in the realm of these equations. Some of those properties have been "capitalized" along several decades of the end of the XXth century. Let us mention a few: emergency of discontinuous solutions from continuous (even smooth) initial data; propagation of singularities, shock and rarefaction waves; many other...

Concerning the development of the studies on conservation laws it is worth mentioning an impressive list of monographs on this subject authored by T.-T. Li and his co-workers [21]–[24], D. Serre [34], A. Bressan [3], S. K. Godunov and E. I. Romenskii [10], P. D. Lax [17], [18], P. G. LeFloch [19], T. P. Liu [25]. This shows the importance of the subject from the mathematical point of view.

In the most interesting survey paper of D. Serre [35] it is mentioned that "fully nonlinear problems or even quasilinear ones are harder to deal and require other ideas. Solutions to reasonable problems might have poor regularity in which case they are called *weak*. Then an "entropy" criterion may be needed to select a unique, relevant solution among the weak ones. In most scalar cases, a comparison principle holds and monotonicity encodes the entropy condition". And to continue with "It turns out that the important class of hyperbolic systems of conservation laws, which includes Euler equations of gas dynamics, gather all the difficulties mentioned above. Their nonlinearity generally forbids the existence of classical solutions." And: "Amazingly, we do not even know a functional space where the Cauchy problem might be well-posed".

1.2. Theory versus control applications

The theory of the conservation laws as a mathematical object encompasses three classes of problems:

i) the Cauchy problem defined by initial data;
ii) the boundary value and initial boundary value problems;
iii) the Riemann problem of the existence of self-similar solutions.

On the other hand, there were identified in the last decades several applications where automatic control occurs. These are the so-called controlled systems with distributed parameters where the boundary control is used for control and stabilization for some special classes of conservation laws arising from fluid mechanics. Due to the dynamics of the devices at the boundaries, the boundary value problems thus defined are non-standard: the usual boundary conditions are controlled by a system of ordinary differential equations which is controlled by them; this kind of internal feedback is able to generate instabilities as pointed out in [26]. The mathematical problems posed by the engineering problems for conservation laws are somehow different: in addition to the standard well-posedness aspects (existence, uniqueness, continuous data dependence, i.e., the three basic components of the well-posedness in the sense of Hadamard) one must add positiveness of some variables accounting for pressures, temperatures, other (in fact existence of some invariant sets), inherent stability of some important steady states (which otherwise would not be noticeable and/or measurable) according to the *Stability Postulate* of N. G. Četaev [4] and feedback control which ensures asymptotic stability, robustness with respect to uncertainties and other "good" properties required by practice.

The aforementioned properties define what was called *augmented model validation* [32]. In the following we shall consider some applications where control of the conservation laws occurs. Worth mentioning that all of them arise from fluid mechanics: shallow water, hydroelectric power plant, energy co-generation.

2. Description of three control applications

2.1. Control of the open channels

The first application where conservation laws are mentioned explicitly is concerned with level and flow control of the open channels. They are modeled by the shallow water equations a.k.a. Saint Venant equations. A most general deduction of these equations is to be found in [20] but we shall give here the simpler model of the horizontal prismatic channels. This model is to be met in a series of papers dedicated to this subject [6], [14]

$$\begin{aligned}
&\partial_t H + \partial_x (VH) = 0, \;\; \partial_t V + \partial_x \left(gH + \frac{V^2}{2} \right) = 0; \quad t > 0,\; 0 \le x \le L, \\
&V(0,t)|V(0,t)|H^2(0,t) = u_1(t)\big(H_{up} - H(0,t)\big), \\
&V(L,t)|V(L,t)|H^2(L,t) = u_2(t)\big(H(L,t) - H_{do}\big),
\end{aligned} \tag{2}$$

where L is the length of the reach and $V(x,t)$, $H(x,t)$ are the water velocity and water depth, respectively. The control is ensured by the underflow gates located at $x = 0$ (upstream) and $x = L$ (downstream). The control parameters are the openings of the gates $u_1(t)$ – upstream and $u_2(t)$ – downstream. The levels outside the reach H_{up} (upstream) and H_{do} (downstream) are assumed constant and subject to $H_{up} > H_{do}$. We just precise that the boundary conditions in (2) are in fact flow conditions (for prismatic channels).

Worth mentioning that the aforementioned references on the subject do not take into account the dynamics of the actuating systems which nevertheless might be important due to the rather high actuating power.

2.2. Co-generation

Another engineering application of the control of the conservation laws is *co-generation*, i.e., combined heat electricity generation by steam turbines having regulated steam extractions. If the thermal energy consumer is located at some distance from the generating plants, the propagation of the steam along the pipe has to be taken into account. While the problem is known for more than 75 years [15], it appeared quite recently that the basic nonlinear model describing the flow of a barotropic fluid is a system of conservation laws. Since this system has been discussed in several papers of the author along the last three decades, we shall give here one of the last—almost definitive form of the model as it occurs in [33]

$$\begin{aligned}
&T_c\partial_t\xi_\rho + \partial_\lambda\xi_w = 0;\ \ \psi_c^2 T_c\partial_t\xi_w + \partial_\lambda\left(\xi_\rho + \psi_c^2\frac{\xi_w^2}{\xi_\rho}\right) = 0,\\
&\xi_w(0,t) = \pi_s(t)\Phi\big(\pi_s(t)/\xi_\rho(0,t)\big);\ \ \xi_w(1,t) = \psi_s\xi_\rho(1,t),\\
&T_a\frac{\mathrm{d}s}{\mathrm{d}t} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_g,\ \ T_1\frac{\mathrm{d}\pi_1}{\mathrm{d}t} = \mu_1 - \pi_1, \qquad 0 \le \mu_1 \le 1, \qquad (3)\\
&T_p\frac{\mathrm{d}\pi_s}{\mathrm{d}t} = \pi_1 - \beta_s\mu_2\pi_s - (1-\beta_s)\xi_w(0,t),\\
&T_2\frac{\mathrm{d}\pi_2}{\mathrm{d}t} = \mu_2\pi_s - \pi_2,\ \ 0 < \mu_{2min} \le \mu_2 \le 1.
\end{aligned}$$

Here all the state variables are scaled/rated to some steady state values of them; these values are chosen to make the model's parameters constant regardless the imposed steady states. (A remark on terminology: the engineers prefer to say "rated" but "scaled" is also adequate). We recall in the following some of the physical and engineering assumptions that define the aforementioned model.

The flows are considered isothermal, motivated by the rather high speed of the flow transients with respect to the temperature transients. Next the steam flow at the turbine extraction is subcritical or critical (during the transients) and subject to the Saint Venant law [33]. The steam flow at the steam consumer is critical what explains the linear boundary condition at $\lambda = 1$ in (3).

Unlike the previous model where the differential equations were in cascade (series) with the standard ones, here the differential equations are in some kind of internal feedback with the boundary conditions. According to [26], this internal feedback can induce instability.

2.3. The hydroelectric power plant

The third model considered here is the simplest case of a hydroelectric power plant without surge tank. The most complete model is given in a by now classical paper [1] but we shall be using the earlier model of [36] where the dynamics of the turbine penstock as well as that of the turbine air dome are neglected. Worth mentioning that the aforementioned models coincide with the more recent ones [2], [28], [29]. The equations are as follows

$$\begin{aligned} &\partial_t V + \partial_x \left(gH + \frac{V^2}{2} \right) + \mathcal{D}(V) = 0; \;\; \partial_t H + \frac{a^2}{g} \partial_x V = 0, \\ &H(0,t) \;=\; H_0; \;\; V(L,t) = k_0 F(t) \sqrt{H(L,t)}, \\ &J\Omega_0 \frac{\mathrm{d}\Omega}{\mathrm{d}t} = \eta_t \frac{\gamma}{2g} \Omega_0 F(t) V^3(L,t) - N_g. \end{aligned} \tag{4}$$

We give some explanations concerning this model: the hydroelectric power plant is connected to the end $x = L$ of the water gallery (penstock) uniting it with the lake at $x = 0$. As in the model of the open channel (2), $V(x,t)$ denotes the water velocity but $H(x,t)$ is here the hydraulic head; $F(t)$ is the water cross-section of turbine's control mechanism (the steering vanes). The model parameters are as follows: a is the sound velocity in water, g – the gravity acceleration, k_0 – a constant. The term $\mathcal{D}(V)$ is called *hydraulic resistance slope* and accounts for the so-called Darcy losses; in its absence the partial differential equations (4)—which are very much alike to (2) and similar to (3)—describe a system of conservation laws. The boundary condition at $x = L$ is "controlled" by the hydraulic turbine equation; this equation is as in (4) if the dynamics of the turbine penstock and that of the turbine air-dome are neglected. What is left is a mechanical power balance: Ω is the rotating speed, J – the inertia momentum, η_t – the turbine efficiency, γ – the specific weight of the water, Ω_0 – the steady state synchronous speed, N_g – the mechanical power delivered to the hydro-generator.

We gave in some detail the aforementioned three models for the following reasons:

i) to illustrate modeling by conservation laws *via* those arising from power engineering, with incompressible—the cases (2) and (4), "dealing" with water—and compressible fluids also—the case (3), "dealing" with steam;

ii) to show non-standard boundary value problems—controlled by systems of ordinary differential equations;

iii to illustrate nonlinear boundary conditions.

Worth mentioning that none of the three models can be reduced to another or deduced from it. Even the simplifying assumptions are different from one model to another.

The first two applications were already considered in our previous papers: the first case—of the open channel—in [27]; the second case—of the co-generation––in [12], [30], [31]—the linear boundary conditions, in [33]—the nonlinear boundary conditions.

We shall therefore focus, in what remains of this paper, on the hydraulic transients described by (4). However, here also one has to start from some more physical aspects. Usually the distributed parameters of the gallery are taken into account for water hammer studies [2], [28], [29] and design of the surge tanks. The transients of the turbine are considered for *Electric Grid* frequency control which is realized mainly through the hydraulic power plants: at the signal from the general *Grid dispatcher*, a power level N_g for the hydraulic turbine is prescribed and the control mechanism will ensure this level *via* the corresponding water flow, i.e., some steady state cross-section $\bar{F}$ and steady state water speed $\bar{V}$. The steady state—which defines the operating point of the plant—must be stabilized using feedback stabilization.

The aforementioned considerations define the object of the mathematical development that follows. What will make the difference with respect to other studies will be precisely the consideration of the distributed parameters of the gallery in the stabilization and stability analysis; however, unlike in [1], [36], nonlinear boundary conditions will be also taken into account.

3. Statics and scaled/rated variables in the hydroelectric transients

In tackling engineering applications models, there is considered useful to have the state variables scaled/rated to some reference values. This approach has several outcomes even for the mathematical treatment: independence with respect to the units of measure, reducing the numerical ill conditioning and some reduction of the number of parameters. Let us consider (4) under the simplifying assumptions of negligible kinetic term $V^2/2$ and negligible Darcy losses $\mathcal{D}(V)$. This negligibility has some background in the Bernoulli Law for incompressible fluids. Registered data for several hundreds of hydro power plants of former USSR [2] show that this is indeed the case (unlike the quite similar case of the water channel (2), where this is not true, possibly because of the low hydraulic head).

Consequently, we shall have the steady state

$$\begin{aligned} &\bar{H}(x) \equiv \text{const}, \;\; \bar{V}(x) \equiv \text{const}; \;\; \bar{H}(0) = H_0, \;\; \bar{V}(L) = k_0 \bar{F}\sqrt{\bar{H}(L)}, \\ &\eta_t \frac{\gamma}{2g}\Omega_0 \bar{F}\bar{V}(L)^3 = N_g. \end{aligned} \tag{5}$$

We deduce

$$\bar{H} = H_0, \;\; \bar{V} = k_0 \bar{F}\sqrt{H_0}, \;\; \eta_t \frac{\gamma}{2g}\Omega_0 k_0^3 \bar{F}^4 H_0^{3/2} = N_g \tag{6}$$

and the imposed load N_g will determine in a unique way the control parameter––the cross section $\bar{F}$ of the turbine steering vanes. Since usually the rating values correspond to the maximal load power, this will imply the maximal cross section F_{max} and this, at its turn, will give the maximal speed $V_{max} = F_{max}\sqrt{H_0}$ (k_0 is such that it equals 1 for maximal flow speed and maximal cross section).

We are now in position to introduce the scaled/rated variables and parameters

$$\begin{aligned} &h(x,t) = H(x,t)/H_0, \;\; v(x,t) = V(x,t)/V_{max} = V(x,t)/(F_{max}\sqrt{H_0}); \\ &f(t) = F(t)/F_{max}, \;\; s(t) = (\Omega - \Omega_0)/\Omega_0; \\ &T_a = \frac{J\Omega_0}{\eta_t \frac{\gamma}{2g} F_{max}^4 H_0^{3/2}}, \;\; \nu_g = \frac{N_g}{\eta_t \frac{\gamma}{2g} F_{max}^4 H_0^{3/2}\Omega_0}. \end{aligned} \tag{7}$$

Consequently, if the rated variables and parameters are introduced, the system (4)— where the kinetic term and the Darcy losses have been neglected— gets the following form

$$\begin{aligned} &\partial_t v + \frac{a}{\delta_0}\partial_x h = 0, \;\; \partial_t h + a\delta_0 \partial_x v = 0, \\ &h(0,t) = 1, \;\; v(L,t) = f(t)\sqrt{h(L,t)}; \;\; T_a \frac{\mathrm{d}s}{\mathrm{d}t} = f(t)v^3(L,t) - \nu_g, \end{aligned} \tag{8}$$

where we denoted, in order to simplify further notations

$$\delta_0 = \frac{aF_{max}}{g\sqrt{H_0}}. \tag{9}$$

Equations (8) define a boundary value problem for a linear system of hyperbolic partial differential equations. The boundary conditions are nonlinear (more precisely, one of them) and controlled by a differential equation—itself controlled by the boundary condition. The entire system (8) is a controlled system, the control signal being $f(t)$ while ν_g is what is called in Control Theory a perturbation signal. To (8) one has to add the initial conditions

$$h(x,0) = h_0(x), \;\; v(x,0) = v_0(x), \;\; 0 \le x \le L; \;\; s(0) = s_0 \tag{10}$$

with $h_0(x)$, $v_0(x)$ sufficiently smooth.

In what follows, we shall focus on system (8) with the initial conditions (10).

4. The basic theory. Equilibria and inherent stability

4.1. Association of the functional differential equations

We shall discuss here the *augmented model validation* [32] as mentioned in Section 1. The approach will be that of integrating the Riemann invariants of (8) along the characteristics in order to associate to (8) a system of functional differential equations. The Riemann invariants of (8) are defined by

$$u^{\pm}(x,t) = h(x,t) \pm \delta_0 v(x,t) \tag{11}$$

and their reverse

$$h(x,t) = \frac{1}{2}\big[u^{+}(x,t) + u^{-}(x,t)\big], \ \ v(x,t) = \frac{1}{2\delta_0}\big[u^{+}(x,t) - u^{-}(x,t)\big]. \tag{12}$$

We re-write (8) using them

$$\begin{aligned} &\partial_t u^{\pm} \pm a\partial_x u^{\pm} = 0; \ \ u^{+}(0,t) + u^{-}(0,t) = 2, \\ &u^{+}(L,t) - u^{-}(L,t) = \delta_0\sqrt{2}f(t)\sqrt{u^{+}(L,t) + u^{-}(L,t)}, \\ &T_a\frac{\mathrm{d}s}{\mathrm{d}t} = \left(\frac{1}{2\delta_0}\right)^3 f(t)\big(u^{+}(L,t) - u^{-}(L,t)\big)^3 - \nu_g. \end{aligned} \tag{13}$$

In the same way we express the initial conditions

$$u^{\pm}(x,0) := u_0^{\pm}(x) = h_0(x) \pm \delta_0 v_0(x). \tag{14}$$

The Riemann invariants are constant along the characteristics which are defined by

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \pm\frac{1}{a}. \tag{15}$$

Let $t^{\pm}(\xi;x,t) = t \pm (\xi - x)/a$ be the two characteristic curves crossing some point $(x,t) \in [0,L] \times \mathbb{R}^{+}$; we integrate $u^{+}(x,t)$ along $t^{+}(\xi;x,t)$ from $\xi = x$ to $\xi = L$ and $u^{-}(x,t)$ along $t^{-}(\xi;x,t)$ from $\xi = 0$ to $\xi = x$ to find

$$\begin{aligned} u^{+}(x,t) &= u^{+}\big(x,t^{+}(x;x,t)\big) \equiv u^{+}\big(L,t^{+}(L;x,t)\big) = u^{+}\big(L,t + (L-x)/a\big), \\ u^{-}(x,t) &= u^{-}\big(x,t^{-}(x;x,t)\big) \equiv u^{-}\big(0,t^{-}(0;x,t)\big) = u^{-}(0,t + x/a). \end{aligned} \tag{16}$$

In particular, if $t^{+}(\cdot;x,t)$ can be extended "to the left" up to $\xi = 0$ and $t^{-}(\cdot;x,t)$ "to the right" up to $\xi = L$ we deduce that

$$u^{+}(0,t) = u^{+}(L,t + L/a), \quad u^{-}(L,t) = u^{-}(0,t + L/a). \tag{17}$$

Denoting

$$y^{+}(t) := u^{+}(L,t), \quad y^{-}(t) := u^{-}(0,t) \tag{18}$$

we deduce from (17)

$$u^{+}(0,t) = y^{+}(t + L/a), \quad u^{-}(L,t) = y^{-}(t + L/a) \tag{19}$$

the time delay L/a accounting for *lossless and distortionless propagation* along the penstock (water gallery). With the function change

$$w^{\pm}(t) := y^{\pm}(t + L/a) \tag{20}$$

the following system of functional differential equations is obtained

$$\begin{aligned} &w^{+}(t) + w^{-}(t - L/a) = 2, \\ &w^{+}(t - L/a) - w^{-}(t) = \delta_0\sqrt{2}f(t)\sqrt{w^{-}(t) + w^{+}(t - L/a)}, \\ &T_a\frac{\mathrm{d}s}{\mathrm{d}t} = \left(\frac{1}{2\delta_0}\right)^3 f(t)\big(w^{+}(t - L/a) - w^{-}(t)\big)^3 - \nu_g. \end{aligned} \tag{21}$$

The solution to (21) can be constructed by steps on intervals $(kL/a, (k+1)L/a)$, $k = 0, 1, 2, \ldots$ starting from the initial condition (14) as follows. The initial conditions $w_0^{\pm}(t)$, $t \in [-L/a, 0)$ are constructed by integrating along those characteristics which cannot be extended “to the left” or “to the right”. For instance, let $t^{+}(\xi; x, t) = t + (\xi - x)/a$ be such that there exists some $\hat{\xi} \in [0, L]$ in order that $t + (\hat{\xi} - x)/a = 0$; this means $\hat{\xi} = x - at$ hence the aforementioned property holds for those characteristics defined for $0 < x - at < L$. Likewise, let $t^{-}(\xi; x, t) = t - (\xi - x)/a$ be such that there exists some $\hat{\xi} \in [0, L]$ in order that $t - (\hat{\xi} - x)/a = 0$; this means that $\hat{\xi} = x + at$ and the aforementioned property holds for those characteristics defined for $0 < x + at < L$. Instead of (16) we shall have, by integrating between $x - at$ and L and between 0 and $x + at$, respectively

$$\begin{aligned} u^{+}(x - at, 0) &= u_0^{+}(x - at) = u^{+}(L, t + (L - x)/a), \\ u^{-}(x + at, 0) &= u_0^{-}(x + at) = u^{-}(0, t + x/a). \end{aligned} \tag{22}$$

If (19) and (20) are to be taken into account, we can write

$$\begin{aligned} y^{+}(t + (L - x)/a) &= w^{+}(t - x/a) = u_0^{+}(x - at) \\ &= h_0(x - at) + \delta_0 v_0(x - at), \quad 0 < x - at < L, \\ y^{-}(t + x/a) &= w^{-}(t + (x - L)/a) = u_0^{-}(x + at) \\ &= h_0(x + at) - \delta_0 v_0(x + at), \quad 0 < x - at < L. \end{aligned} \tag{23}$$

With a simple change of variables, we obtain

$$\begin{aligned} w_0^{+}(\theta) &= u_0^{+}(-a\theta) = h_0(-a\theta) + \delta_0 v_0(-a\theta), \\ w_0^{-}(\theta) &= u_0^{-}(L + a\theta) = h_0(L + a\theta) - \delta_0 v_0(L + a\theta), \end{aligned} \tag{24}$$

where $-L/a \le \theta < 0$.

4.2. Basic theory and invariant sets

We shall consider now system (21) with the initial conditions given by (24) and by some $s(0) = s_0$. Our aim is to construct the solution to (21) by steps, starting from (24) thus showing existence, uniqueness, data dependence and some invariant sets. Especially this last aspect is important for model validation—taking into account the physical significance of the state variables. Assume the following to be true for the solution to (8)

$$0 < h_0(x) < 2, \quad v_0(x) > 0. \tag{25}$$

These inequalities have an obvious physical significance. We deduce from (24) that $w_0^-(t) \geq 0$ for $-L/a \leq t < 0$. We show now the construction by steps of the solution: let $t \in (0, L/a)$. From the first equation of (21), taking into account (25) it follows that

$$w^+(t) = 2 - w_0^-(t - L/a) = 2 - h_0(at) + \delta_0 v_0(at) > 0, \quad 0 < t < L/a.$$

Consider now the second equation of (21): the square root would require $w^-(t) + w^+(t - L/a) \geq 0$ on this interval. Now, the function $\Phi(X) = X + 2\delta_0 f\sqrt{2X}$ is strictly increasing for $X > 0$. Consequently, a rather elementary and straightforward manipulation will replace the second equation of (21) by

$$w^-(t) = -\left(\frac{\delta_0 f(t)}{\sqrt{2}} + w^+(t - L/a)\right) + \sqrt{\left(\frac{\delta_0 f(t)}{\sqrt{2}}\right)^2 + 2w^+(t - L/a)} \tag{26}$$

and $w^-(t) + w^+(t - L/a) > 0$ will follow for all $t > 0$. Let now $t \in (L/a, 2L/a)$. We substitute (26) in the first equation of (21) to find

$$w^+(t) = 2 + \frac{\delta_0 f(t)}{\sqrt{2}} + w^+(t - L/a) - \sqrt{\left(\frac{\delta_0 f(t)}{\sqrt{2}}\right)^2 + 2w^+(t - L/a)} \tag{27}$$

which gives $w^+(t) > 0$ for all $t > 0$. The solution to the new difference system obtained from (21) that is

$$\begin{aligned} w^+(t) &= -\,w^-(t - L/a) + 2, \\ w^-(t) &= -\left(\frac{\delta_0 f(t)}{\sqrt{2}} + w^+(t - L/a)\right) \\ &\quad + \sqrt{\left(\frac{\delta_0 f(t)}{\sqrt{2}}\right)^2 + 2w^+(t - L/a)} \end{aligned} \tag{28}$$

can be constructed by steps and has the invariant set

$$w_t^+(\theta) = w^+(t + \theta) > 0, \quad -L/a \leq \theta < 0 \tag{29}$$

provided $w_0^+(\theta) \geq 0$, $-L/a \leq \theta < 0$. Worth mentioning that $w^{\pm}(t)$ have finite discontinuities at kL/a where k is a positive integer.

Conversely, let $\big(w^{\pm}(t),\ s(t)\big)$ be some solution to (28) hence of (21) with $\big(w_0^{\pm}(t),\ -L/a \le t < 0;\ s(0)\big)$ as initial conditions satisfying $w_0^{+}(t) > 0$. Then, defining

$$u^{+}(x,t) = w^{+}(t - x/a),\ \ u^{-}(x,t) = w^{-}(t + (x - L)/a) \tag{30}$$

the functions $\big(u^{\pm}(x,t), s(t)\big)$ are a classical (possibly discontinuous) solution to (13) with $u^{+}(x,t) > 0$ These assertions can be checked by direct computation.

We have obtained in fact the following result.

Theorem 1. *Consider the system* (8) *together with the system* (13)*, expressed in the Riemann invariants—with the initial conditions* $\big(h_0(x), v_0(x), s(0)\big)$ *or* $\big(u_0^{\pm}(x), s(0)\big)$*—accordingly. Let* $\big(u^{\pm}(x,t), s(t)\big)$ *be some classical solution to* (13) *defined by sufficiently smooth initial conditions. Define* $w^{\pm}(t)$ *via* (16)–(20)*. Then* $w^{\pm}(t), s(t)$ *are a solution to the system of functional differential equations* (21) *with the initial conditions* $\big(w_0^{\pm}(t), -L/a \le t < 0; s(0)\big)$*, where* $w^{\pm}(t)$ *are defined by* (24)*. Conversely, let* $w^{\pm}(t), s(t)$ *be a solution to* (21) *with differentiable initial conditions* $w_0^{\pm}(t), -L/a \le t < 0$ *and having possible discontinuities at* $t = kL/a$ *with* k *a positive integer. Then* $\big(u^{\pm}(x,t), s(t)\big)$ *where* $u^{\pm}(x,t)$ *are defined by* (27) *is a (possibly discontinuous) classical solution to* (13) *with corresponding initial conditions.*

The theorem should be completed by the following two short remarks. First, the solution to (21) can be constructed by steps *via* the system (28). Observe also that system (13) is a system of coupled delay differential and difference equations and, as mentioned in Theorem 1, the variables of the difference part, i.e., $w^{\pm}(t)$ can have discontinuities, their smoothness remaining the same along the "time" (independent variable t). This suggests that, viewed as a system of equations with deviated argument, (21) must be considered of *neutral type*; the assertion is in accordance also with the classical (by now) classification of G. A. Kamenskii [8]. Moreover, in [5]—one of the papers developing the association of the equations with deviated argument to boundary value problems by integration along the characteristics—there is given a simple classification criterion for the resulting equations and, according to it, (21) results again of neutral type. Next, the representation formulae (30) allow representation of the solutions to (8)

$$h(x,t) = \frac{1}{2}\big[w^{+}(t - x/a) + w^{-}(t + (x - L)/a)\big],$$

$$v(x,t) = \frac{1}{2\delta_0}\big[w^{+}(t - x/a) - w^{-}(t + (x - L)/a)\big]. \tag{31}$$

In this way, as stated elsewhere [32], a one-to-one correspondence between the solutions to (13)- or (8)- and (21) has been established and all properties obtained for one mathematical object are projected back on the other one.

With respect to this, let us observe that the following result has been obtained as a by-product during the construction by steps of the solution to (21) *via* system (28).

Theorem 2. *Consider the system* (21) *with the initial condition satisfying* $w_0^+(t) \geq 0$ *on* $[-L/a, 0)$. *Then* $w^+(t) \geq 0$ *and* $w^-(t) + w^+(t - L/a) > 0$, $w^-(t) - w^+(t - L/a) < 0$ *for all* $t > 0$ *for which these functions are defined.*

The significance of this theorem is existence of an invariant set with physical meaning. If we refer to system (13) this means that $u^+(x,t) \geq 0$ for all $t > 0$ and $0 \leq x \leq L$ provided $u_0^+(x) \geq 0$ for $0 \leq x \leq L$. For system (8) this means

$$h(x,t) \geq 0;\ \ h(x,t) + \delta_0 v(x,t) \geq 0; \quad \text{for all} \quad t > 0;\ \ 0 \leq x \leq L. \tag{32}$$

At the boundaries we shall have $v(L,t) > 0$, $h(L,t) > 0$.

4.3. Equilibria. Deviations. Linearization

As discussed in the previous sections, equilibria, i.e., constant solutions are important since they represent the so-called *operating points* of the hydroelectric plant. We consider system (13): for a given level $\bar{\nu}_g$ of the required power, the constant solution to (13) is obtained from

$$\bar{u}^+ + \bar{u}^- = 2,\ \ \bar{u}^+ - \bar{u}^- = \delta_0\sqrt{2}\bar{f}\sqrt{\bar{u}^+ + \bar{u}^-};\ \ \bar{f}(\bar{u}^+ - \bar{u}^-) = (2\delta_0)^3\bar{\nu}_g.$$

We deduce

$$\bar{w}^\pm = \bar{u}^\pm = 1 \pm \delta_0\sqrt[4]{\bar{\nu}_g};\ \ \bar{f} = \sqrt[4]{\bar{\nu}_g}.$$

It is obvious that the operating point is fully determined by the power demand $\bar{\nu}_g$.

We introduce now the *system in deviations*—a further step towards *stability analysis*. Define the deviations

$$\nu^\pm(x,t) = u^\pm(x,t) - \bar{u}^\pm,\ \ \zeta^\pm(t) = w^\pm(t) - \bar{u}^\pm;\ \ \mu(t) = f(t) - \bar{f}. \tag{33}$$

With these notations system (21) becomes

$$\begin{aligned} &\zeta^+(t) + \zeta^-(t - L/a) = 0,\\ &\zeta^+(t - L/a) - \zeta^-(t) + 2\delta_0\bar{f} = \delta_0\sqrt{2}(\mu(t) + \bar{f})\\ &\qquad\qquad\qquad\qquad\qquad\qquad \times\sqrt{\zeta^-(t) + \zeta^+(t - L/a) + 2},\\ &(2\delta_0)^3 T_a\frac{\mathrm{d}s}{\mathrm{d}t} = \big(\mu(t) + \bar{f}\big)\big(\zeta^+(t - L/a) - \zeta^-(t) + 2\delta_0\bar{f}\big)^3 - (2\delta_0)^3\nu_g. \end{aligned} \tag{34}$$

Linearization around the zero solution will give

$$\begin{aligned} &\zeta^+(t) + \zeta^-(t - L/a) = 0,\\ &\zeta^+(t - L/a) - \zeta^-(t) = (\delta_0/2)\bar{f}\big(\zeta^-(t) + \zeta^+(t - L/a)\big) + 2\delta_0\mu(t),\\ &(T_a/\bar{f}^3)\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{3}{2\delta_0}\big(\zeta^+(t - L/a) - \zeta^-(t)\big) + \mu(t). \end{aligned} \tag{35}$$

This system can be given in the standard form, see, e.g., [32],

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 y(t-\tau) + b_1 u(t),\\ y(t) &= A_2 x(t) + D y(t-\tau) + b_2 u(t)\end{aligned} \tag{36}$$

that is

$$\begin{aligned}&(T_a/\bar{f}^3)\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{3}{2\delta_0}(1-\rho)\zeta^+(t-L/a) + \left(1+\frac{3}{2}(1+\rho)\mu(t)\right),\\ &\zeta^+(t) = \zeta^-(t-L/a),\\ &\zeta^-(t) = \rho\zeta^+(t-L/a) - \delta_0(1+\rho)\mu(t),\end{aligned} \tag{37}$$

where we denoted $\rho = (1-\delta_0\bar{f}/2)(1+\delta_0\bar{f}/2)^{-1}$. With respect to (36) we shall have

$$A_0 = 0; \quad A_1 = \frac{1}{T_a/\bar{f}^3}\big((3/(2\delta_0))(1-\rho)\;0\big)\,; \quad b_1 = \frac{1+(3/2)(1+\rho)}{T_a/\bar{f}^3},$$

$$A_2 = 0; \quad D = \begin{pmatrix} 0 & -1\\ \rho & 0\end{pmatrix}; \quad b_2 = \begin{pmatrix} 0\\ -\delta_0(1+\rho)\end{pmatrix}.$$

4.4. Inherent stability

We shall discuss now inherent stability of (37): since it is a system in deviations, the discussion concerns the stability of the zero solution to (37) with $\mu(t) \equiv 0$. The characteristic equation reads

$$\begin{aligned}\pi(\lambda) &= \begin{vmatrix} (T_a/\bar{f}^3)\lambda & -\big(3/(2\delta_0)\big)(1-\rho)\mathrm{e}^{-\lambda L/a} & 0\\ 0 & 1 & \mathrm{e}^{-\lambda L/a}\\ 0 & -\rho\mathrm{e}^{-\lambda L/a} & 1\end{vmatrix}\\ &= (T_a/\bar{f}^3)\lambda(1+\rho\mathrm{e}^{-2\lambda L/a}).\end{aligned}$$

The following properties of $\pi(\lambda)$, accounting for the aforementioned stability, are true:

i) $\lambda = 0$ is a simple root of $\pi(\lambda)$;

ii) the roots of the second factor of $\pi(\lambda)$ are connected to the unique root of $z + \rho = 0$ as follows.

Observe first that $|\rho| < 1$. This means that the roots of $1 + \rho\mathrm{e}^{-2\lambda L/a}$ are always in the left half plane of $\mathbb{C}$: if $\lambda = \sigma + \imath\omega$ then all the roots are on the vertical $\sigma = -(a/2L)\ln|\rho|^{-1} < 0$.

This aspect requires some additional comments. The factor $1 + \rho e^{-2\lambda L/a}$ accounts for the characteristic equation of the difference subsystem in (37); for $\mu(t) \equiv 0$ this subsystem is decoupled of the differential equation. Since its characteristic equation has its roots subject to $\Re e(\lambda) \leq -\alpha$ for some $\alpha > 0$, the difference operator is strongly stable [13]—a basic necessary condition for many results concerning neutral systems with deviated argument.

From the engineering point of view, inherent non-asymptotic stability is unacceptable and requires feedback stabilization; feedback is used not only for stabilization but also for ensuring other properties such as robustness and disturbance rejection; however, these last aspects are outside the purpose of the present paper.

5. Hints for simple stabilizing feedback

5.1. Lyapunov functionals

For methodological reasons we shall neglect here the dynamics of the control structure (actuator, slide-valve, corrector) and focus on feedback synthesis structure, more precisely on some hints concerning its synthesis. As mentioned, e.g., in [32] the feedback control synthesis can be performed at a formal level, being viewed as some sort of inference. Once the closed loop structure obtained, the rigorous analysis—in fact a model *revalidation*—will be concerned with the aforementioned closed loop structure.

We shall start here from the equations (8) with the notation (9) and recall here the equations for the steady-state (equilibria)

$$\bar{h}(x) \equiv \text{const} = 1, \;\; \bar{v} = \bar{f} = \sqrt[4]{\nu_g} \tag{38}$$

for a given required power level; the steady state synchronous speed has to be imposed by the control system thus leading to $\bar{s} = 0$. If the deviations are considered

$$\chi(x,t) = h(x,t) - 1, \;\; \nu(x,t) = v(x,t) - \bar{v}; \;\; \mu(t) = f(t) - \bar{f}, \tag{39}$$

the system in deviation is obtained

$$\begin{aligned}
&\partial_t \nu + \frac{a}{\delta_0}\partial_x \chi = 0, \;\; \partial_t \chi + a\delta_0 \partial_x \nu = 0, \\
&\chi(0,t) = 0, \;\; \nu(L,t) + \bar{v} = \big(\mu(t) + \bar{f}\big)\sqrt{\chi(L,t) + 1}, \\
&T_a \frac{\mathrm{d}s}{\mathrm{d}t} = \big(\mu(t) + \bar{f}\big)\big(\nu(L,t) + \bar{v}\big)^3 - \nu_g.
\end{aligned} \tag{40}$$

The difficulty is given here by the nonlinear boundary conditions. The suggested approach will rely on the inference of a control Lyapunov functional.

We give here two possible control Lyapunov functional to adopt. The first one is deduced from the energy identity [9]

$$\frac{\delta_0}{2a}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\nu^2(x,t)+\frac{1}{\delta_0^2}\chi^2(x,t)\right)\mathrm{d}x+\chi(x,t)\nu(x,t)\,\Big|_0^L\equiv 0 \tag{41}$$

and might be (written as a state function on a function space)

$$\mathcal{V}_1\big(\nu(\cdot),\chi(\cdot),s\big)=\frac{1}{2}\left(\gamma_0 T_a s^2+\frac{\delta_0}{a}\int_0^L\left(\nu^2(x)+\frac{1}{\delta_0^2}\chi^2(x)\right)\mathrm{d}x\right). \tag{42}$$

Another Lyapunov functional, supposed to ensure (possibly) exponential stabilization is suggested by [6] and might be

$$\mathcal{V}_2\big(\nu(\cdot),\chi(\cdot),s\big)=\frac{1}{2}\gamma_0 T_a s^2+U_1\big(\nu(\cdot),\chi(\cdot)\big)+U_2\big(\nu(\cdot),\chi(\cdot)\big), \tag{43}$$

where

$$U_1\big(\nu(\cdot),\chi(\cdot)\big)=\frac{\gamma_1}{a}\int_0^L\big(\chi(x)+\delta_0\nu(x)\big)^2\mathrm{e}^{-(\alpha/a)x}\,\mathrm{d}x,$$

$$U_2\big(\nu(\cdot),\chi(\cdot)\big)=\frac{\gamma_2}{a}\int_0^L\big(\chi(x)-\delta_0\nu(x)\big)^2\mathrm{e}^{(\alpha/a)x}\,\mathrm{d}x \tag{44}$$

with γ_0, γ_1, γ_2, α some free positive parameters.

5.2. Stability by the first approximation

In the sequel we shall consider stability by the first approximation of (39)–(40). Linearizing also the boundary conditions we obtain the linear system

$$\begin{aligned}&\partial_t\nu+\frac{a}{\delta_0}\partial_x\chi=0,\ \ \partial_t\chi+a\delta_0\partial_x\nu=0,\\&\chi(0,t)=0,\ \ \nu(L,t)=\frac{\bar f}{2}\chi(L,t)+\mu(t),\\&T_a'\frac{\mathrm{d}s}{\mathrm{d}t}=3\nu(L,t)+\mu(t)\end{aligned} \tag{45}$$

with $T_a'=T_a/\bar f^3$. Based on the energy identity in this case

$$\frac{\delta_0}{2a}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\nu^2(x,t)+\frac{1}{\delta_0^2}\chi^2(x,t)\right)\mathrm{d}x+\chi(L,t)\left(\frac{\bar f}{2}\chi(L,t)+\mu(t)\right)\equiv 0 \tag{46}$$

we shall consider the Lyapunov functional

$$\mathcal{V}\big(\nu(\cdot),\chi(\cdot),s\big) = \frac{1}{2}\left(\gamma_0 T'_a s^2 + \frac{\delta_0}{a}\int\limits_0^L \left(\nu^2(x) + \frac{1}{\delta_0^2}\chi^2(x)\right)\mathrm{d}x\right) > 0 \tag{47}$$

(positiveness is viewed in the sense of its own metrics). Here $\gamma_0 > 0$ is a free parameter. The derivative of $\mathcal{V}$ along the solutions to (45) will be

$$\mathcal{W}\big(\chi(L,t),s\big) = -\frac{\bar f}{2}\chi^2(L,t) + \frac{3\bar f}{2}\gamma_0 s\chi(L,t) + \mu(t)\big(-\chi(L,t) + 4\gamma_0 s\big).$$

By choosing

$$\mu(t) = -\gamma_1\big(4\gamma_0 s - \chi(L,t)\big) \tag{48}$$

with $\gamma_1 > 0$ another free parameter, we obtain

$$\begin{aligned}&\mathcal{W}\big(\chi(L,t),s\big)\\ &= -\left[\left(\gamma_1 + \frac{\bar f}{2}\right)\chi^2(L,t) - \gamma_0\left(\frac{3\bar f}{2} + 8\gamma_1\right)s\chi(L,t) + 16\gamma_1\gamma_0^2 s^2\right] < 0\end{aligned} \tag{49}$$

provided $\gamma_1 > 9/32 > (9\bar f)/32$.

Consequently, the closed loop system is obtained (the linearized system)

$$\begin{aligned}&\partial_t\nu + \frac{a}{\delta_0}\partial_x\chi = 0,\ \ \partial_t\chi + a\delta_0\partial_x\nu = 0,\\ &\chi(0,t) = 0,\ \ \nu(L,t) = \left(\gamma_1 + \frac{\bar f}{2}\right)\chi(L,t) - 4\gamma_0\gamma_1 s,\\ &T'_a\frac{\mathrm{d}s}{\mathrm{d}t} = 3\nu(L,t) + \gamma_1\chi(L,t) - 4\gamma_0\gamma_1 s.\end{aligned} \tag{50}$$

The aforementioned Lyapunov functional (47) has its derivative (49) negative semi-definite. Application of the Barbashin Krasovskii LaSalle invariance principle is thus necessary but this principle is valid for functional differential equations. We turn thus back to the representation formulae (18)–(20) written in deviations, see (33), to associate the system of differential and difference equations

$$\begin{aligned}&T'_a\frac{\mathrm{d}s}{\mathrm{d}t} = -4\gamma_0\gamma_1 s + \left(\gamma_1 - \frac{3}{2\delta_0}\right)\zeta^-(t) + \left(\gamma_1 + \frac{3}{2\delta_0}\right)\zeta^+(t - L/a),\\ &\zeta^+(t) + \zeta^-(t - L/a) = 0,\\ &\left(\gamma_1 + \frac{1}{2\delta_0} + \frac{\bar f}{2}\right)\zeta^-(t) + \left(\gamma_1 + \frac{1}{2\delta_0} - \frac{\bar f}{2}\right)\zeta^+(t - L/a) = 4\gamma_0\gamma_1 s.\end{aligned}$$

Eliminating $\zeta^+(t)$, for stability studies, the following system is obtained

$$T_a''\frac{\mathrm{d}s}{\mathrm{d}t} = -4\gamma_0\gamma_1\left(\frac{2}{\delta_0}+\frac{\bar f}{2}\right) - \left[\gamma_1\bar f + \frac{3}{\delta_0}\left(\gamma_1+\frac{1}{2\delta_0}\right)\right]\zeta^-(t-2L/a),$$

$$\zeta^-(t) = \frac{\gamma_1+1/(2\delta_0)-\bar f/2}{\gamma_1+1/(2\delta_0)+\bar f/2}\zeta^-(t-2L/a) + \frac{4\gamma_0\gamma_1}{\gamma_1+1/(2\delta_0)+\bar f/2}s\,, \tag{51}$$

where we denoted

$$T_a'' = T_a'(\gamma_1+1/(2\delta_0)+\bar f/2).$$

On the other hand, using the representation formulae (31) and (33) we obtain

$$\chi(x,t) = \frac{1}{2}\left[\zeta^+(t-x/a)+\zeta^-\big(t+(x-L)/a\big)\right],$$

$$\nu(x,t) = \frac{1}{2\delta_0}\left[\zeta^+(t-x/a)-\zeta^-\big(t+(x-L)/a\big)\right] \tag{52}$$

and re-write the Lyapunov functional (47) along the solutions to (51) as

$$\mathcal{V}(\zeta_t^-(\cdot),s) = \frac{1}{2}\left[\gamma_0'T_a''s^2 + \frac{1}{2\delta_0}\int\limits_{-2L/a}^{0}\zeta^-(t+\theta)^2\,\mathrm{d}\theta\right], \tag{53}$$

where $\gamma_0'T_a'' = \gamma_0T_a'$. The derivative (49) is re-written as

$$\begin{aligned}\mathcal{W}\big(\zeta_t^-,s(t)\big) =& -16\gamma_1\gamma_0^2s(t)^2 + \gamma_0\left(8\gamma_1+\frac{3\bar f}{2}\right)s(t)\big(\zeta^-(t)-\zeta^-(t-2L/a)\big)\\ &-\left(\gamma_1+\frac{\bar f}{2}\right)\big(\zeta^-(t)-\zeta^-(t-2L/a)\big)^2.\end{aligned} \tag{54}$$

Its kernel will be defined by $s=0$, $\zeta^-(t)=\zeta^-(t-2L/a)$ and it follows easily that the only invariant set in it is $\zeta^-(t)\equiv 0$, $s(t)\equiv 0$. Application of the Barbashin Krasovskii LaSalle invariance principle ([13, Theorem 9.8.2]) will give asymptotic stability of (51). Since this system is linear, the stability is exponential, due to the fact that in the second equation of (51) we have

$$\left|\frac{\gamma_1+1/(2\delta_0)-\bar f/2}{\gamma_1+1/(2\delta_0)+\bar f/2}\right| < 1$$

with the significance that the difference operator is strongly stable. As a consequence, the characteristic equation of (51) will have all its roots satisfying $\Re e(\lambda) \le -\alpha < 0$ for some $\alpha > 0$ sufficiently small. From (52) we deduce exponential stability of (50). We have thus obtained the following mathematical result.

Theorem 3. *Consider the linear controlled system* (45). *If the feedback control signal $\mu(t)$ is taken from the feedback law* (48)*, the linear closed loop* (50) *is exponentially stable.*

To conclude this section, we just mention that exponential stability of the linear first approximation system normally implies exponential stability of the nonlinear system in some bounded domain of the state space. The aforementioned statement should be valid for the system of functional differential equations hence for the system of the mixed problem (39) also (considered with the linear feedback (48)). However, the classical monographs of N. N. Krasovskii [16] and A. Halanay [11], while giving theorems on stability by the first approximation, do not consider the case of the neutral equations. Unlike these references, the monograph [13] deals with stability for neutral systems but not with stability by the first approximation. Nevertheless, the proof of such a theorem should not be difficult when a Lyapunov functional is available; moreover, this proof is usually accompanied by some estimate of the stability domain in the state space. These problems will be tackled elsewhere.

6. Conclusion and perspective

The aim of this paper has been that of emphasizing some applications modeled by non-standard boundary problems for systems of conservation laws in one space dimension. Their specificity is that neglecting some nonlinear terms is sometimes acceptable from the point of view of the applications (physics and engineering). In this way their models are reduced to non-standard boundary value problems for the linear hyperbolic equations of the lossless propagation thus allowing a one-to-one correspondence between their solutions and the solutions to the system of functional differential and difference equations of neutral type.

Besides standard model validation, i.e., well-posedness in the sense of J. Hadamard, this one-to-one correspondence allows to obtain existence of some useful invariant sets and inherent stability for the equilibria of the uncontrolled system. The paper contains some suggestions for the choice of a control Lyapunov functional allowing feedback control synthesis. For one of the considered applications (the hydroelectric power plant) a stabilizing feedback synthesis is performed in the linearized case. At its turn this linear stability result should imply local stability of the nonlinear system—stability by the first approximation. Since theorems on stability by the first approximation are not explicitly known for neutral equations, two ways are thus possible: either a direct proof based on the available Lyapunov functionals or proving the aforementioned theorems of stability by the first approximation in the general case. Normally these theorems are accompanied by estimates of the "stability" (attraction) domain.

The relatively small steps accomplished for the aforementioned application contain nevertheless many suggestions for future development: some quadratic control Lyapunov functionals that might be adapted to the nonlinear cases (i.e., with nonlinear boundary conditions) as well as possible approaches aiming to global stability for the systems with nonlinear boundary conditions. These suggestions concern all three applications of the paper as well as other linearizable conservation laws. They might constitute a research programme.

REFERENCES

[1] ARONOVICH, G. V.: *Waterhammer influence on the control stability for hydraulic turbines*, Avtom. i telemekhanika **9** (1948), 204–232. (In Russian)

[2] ARONOVICH, G. V.—KARTVELISHVILI, N. A—LYUBIMTSEV, YA. K.: *Waterhammer and Surge Tanks*. Nauka Publ. House, Moscow, 1968. (In Russian)

[3] BRESSAN, A.: *Hyperbolic Systems of Conservation Laws. The One-dimensional Cauchy Problem*. Oxford University Press, Oxford, 2000.

[4] ČETAEV, N. G.: *Stability and the classical laws*, Coll. Sci. Works Kazan Aviation Inst. **5** (1936), 3–18. (In Russian)

[5] COOKE, K. L.: *A linear mixed problem with derivative boundary conditions*, in: Seminar on Differential Equations and Dynamical Systems (III) (D. Sweet and J. A. Yorke, eds.), Lecture Series, Vol. 51, University of Maryland, College Park, 1970, pp. 11–17.

[6] CORON, J.-M.—D'ANDRÉA-NOVEL, B.—BASTIN, G.: *A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws*, IEEE Trans. on Autom. Control **52** (2007), 2–11.

[7] DAFERMOS, C. M.: *Hyperbolic Conservation Laws in Continuum Physics*. In: Grundlehren der mathematischen Wissenschaften, Vol. 325, Springer-Verlag, Berlin, 2005.

[8] EL'SGOL'TS, L. E.—NORKIN, S. B.: *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*. In: Mathematics in Science and Engineering, Vol. 105, Academic Press, New York, 1973.

[9] GODUNOV, S. K.: *Équations de la physique mathématique*. Éditions Mir, Moscou, 1978.

[10] GODUNOV, S. K.—ROMENSKII, E. I.: *Elements of Continuum Mechanics and Conservation Laws*. Springer, New York, 2003.

[11] HALANAY, A.: *Differential Equations. Stability. Oscillations. Time Lags*. In: Mathematics in Science and Engineering, Vol. 23, Academic Press, New York, 1966.

[12] HALANAY, A.—RĂSVAN, V.: *Stabilization of a class of bilinear control systems with application to steam turbine regulation*, Tohoku Math. J. (2) **32** (1980), 299–308.

[13] HALE, J. K.—VERDUYN LUNEL, SJOERD M.: *Introduction to Functional Differential Equations*. In: Appl. Math. Sci., Vol. 99, Springer-Verlag, New York, 1993.

[14] DE HALLEUX, J.—PRIEUR, C.—CORON, J.-M.—D'ANDRÉA–NOVEL, B.—BASTIN, G.: *Boundary feedback control in networks of open channels*, Automatica **39** (2003), 1365–1376.

[15] KABAKOV, I. P.: *Concerning the control process for the steam pressure*, Inzh. sbornik **2** (1946), 27–60. (In Russian)

[16] KRASOVSKII, N. N.: *Stability of Motion*. Stanford University Press, Stanford, 1963.

[17] LAX, P. D.: *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. In: CBMS Regional Conference Series in Mathematics, Vol. 11, SIAM Publications, Philadelphia, 1973.

[18] LAX, P. D.: *Hyperbolic Partial Differential Equations*. In: Courant Lecture Notes in Mathematics, Vol. 14, AMS-Courant Institute of Mathematical Sciences, Providence, 2006.
[19] LEFLOCH, P. G.: *Hyperbolic Systems of Conservation Laws*. Birkhäuser, Basel, 2002.
[20] LEUGERING, G.—SCHMIDT, E. J. P. G.: *On the modeling and stabilization of flows in networks of open canals*, SIAM J. Contr. Optim. **41** (2002), 164–180.
[21] LI, T.-T.—YU, W.-C.: *Boundary Value Problems for Quasilinear Hyperbolic Systems*. Duke University Math. Series V, Durham, 1985.
[22] LI, T.-T.: *Global Classical Solutions for Quasilinear Hyperbolic Systems*. In: Research Appl. Math., Vol. 32, Wiley, Chichester, 1994.
[23] LI, T.-T.—WANG, L.-B.: *Global Propagation of Regular Nonlinear Hyperbolic Waves*. In: Progr. Nonlinear Differential Equations Appl., Vol. 76, Birkhäuser, Berlin, 2009.
[24] LI, T.-T.—QIN, T.-H.: *Physics and Partial Differential Equations, Vol. I-II*. SIAM Publications, Philadelphia, 2012.
[25] LIU, T. P.: *Hyperbolic and Viscous Conservation Laws*. In: CBMS-NSF Regional Conference Series in Mathematics, Vol. 72, SIAM Publications, Philadelphia, 2000.
[26] NEYMARK, YU. I.: *Dynamical Systems and Controlled Processes*. Nauka Publishing House, Moscow, 1978. (In Russian)
[27] PETRE, E.—RĂSVAN, V.: *Feedback control of conservation laws systems. Part I: Models*, Rev. Roum. Sci. Techn. Sér. Electr. Energ. **54** (2009), 311–320.
[28] POPESCU, M.—ARSENIE, D.—VLASE, P.: *Applied Hydraulic Transients: For Hydropower Plants and Pumping Stations*. Taylor & Francis, Oxford, 2003.
[29] POPESCU, M.: *Hydroelectric Plants and Pumping Stations*. Editura Universitară, Bucharest, 2008. (In Romanian)
[30] RĂSVAN, V.: *Stability of bilinear control systems occurring in combined heat electricity generation I: The mathematical models and their properties*, Rev. Roum. Sci. Techn. Sér. Electr. Energ. **26** (1981), 455–465.
[31] ______ *Stability of bilinear control systems occurring in combined heat electricity generation II: Stabilization of the reduced models*, Rev. Roum. Sci. Techn. Sér. Electr. Energ. **29** (1984), 423–432.
[32] ______ *Augmented validation and a stabilization approach for systems with propagation*, in: Systems Theory: Perspectives, Applications and Developments (F. Miranda, ed.), Systems Science Series, Vol. 1, Nova Science Publ., New York, 2014, pp. 125–170.
[33] ______ *The stability postulate of N. G. Četaev and the augmented model validation*, IFAC PapersOnLine **50** (2017), 7450–7455.
[34] SERRE, D.: *Systèmes de Lois de Conservation. Vol. I-II*. Diderot, Paris, 1996.
[35] ______ *Systems of conservation laws: A challenge for the XXIst century*, in: Mathematics Unlimited – 2001 and Beyond (B. Engquist and W. Schmid, eds.), Springer, Berlin, 2001, pp. 1061–1080.
[36] SOLODOVNIKOV, V.V.: *Operational method application to the study of the speed control process of the hydraulic turbines* Avtom. i telemekhanika 6 (1941) 5–20. (In Russian)

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