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ON SOME DISCRETE POTENTIAL LIKE OPERATORS

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ABSTRACT. We consider some discrete pseudo-differential equations in discrete Sobolev–Slobodetskii spaces. For a discrete half-space and certain values of an index of periodic factorization for an elliptic symbol we introduce additional potential-like unknowns and prove existence and uniqueness theorem in appropriate discrete Sobolev–Slobodetskii spaces.

1. Introduction

We will consider model discrete elliptic pseudo-differential equations in special discrete canonical domain $D_d = D \cap h\mathbb{Z}^m$, where $D \subset \mathbb{R}^m$ is a domain, now it will be the discrete half-space

$$D_d = h\mathbb{Z}^m_+ = h\mathbb{Z}^m \cap \mathbb{R}^m_+, \quad \mathbb{R}^m_+ = \big\{x \in \mathbb{R}^m \colon x = (x', x_m),\ x_m > 0\big\}, \quad h > 0.$$

Our main goal is to describe invertibility conditions for an elliptic model pseudo-differential operator in such a canonical discrete domain. It was shown earlier in authors' papers that these conditions depend on the index of periodic factorization of an elliptic symbol. There arise different discrete boundary value problems with (co)boundary conditions determined by the index of factorization and the order of Sobolev–Slobodetskii space. This paper is devoted to the special case when we need to enlarge the number of unknown functions and introduce under consideration certain discrete potential like operators.

The basic object under consideration is the following operator equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \qquad \tilde{x} \in D_d, \tag{1}$$

where A_d is a discrete (or digital) pseudo-differential operator, D_d is a special canonical discrete domain in m-dimensional space $\mathbb{R}^m$.

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The equation (1) for continuous situation was studied earlier [5], but there are a lot of problems related to discrete and computational aspects. To construct good computational algorithms we need a good discrete solvability theory for the equation (1). For some simplest operators A_d generated for example by Calderon–Zygmund operators it was done in authors' papers [12], [13] using the theory of periodic Riemann boundary value problem [14]. This statement of the periodic Riemann boundary value problem was initiated by solving discrete equations like (1) and it was based on classical results [1]–[4].

We widely use a factorization idea which was very useful for many cases and similar discrete and continuous equations [9]–[11], [15] and hope it will help us one more time.

The main difference from other papers related to similar problems (see, for example, [6], [7]) is the following. They consider the given boundary value problem, construct a discrete approximation (or first reduce the boundary value problem to discrete variants of integral equations), prove a solvability for this discrete problem and give error estimates. We try to construct a discrete theory of pseudo-differential equations like [5] so that there is a certain correspondence between a solvability of discrete and continuous equations (see, e. g., [13]) and boundary value problems for them, and after it obtain error estimates for discrete solutions.

Some preliminary results (certain ones for another canonical domain) are given in [16]–[20].

2. Preliminaries

2.1. Notations and definitions

2.1.1. Discrete and periodic objects

We will use the following notations. Let $\mathbb{T}^m$ be the m-dimensional cube $[-\pi,\pi]^m$, $h>0$, $\hbar=h^{-1}$. We will consider all functions defined on a cube as periodic functions in $\mathbb{R}^m$ with the same cube of periods.

If $u_d(\tilde{x})$, $\tilde{x}\in h\mathbb{Z}^m$, is a function of a discrete variable, then we call it "discrete function". For such discrete functions one can define the discrete Fourier transform

$$(F_d u_d)(\xi)\equiv \tilde{u}_d(\xi)=\sum_{\tilde{x}\in h\mathbb{Z}^m} e^{-i\tilde{x}\cdot\xi}u_d(\tilde{x})h^m, \qquad \xi\in\hbar\mathbb{T}^m,$$

if the last series converges, and the function $\tilde{u}_d(\xi)$ is a periodic function on $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$. The discrete Fourier transform is a one-to-one correspondence between the spaces $L_2(h\mathbb{Z}^m)$ and $L_2(\hbar\mathbb{T}^m)$ with norms

$$||u_d||_2 = \left(\sum_{\tilde{x}\in h\mathbb{Z}^m} |u_d(\tilde{x})|^2 h^m\right)^{1/2}$$

and

$$||\tilde{u}_d||_2 = \left(\int\limits_{\xi\in\hbar\mathbb{T}^m} |\tilde{u}_d(\xi)|^2 \mathrm{d}\xi\right)^{1/2}.$$

We will use the discrete Fourier transform to introduce special discrete Sobolev–Slobodetskii spaces which are very convenient for studying discrete pseudo-differential operators and related equations.

2.1.2. Discrete functions and functional spaces

Since the definition for Sobolev–Slobodetskii spaces includes partial derivatives, we use their discrete analogue, i.e., divided difference of first order

$$\big(\Delta_k^{(1)} u_d\big)(\tilde{x}) = h^{-1}\big(u_d(x_1,\dots,x_k+h,\dots,x_m) - u_d(x_1,\dots,x_k,\dots,x_m)\big),$$

for which its discrete Fourier transform looks as follows

$$\big(\widetilde{\Delta_k^{(1)} u_d}\big)(\xi) = h^{-1}(e^{-ih\cdot\xi_k}-1)\tilde{u}_d(\xi).$$

Further, for the divided difference of second order we have

$$\begin{aligned}\big(\Delta_k^{(2)} u_d\big)(\tilde{x}) = h^{-2}\big(&u_d(x_1,\dots,x_k+2h,\dots,x_m)\\ &- 2u_d(x_1,\dots,x_k+h,\dots,x_m) + u_d(x_1,\dots,x_k+h,\dots,x_m)\big)\end{aligned}$$

and its discrete Fourier transform

$$\big(\widetilde{\Delta_k^{(2)} u_d}\big)(\xi) = h^{-2}\big(e^{-ih\cdot\xi_k}-1\big)^2\tilde{u}_d(\xi).$$

Thus, for the discrete Laplacian we have

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m}\big(\Delta_k^{(2)} u_d\big)(\tilde{x}),$$

so that

$$(\widetilde{\Delta_d u_d})(\xi) = h^{-2}\sum_{k=1}^{m}\big(e^{-ih\cdot\xi_k}-1\big)^2\tilde{u}_d(\xi).$$

Let us denote $\zeta^2 = h^{-2}\sum_{k=1}^{m}(e^{-ih\cdot\xi_k}-1)^2$ and introduce the following

Definition 1. The space $H^s(h\mathbb{Z}^m)$ consists of discrete functions $u_d(\tilde{x})$ for which the norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbb{T}^m} (1+|\zeta^2|)^s |\tilde{u}_d(\xi)|^2 \mathrm{d}\xi\right)^{1/2}$$

is finite.

Let us note that such spaces were systematically studied in [8], and we will use some their properties.

Further, let $D \subset \mathbb{R}^m$ be a domain, and $D_d = D \cap h\mathbb{Z}^m$ be a discrete domain.

Definition 2. The space $H^s(D_d)$ consists of discrete functions from $H^s(h\mathbb{Z}^m)$ which supports belong to $\overline{D_d}$. A norm in the space $H^s(D_d)$ is induced by a norm of the space $H^s(h\mathbb{Z}^m)$. The space $H^s_0(D_d)$ consists of discrete functions u_d with a support in D_d, and these discrete functions should admit a continuation into the whole $H^s(h\mathbb{Z}^m)$. A norm in the $H^s_0(D_d)$ is given by the formula

$$||u_d||_s^+ = \inf ||\ell u_d||_s,$$

where infimum is taken over all continuations ℓ.

The Fourier image of the space $H^s(D_d)$ will be denoted by $\widetilde{H}^s(D_d)$, and $\widetilde{H}^s(h\mathbb{Z}^m) \equiv H^s(\hbar\mathbb{T}^m)$. For the space $\widetilde{H}^s(D_d)$ one has a description like the Wiener–Paley theorem [5], it can be obtained by the same methods.

2.1.3. Digital pseudo-differential operators

Let $\widetilde{A}_d(\xi)$ be a periodic function in $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$. Such functions are called symbols. As usual we will define a digital pseudo-differential operator by its symbol.

Definition 3. A digital pseudo-differential operator A_d in a discrete domain D_d is called an operator of the following kind

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m} \int\limits_{\hbar\mathbb{T}^m} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi)\,\mathrm{d}\xi\,, \qquad \tilde{x}\in D_d\,,$$

An operator A_d is called an elliptic operator if

$$ess \inf_{\xi\in\hbar\mathbb{T}^m} |\widetilde{A}_d(\xi)| > 0.$$

Remark 1. One can introduce the symbol $\widetilde{A}_d(\tilde{x},\xi)$ depending on a spatial variable $\tilde{x}$ and define a general pseudo-differential operator by the formula

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m} \int\limits_{\hbar\mathbb{T}^m} \widetilde{A}_d(\tilde{x},\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi)\,\mathrm{d}\xi\,, \qquad \tilde{x}\in D_d\,.$$

For studying such operators and related equations, one needs to use more fine and complicated technique.

Definition 4. By definition the class E_α includes symbols satisfying the following condition

$$c_1\big(1+|\zeta^2|\big)^{\alpha/2} \le |A_d(\xi)| \le c_2\big(1+|\zeta^2|\big)^{\alpha/2} \tag{2}$$

with positive constants c_1, c_2 non-depending on h.

The number $\alpha \in \mathbb{R}$ is called an order of a digital pseudo-differential operator A_d. Roughly speaking the order of a digital pseudo-differential operator is the power of h with the sign "minus".

Using the last definition, one can easily get the following property.

Lemma 1. *A digital pseudo-differential operator $A_d \in E_\alpha$ is a linear bounded operator $H^s(h\mathbb{Z}^m) \to H^{s-\alpha}(h\mathbb{Z}^m)$.*

2.1.4. Discrete pseudo-differential equations

We study the equation (1) assuming that we are interested in a solution $u_d \in H^s(D_d)$ taking into account $v_d \in H_0^{s-\alpha}(D_d)$.

Main difficulty for this problem is related to a geometry of the domain D. Indeed, if $D = \mathbb{R}^m$, then the condition (2) guarantees the unique solvability for the equation (1). We will consider here only so-called canonical domains and simplest digital pseudo-differential operators with symbols non-depending on a spatial variable $\tilde{x}$. This fact is dictated by using in future the local principle. The last asserts that for a Fredholm solvability of the general equation (1) with symbol $A_d(\tilde{x}, \xi)$ in an arbitrary discrete domain D_d, one needs to obtain invertibility conditions for so-called local representatives of the operator A_d, i.e., for an operator with symbol $A_d(\cdot, \xi)$ in a special canonical domain.

Earlier authors have extracted some canonical domains, namely $D = \mathbb{R}^m$, $\mathbb{R}^m_+, C^a_+$, where $\mathbb{R}^m_+ = \big\{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\big\}, C^a_+ = \big\{x \in \mathbb{R}^m : x_m > a|x'|, a > 0\big\}$. Methods for studying two last cases are related to special boundary value problems for holomorphic functions.

2.2. Technical tools and periodic Riemann boundary value problem

Let us denote P_+, P_- projection operators on $D_d, h\mathbb{Z}^m \setminus D_d$, respectively, i.e.,

$$(P_+u_d)(\tilde{x}) = \begin{cases} u_d(\tilde{x}), & \tilde{x} \in D_d\,; \\ 0, & \tilde{x} \in h\mathbb{Z}^m \setminus D_d\,, \end{cases}$$

and analogously,

$$(P_-u_d)(\tilde{x}) = \begin{cases} u_d(\tilde{x}), & \tilde{x} \in h\mathbb{Z}^m \setminus \overline{D_d}\,; \\ 0, & \tilde{x} \in \overline{D_d}\,. \end{cases}$$

To apply the discrete Fourier transform F_d to the equation (1) we need to know what are the operators FP_+, FP_-. For the case $D = \mathbb{R}^m_+$ it was done in papers [12], [14] in the space $L_2(h\mathbb{Z}^m_+)$, and here we will briefly describe these constructions. Let us introduce the following operators which are generated by periodic analogue of the Hilbert transform, $\xi = (\xi', \xi_m)$,

$$(H^{per}_{\xi'}\tilde{u}_d)(\xi) = \frac{h}{2\pi i}\, v.p.\int\limits_{-h^{-1}\pi}^{h^{-1}\pi} \cot\frac{h(\xi_m-\eta_m)}{2}\,\tilde{u}_d(\xi',\eta_m)\,\mathrm{d}\eta_m\,, \qquad \xi' \in \hbar\mathbb{T}^{m-1},$$

$$P^{per}_{\xi'} = 1/2(I + H^{per}_{\xi'})\,, \quad Q^{per}_{\xi'} = 1/2(I - H^{per}_{\xi'})\,.$$

We have the following relations

$$FP_+ = P^{per}_{\xi'}F, \quad FP_- = Q^{per}_{\xi'}F.$$

Thus the equation (1) in the space $L_2(h\mathbb{Z}^m_+)$ is equivalent to the following equation

$$\big(\widetilde{A}_d(\xi)P^{per}_{\xi'} + \ Q^{per}_{\xi'}\big)\tilde{U}_d = \tilde{V}_d\,,$$

in the space $L_2(h\mathbb{Z}^m)$.

One can rewrite the last equation as a one-dimensional singular integral equation with a parameter $\xi' \in \hbar\mathbb{T}^{m-1}$

$$\frac{\widetilde{A}_d(\xi',\xi_m)+1}{2}\,\tilde{U}_d(\xi)+\frac{\widetilde{A}_d(\xi)-1}{2}\cdot\frac{h}{2\pi i}\,v.p.\int\limits_{-h^{-1}\pi}^{h^{-1}\pi}\cot\frac{h(\xi_m-\eta_m)}{2}\,\tilde{U}_d(\xi',\eta_m)\,\mathrm{d}\eta_m=\tilde{V}_d(\xi)\,. \tag{3}$$

The equation (3) is closely related to the so-called periodic Riemann boundary value problem [14]. We will formulate it for our situation. Let us denote $\Pi_\pm$ the upper and lower half-strips in a complex plane $\mathbb{C}$,

$$\Pi_\pm = \big\{z \in \mathbb{C}\colon z = \xi_m + i\tau, \xi_m \in [-h^{-1}\pi, h^{-1}\pi],\ \pm\tau > 0\big\}.$$

We formulate the problem as follows. Finding two functions $\Phi^\pm(\xi', \xi_m)$, $\xi_m \in [-h^{-1}\pi, h^{-1}\pi]$, which admit an analytical continuation into $\Pi_\pm$ on the variable ξ_m for almost all $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the linear relation

$$\Phi^+(\xi', \xi_m) = G(\xi)\Phi^-(\xi', \xi_m) + g(\xi),$$

where $G(\xi), g(\xi)$ are given functions on $\hbar\mathbb{T}^m$. For our case the corresponding periodic Riemann problem with a parameter ξ' will be the following

$$P^{per}_{\xi'}\tilde{U}_d = -\widetilde{A}^{-1}_d(\xi)Q^{per}_{\xi'}\tilde{U}_d + \widetilde{A}^{-1}_d\tilde{V}_d\,, \tag{4}$$

because the operators $P^{per}_{\xi'}, Q^{per}_{\xi'}$ are projectors on subspaces of $L_2(\hbar\mathbb{T}^m)$ consisting of functions admitting bounded analytical continuation on the last variable ξ_m into $\Pi_\pm$ under almost all $\xi' \in \hbar\mathbb{T}^{m-1}$.

2.3. The periodic factorization and a general solution of discrete equation

To study the general Riemann boundary value problem we will use the following concept.

Definition 5. Periodic factorization of an elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form

$$A_d(\xi) = A_{d,+}(\xi) A_{d,-}(\xi)\,,$$

where the factors $A_{d,\pm}(\xi)$ admit an analytical continuation into half-strips $\hbar\Pi_\pm$ on the last variable ξ_m for all fixed $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the estimates

$$\left|A_{d,+}^{\pm 1}(\xi)\right| \le c_1\big(1+|\hat{\zeta}^2|\big)^{\pm\frac{æ}{2}}, \quad \left|A_{d,-}^{\pm 1}(\xi)\right| \le c_2\big(1+|\hat{\zeta}^2|\big)^{\pm\frac{\alpha-æ}{2}},$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2 \equiv \hbar^2\left(\sum_{k=1}^{m-1}\big(e^{-ih\xi_k}-1\big)^2 + \big(e^{-ih(\xi_m+i\tau)}-1\big)^2\right), \qquad \xi_m + i\tau \in \hbar\Pi_\pm\,.$$

The number $æ \in \mathbb{R}$ is called an index of periodic factorization.

For some simple cases one can use the topological formula

$$æ = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi} \mathrm{d}\arg A_d(\cdot,\xi_m)\,,$$

where $A_d(\cdot,\xi_m)$ means that $\xi' \in \hbar\mathbb{T}^{m-1}$ is fixed, and the integral is the integral in the Stieltjes sense. It means that we need to calculate divided by 2π variation of the argument of the symbol $A_d(\xi)$ when ξ_m varies from $-\hbar\pi$ to $\hbar\pi$ under fixed ξ'.

Example 1. Let $A_d(\xi) = k^2 + \hat{\xi}^2$, $k \in \mathbb{R}$, such that the condition (3) is satisfied, in other words, A_d is the discrete Laplacian plus $k^2 I$. The variation of an argument mentioned above can be calculated immediately, and it equals to 1.

As we will see the index of factorization very influences on the solvability picture of the equation (2).

2.3.1. Existence and uniqueness theorem

For special case we have the following result.

Theorem 1. *If the elliptic symbol $\tilde{A}_d(\xi) \in E_\alpha$ admits periodic factorization with index æ so that $|æ - s| < 1/2$, then the equation* (1) *has unique solution in the space $H^s(hD_d)$ for an arbitrary right-hand side $v_d \in H^{s-\alpha}(hD_d)$,*

$$\tilde{u}_d(\xi) = \tilde{A}_{d,+}^{-1}(\xi) P_{\xi'}^{per}\big(\tilde{A}_{d,-}^{-1}(\xi)\widetilde{\ell v_d}(\xi)\big)\,. \tag{5}$$

Remark 2. It is easy to see that the solution does not depend on choice of continuation ℓv_d

2.3.2. A general solution of the discrete equation

Here we consider a more complicated case when the condition $|æ - s| < 1/2$ does not hold. There are two possibilities in this situation, and we consider one case which leads to typical boundary value problems.

Theorem 2. *Let $æ - s = n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$. Then a general solution of the equation* (1) *in Fourier images has the following form*

$$\tilde{u}_d(\xi) = \tilde{A}_{d,+}^{-1}(\xi) X_n(\xi) P_{\xi'}^{per}\big(X_n^{-1}(\xi)\tilde{A}_{d,-}^{-1}(\xi)\widetilde{\ell v_d}(\xi)\big) + \tilde{A}_{d,+}^{-1}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}_m^k\,,$$

where $X_n(\xi)$ is an arbitrary polynomial of order n of variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1)$, $k = 1, \dots, m$, satisfying the condition (3), *$c_k(\xi'), j = 0, 1, \dots, n-1$, are arbitrary functions from $H_{s_k}(h\mathbb{T}^{m-1})$, $s_k = s - æ + k - 1/2$.*

The theorem 2 implies that if we want to have a unique solution in the case $æ - s = n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$, we need some additional conditions to determine uniquely unknown functions $c_k(\xi')$, $k = 0, 1, \dots, n-1$. This case we will discuss in the next section.

Corollary 1. *Let $æ - s = n + \delta$, $n \in \mathbb{N}, |\delta| < 1/2$, $v_d \equiv 0$. A general solution of the equation* (1) *has the following form*

$$\tilde{u}_d(\tilde{x}', \tilde{x}_m) = \tilde{A}_{d,+}^{-1}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}_m^k\,.$$

3. Over-determined case

Here we consider the left case $æ - s = -n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$.

3.1. Special representation for a periodic kernel

Since the solving of the last problem is based on properties of the Cauchy type integral, we need to use for negative n the following expansion for the kernel

$$\frac{1}{\tau - z} = -\sum_{j=1}^{n}\frac{\tau^{j-1}}{z^j} + \frac{\tau^n}{z^n(\tau - z)}\,, \qquad z \in D, \tag{6}$$

where D is a bounded domain with a smooth closed boundary γ.

Since solutions of continuous [5] and discrete problems are written by singular integral with the Cauchy kernel and the kernel $\cot \frac{ht}{2}$, respectively, we take into account the following correlation between these two integrals. Indeed, if

$$\xi + 1 = e^{ih\eta}, \qquad \tau + 1 = e^{iht},$$

then we have

$$\int\limits_{\mathbb{S}^1} \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau - \xi} = ih\int\limits_{-\hbar\pi}^{\hbar\pi} \frac{\phi(t)e^{iht}\,\mathrm{d}t}{e^{iht} - e^{ih\eta}} = ih\int\limits_{-\hbar\pi}^{\hbar\pi} \frac{\phi(t)(e^{iht} + e^{ih\eta})\,\mathrm{d}t}{e^{iht} - e^{ih\eta}} - ihe^{ih\eta}\int\limits_{-\hbar\pi}^{\hbar\pi} \frac{\phi(t)\,\mathrm{d}t}{e^{iht} - e^{ih\eta}}\,,$$

where $\phi(t) = \varphi(-1 + e^{iht})$, and further taking into account

$$i\frac{e^{iht} + e^{ih\eta}}{e^{iht} - e^{ih\eta}} = \cot\frac{h(t-\eta)}{2}, \qquad \frac{1}{(\tau+1)(\tau-\xi)} = -\frac{1}{\xi+1}\left(\frac{1}{\tau+1} - \frac{1}{\tau-\xi}\right),$$

we have

$$\int\limits_{\mathbb{S}^1} \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau - \xi} = \frac{h}{2}\int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(t-\eta)}{2}\phi(t)\,\mathrm{d}t + \frac{1}{2}\int\limits_{\mathbb{S}^1} \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau+1}\,. \tag{7}$$

Thus each expansion (6) generates the corresponding expansion

$$\int\limits_{\gamma} \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau - z} = -\sum_{j=1}^{n}\int\limits_{\gamma} \frac{\tau^{j-1}\varphi(\tau)\,\mathrm{d}\tau}{z^j} + \int\limits_{\gamma} \frac{\tau^n\varphi(\tau)\,\mathrm{d}\tau}{z^n(\tau - z)}\,.$$

Let γ be a circumference of the following type

$$\gamma = \left\{z \in \mathbb{C}\colon\, z = 1 + e^{ih\tau}\right\}.$$

We put $z = e^{ih\zeta} - 1$ and write

$$\int\limits_{\mathbb{S}^1} \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau - \xi} = -\sum_{j=1}^{n} c_j (e^{ih\zeta} - 1)^{1-j}$$
$$+ (e^{ih\zeta} + 1)^{-n}\frac{h}{2}\int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(t-\zeta)}{2}\phi(t)(e^{ith} - 1)^n\,\mathrm{d}t + 1/2c_n\,,$$

where

$$c_j = \int\limits_{\mathbb{S}^1} \tau^{j-1}\varphi(\tau)\,\mathrm{d}\tau = ih\int\limits_{-\hbar\pi}^{\hbar\pi} (e^{iht} - 1)^{j-1}\phi(t)\,\mathrm{d}t, \qquad j = 1, \ldots, n\,.$$

Analogously, according to the formula (7),

$$\int\limits_{\mathbb{S}^1} \frac{\tau^n\varphi(\tau)\,\mathrm{d}\tau}{\tau - \xi} = \frac{h}{2}\int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(t-\eta)}{2}(e^{iht} - 1)^n\phi(t)\,\mathrm{d}t + \frac{1}{2}\int\limits_{\mathbb{S}^1} \frac{\tau^n\varphi(\tau)}{\tau+1}\,\mathrm{d}\tau.$$

Then

$$\frac{h}{2}\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(t-\eta)}{2}\phi(t)\,\mathrm{d}t+\frac{1}{2}\int\limits_{\mathbb{S}^1}\frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau+1}=$$
$$-\sum_{j=1}^{n}c_j(e^{ih\zeta}+1)^{-j}+(e^{ih\zeta}-1)^{-n}\frac{h}{2}\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(t-\zeta)}{2}\phi(t)(e^{ith}-1)^n\,\mathrm{d}t$$
$$+\frac{1}{2}\int\limits_{\mathbb{S}^1}\frac{\tau^n\varphi(\tau)}{\tau+1}\,\mathrm{d}\tau.$$

Hence we have

$$\frac{h}{2}\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(t-\zeta)}{2}\phi(t)\,\mathrm{d}t=$$
$$-\sum_{j=0}^{n}b_j(e^{ih\zeta}-1)^{-j}+(e^{ih\zeta}-1)^{-n}\frac{h}{2}\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(t-\zeta)}{2}\phi(t)(e^{ith}-1)^n\,\mathrm{d}t,\quad (8)$$

where

$$b_j=c_j,\quad j=1,\dots,n,\quad b_0=\frac{1}{2}\int\limits_{\mathbb{S}^1}\frac{(1-\tau^n)\varphi(\tau)\,\mathrm{d}\tau}{\tau+1}.$$

Of course, all constants b_j depend on the parameter h.

3.2. How a solution looks

Here we remind that the periodic Riemann problem really depends on a parameter ξ'. Thus all coefficients c_j in the formula (8) will depend on ξ'. We can collect our considerations of this section in the following

Lemma 2. *There exists a unique collection of functions*

$$c_j(\xi')\in H^{s_j}(\hbar\mathbb{T}^{m-1}),\quad s_j=s-æ+j+1/2\,,\quad j=0,1,\dots,n\,,$$

such that the following representation

$$\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(\eta_m-\xi_m)}{2}g(\xi',\eta_m)\,\mathrm{d}\eta_m=\sum_{j=0}^{n}c_j(\xi')(e^{ih\xi_m}-1)^{-j}$$
$$+(e^{ih\xi_m}-1)^{-n}\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(\eta_m-\xi_m)}{2}g(\xi',\eta_m)(e^{ih\eta_m}-1)^n\,\mathrm{d}\eta_m\,,$$

where

$$c_j(\xi')=ih\int\limits_{-\hbar\pi}^{\hbar\pi}(e^{ih\xi_m}-1)^j g(\xi',\xi_m)\,\mathrm{d}\xi_m,\qquad j=0,1,\dots,n\,,$$

holds, for all $g(\xi',\xi_m)\in H^{-n-\delta}(\hbar\mathbb{T}^m)$, $n\in\mathbb{N}$, $|\delta|<1/2$.

Theorem 3. *Let* $æ - s = -n + \delta$, $|\delta| < 1/2$. *Then the equation* (1) *has a solution in the space* $H^s(D_d)$ *if and only if*

$$c_j(\xi') = 0, \qquad \text{for all} \quad a.a.\ \xi' \in \hbar\mathbb{T}^{m-1}, \qquad j = 0, 1, \ldots, n\,. \tag{9}$$

P r o o f. Let ℓv_d be an arbitrary continuation of v_d on the whole $H^{s-\alpha}(h\mathbb{Z}^m)$. After periodic factorization of the symbol $A_d(\xi)$ we have

$$A_{d,+}(\xi)\tilde{u}_d(\xi) + A_{d,-}^{-1}(\xi)\tilde{w}_d(\xi) = A_{d,-}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi). \tag{10}$$

According to the properties of digital pseudo-differential operators (see Lemma 1) both summands and the right-hand side belong to the space $H^{s-æ}(\hbar\mathbb{T}^m) = H^{n-\delta}(\hbar\mathbb{T}^m)$. Since

$$H^{n-\delta}(\hbar\mathbb{T}^m) \subset h^{n-\delta}L_2(\hbar\mathbb{Z}^m),$$

we will multiply the equality (10) by h^n

$$h^n A_{d,+}(\xi)\tilde{u}_d(\xi) + h^n A_{d,-}^{-1}(\xi)\tilde{w}_d(\xi) = h^n A_{d,-}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)$$

and will represent the $h^n A_{d,-}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)$ as a direct sum

$$h^n A_{d,-}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) = f_+(\xi) + f_-(\xi),$$

where

$$f_+(\xi) = P_{\xi'}^{per}\big(h^{n-\delta}A_{d,-}^{-1}\widetilde{(\ell v_d)}\big)(\xi), \quad f_-(\xi) = Q_{\xi'}^{per}\big(h^{n-\delta}A_{d,-}^{-1}\widetilde{(\ell v_d)}\big)(\xi),$$

and

$$f_\pm \in H^{-\delta}(\hbar\mathbb{T}^m).$$

Therefore,

$$h^n A_{d,+}(\xi)\tilde{u}_d(\xi) + h^n A_{d,-}^{-1}(\xi)\tilde{w}_d(\xi) = f_+(\xi) + f_-(\xi),$$

and then

$$h^n A_{d,+}(\xi)\tilde{u}_d(\xi) - f_+(\xi) = f_-(\xi) - h^n A_{d,-}^{-1}(\xi)\tilde{w}_d(\xi) = 0$$

according to above considerations and Theorem 1. So, we have the unique solution

$$\tilde{u}_d(\xi) = h^{-n}A_{d,+}^{-1}(\xi)P_{\xi'}^{per}\big(h^n A_{d,-}^{-1}\widetilde{(\ell v_d)}\big)(\xi).$$

Let us denote for brevity $h^n A_{d,-}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) \equiv g(\xi)$ and apply the Lemma 2 to the $\big(P_{\xi'}^{per}g\big)(\xi)$. According to the Lemma 2 we have

$$\big(P_{\xi'}^{per}g\big)(\xi) = \frac{1}{2}g(\xi',\xi_m) + \sum_{j=0}^{n} c_j(\xi')(e^{ih\xi_m}-1)^{-j}$$
$$+ (e^{ih\xi_m}-1)^{-n}\frac{h}{2\pi i}\, v.p.\int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\eta_m-\xi_m)}{2}\, g(\xi',\eta_m)(e^{ih\eta_m}-1)^n \mathrm{d}\eta_m\,,$$

and thus

$$\tilde{u}_d(\xi) = \frac{1}{2}A^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) + h^{-n}A^{-1}_{d,+}(\xi)\sum_{j=0}^{n} c_j(\xi')(e^{ih\xi_m}-1)^{-j}$$
$$+ h^{-n}A^{-1}_{d,+}(\xi)(e^{ih\xi_m}-1)^{-n}$$
$$\times \frac{h}{2\pi i}\, v.p.\int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\eta_m-\xi_m)}{2} g(\xi',\eta_m)(e^{ih\eta_m}-1)^n \mathrm{d}\eta_m\,. \tag{11}$$

Since the first summand in the last formula belongs to $H^s(\hbar\mathbb{T}^m)$, we consider left summands and rewrite the above formula as follows

$$\tilde{u}_d(\xi) = \frac{1}{2}A^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) + h^{-n}A^{-1}_{d,+}(\xi)\sum_{j=0}^{n} c_j(\xi')(e^{ih\xi_m}-1)^{-j}$$
$$+ h^{-n}A^{-1}_{d,+}(\xi)(e^{ih\xi_m}-1)^{-n}$$
$$\times \frac{h}{2\pi i}\, v.p.\int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\eta_m-\xi_m)}{2} g(\xi',\eta_m)(e^{ih\eta_m}-1)^n \mathrm{d}\eta_m\,.$$

Let us consider separately the summand

$$h^{-n}c_j(\xi')(e^{ih\xi_m}-1)^{-j} = h^{-n-j}c_j(\xi')\left(\frac{e^{ih\xi_m}-1}{h}\right)^{-j}$$
$$= h^{-n}\left(\frac{e^{ih\xi_m}-1}{h}\right)^{-j} ih\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^{j} g(\xi',\xi_m)\,\mathrm{d}\xi_m$$
$$= ih^{æ-s-\delta+1}\left(\frac{e^{ih\xi_m}-1}{h}\right)^{-j}\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^{j} g(\xi',\xi_m)\,\mathrm{d}\xi_m\,.$$

We remind $h^nA^{-1}_{d,-}(\xi)\widetilde{(\ell v_d)}(\xi) \equiv g(\xi)$ and then we denote

$$b_j(\xi') \equiv h^{æ-s-\delta+1}\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^{j} g(\xi',\xi_m)\,\mathrm{d}\xi_m$$
$$= h\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^{j} A^{-1}_{d,-}(\xi',\xi_m)\widetilde{(\ell v_d)}(\xi',\xi_m)\,\mathrm{d}\xi_m\,,$$

therefore, since

$$A^{-1}_{d,-}(\xi',\xi_m)\widetilde{(\ell v_d)}(\xi',\xi_m)$$

$$\in H^{s-æ}(\hbar\mathbb{T}^m) \implies \left(\frac{e^{ih\xi_m}-1}{h}\right)^j A^{-1}_{d,-}(\xi)\widetilde{(\ell v_d)}(\xi)$$

$$\in H^{s-æ-j}(\hbar\mathbb{T}^m) \implies \int\limits_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^j A^{-1}_{d,-}(\xi',\xi_m)\widetilde{(\ell v_d)}(\xi',\xi_m)\,\mathrm{d}\xi_m$$

$$\in H^{s-æ+n-j-1/2}\big(\hbar\mathbb{Z}^{m-1}\big) \implies h\int\limits_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^j A^{-1}_{d,-}(\xi',\xi_m)\widetilde{(\ell v_d)}(\xi',\xi_m)\,\mathrm{d}\xi_m$$

$$\in H^{s-æ-j+1/2}\big(\hbar\mathbb{T}^{m-1}\big) \equiv H^{s_j}\big(\hbar\mathbb{T}^{m-1}\big), \qquad \text{because} \quad s-æ-j+1/2\equiv s_j.$$

In other words,

$$||b_j||_{s_j} \le c\,||v_d||^+_{s-\alpha}\,.$$

For the last summand we obtain

$$h^{-n}\big(e^{ih\xi_m}-1\big)^{-n}\times\frac{h}{2\pi i}\,v.p.\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(\eta_m-\xi_m)}{2}g(\xi',\eta_m)\big(e^{ih\eta_m}-1\big)^n\mathrm{d}\eta_m \;=$$

$$h^{-n}\left(\frac{e^{ih\xi_m}-1}{h}\right)^{-n}\times\frac{h}{2\pi i}\,v.p.\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(\eta_m-\xi_m)}{2}g(\xi',\eta_m)\left(\frac{e^{ih\eta_m}-1}{h}\right)^n\mathrm{d}\eta_m\,.$$

Arguing as above we get

$$g\in H^{s-æ+n}(\hbar\mathbb{T}^m)\implies h^{-n}g(\xi',\eta_m)\left(\frac{e^{ih\eta_m}-1}{h}\right)^n\in$$
$$H^{s-æ-n}(\hbar\mathbb{T}^m)\equiv H^{-\delta}(\hbar\mathbb{T}^m),$$

after the periodic Hilbert transform we have

$$\frac{h}{2\pi i}\,v.p.\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(\eta_m-\xi_m)}{2}A^{-1}_{d,-}(\xi',\eta_m)\widetilde{(\ell v_d)}(\xi',\eta_m)\times\left(\frac{e^{ih\eta_m}-1}{h}\right)^n\mathrm{d}\eta_m\in$$
$$H^{-\delta}(\hbar\mathbb{T}^m),$$

and finally, multiplication by

$$\left(\frac{e^{ih\xi_m}-1}{h}\right)^{-n}$$

leads to the space

$$H^{n-\delta}(\hbar\mathbb{T}^m)=H^{s-æ}(\hbar\mathbb{T}^m).$$

So we obtain the following representation rewriting (11)

$$\tilde{u}_d(\xi) = \sum_{j=0}^{n} b_j(\xi') \left(\frac{e^{ih\xi_m}-1}{h}\right)^{-j} A^{-1}_{d,+}(\xi) + \tilde{U}_d(\xi), \tag{12}$$

where

$$\tilde{U}_d(\xi) = \left(\frac{e^{ih\xi_m}-1}{h}\right)^{-n} A^{-1}_{d,+}(\xi) \times \left(P^{per}_{\xi'}\left(A^{-1}_{d,-}(\xi',\eta_m)\widetilde{(\ell v_d)}(\xi',\eta_m)\left(\frac{e^{ih\eta_m}-1}{h}\right)^{n}\right)\right)(\xi',\xi_m)$$

for which from above we have obtained the estimate

$$||U_d||_s \le c\,||v_d||^{+}_{s-\alpha}.$$

□

Remark 3. Conditions (9) might be written in the initial space $H^s(h\mathbb{Z}^m)$. We will use the operators $\Delta^{(1)}_j \colon H^s(h\mathbb{Z}^m) \to H^{s-1}(h\mathbb{Z}^m)$ introduced in the Section 2 and their Fourier images

$$\tilde{\Delta}^{(1)}_j \colon \tilde{u}_d(\xi) \longmapsto \frac{e^{-ih\xi_j}-1}{h}\tilde{u}_d(\xi), \qquad \xi \in \hbar\mathbb{T}^m.$$

We will remind one property of the discrete Fourier transform related to restriction on a discrete hyper-plane. We consider a restriction of the function $u_d(\tilde{x})$ on the discrete hyper-plane $\tilde{x}_m = 0$, i.e., $\mathbb{Z}^{m-1}$. According to the inverse Fourier transform we have

$$u_d(\tilde{x}',\tilde{x}_m) = \frac{1}{(2\pi)^m}\int\limits_{\mathbb{T}^m} e^{i\tilde{x}'\cdot\xi'} e^{i\tilde{x}_m\cdot\xi_m}\tilde{u}_d(\xi',\xi_m)\,\mathrm{d}\xi'\mathrm{d}\xi_m\,,$$

hence

$$\begin{aligned} u_d(\tilde{x}',0) &= \frac{1}{(2\pi)^m}\int\limits_{\hbar\mathbb{T}^m} e^{i\tilde{x}'\cdot\xi'}\tilde{u}_d(\xi',\xi_m)\,\mathrm{d}\xi'\mathrm{d}\xi_m \\ &= \frac{1}{(2\pi)^{m-1}}\int\limits_{\hbar\mathbb{T}^{m-1}} e^{i\tilde{x}'\cdot\xi'}\left(\frac{1}{2\pi}\int\limits_{-\hbar\pi}^{\hbar\pi}\tilde{u}_d(\xi',\xi_m)\,\mathrm{d}\xi_m\right)\mathrm{d}\xi', \end{aligned}$$

and we see that restriction on a hyper-plane corresponds to integration of the Fourier image on the last variable. Taking into account this fact and recalling that multiplication in Fourier images corresponds to a pseudo-differential operator in original discrete space $H^s(h\mathbb{Z}^m)$, we can write the following condition instead of the (9)

$$\Delta^{(j)}_m A^{-1}_{d,-}(\ell v_d)(\tilde{x}',0) = 0, \qquad \text{for all} \quad \tilde{x}' \in h\mathbb{Z}^{m-1}, \quad j = 0,1,\ldots,n\,, \tag{13}$$

where $A^{-1}_{d,-}$ is a digital pseudo-differential operator with the symbol $A^{-1}_{d,-}(\xi)$.

4. Discrete potentials

Starting from the representation (12) we can consider a more general equation than (1) including additional unknown functions. More precisely we will consider the following equation.

In other words, numbers $u_d(\tilde{x})$ are Fourier coefficients of the function $\tilde{u}_d(\xi)$. Now we'll define a discrete analogue of a one-dimensional discrete indicator function in the following way. We put $\delta(\tilde{x}_m)$ as

$$\delta(\tilde{x}_m) = \begin{cases} 1 & \text{if } \tilde{x}_m = 0, \\ 0, & \text{in other cases.} \end{cases}$$

So the one-dimensional discrete Fourier transform of such function is

$$(F\delta)(\xi_m) = 1.$$

Thus for the case æ $- s = -n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$, taking into account the representation (11) we have additional functions defined by the right-hand side, and one consider the equation

$$(A_d u_d)(\tilde{x}) + \sum_{j=0}^{n} K_j\big(\tilde{b}_j(\tilde{x}') \otimes \delta(\tilde{x}_m)\big) = v_d(\tilde{x}), \qquad \tilde{x} \in D_d\,, \tag{14}$$

where we have unknowns $u_d, \tilde{b}_j, j = 0, 1, \ldots, n$, and K_j is a pseudo-differential operator with the symbol $K_j(\xi) \in E_{\alpha_j}$.

Remark 4. We use a term "potential like operator" following [5] because the operator K_j acts as follows. If we denote by $\hat{K}_j(\tilde{x})$ "a kernel" of the pseudo--differential operator K_j, then we have

$$K_j\big(\tilde{b}_j(\tilde{x}') \otimes \delta(\tilde{x}_m)\big) = \sum_{\tilde{y} \in h\mathbb{Z}^{m-1}} \hat{K}_j(\tilde{x}' - \tilde{y}', \tilde{x}_m) b_j(\tilde{y}') h^{m-1}.$$

It is really a discrete potential like operator.

Using the above scheme we introduce ℓv_d and put

$$w_d(\tilde{x}) = (\ell v_d)(\tilde{x}) - (A_d u_d)(\tilde{x}) - \sum_{j=0}^{n} K_j\big(\tilde{b}_j(\tilde{x}') \otimes \delta(\tilde{x}_m)\big)$$

so that $w_d(\tilde{x}) = 0$, for all $\tilde{x} \in D_d$; then

$$w_d(\tilde{x}) + (A_d u_d)(\tilde{x}) = (\ell v_d)(\tilde{x}) - \sum_{j=0}^{n} K_j\big(\tilde{b}_j(\tilde{x}') \otimes \delta(\tilde{x}_m)\big),$$

and after the Fourier transform

$$A_d(\xi)\tilde{u}_d(\xi) + \tilde{w}_d(\xi) = \widetilde{(\ell v_d)}(\xi) - \sum_{j=0}^{n} K_j(\xi', \xi_m)\tilde{b}_j(\xi').$$

Further, we use the periodic factorization, write

$$A_{d,+}(\xi)\tilde{u}_d(\xi) + A_{d,-}^{-1}(\xi)\tilde{w}_d(\xi) = A_{d,-}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) - A_{d,-}^{-1}(\xi)\sum_{j=0}^{n} K_j(\xi', \xi_m)\tilde{b}_j(\xi').$$

Now, we use the conditions (13) for the right-hand side

$$A_{d,-}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) - A_{d,-}^{-1}(\xi)\sum_{j=0}^{n} K_j(\xi', \xi_m)\tilde{b}_j(\xi'), \qquad k = 0, 1, \dots, n\,.$$

Taking into account our above remarks we can write

$$\frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^k A_{d,-}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)\,\mathrm{d}\xi_m = \frac{1}{2\pi}\sum_{j=0}^{n}\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^k \frac{K_j(\xi', \xi_m)\tilde{b}_j(\xi')}{A_{d,-}(\xi', \xi_m)}\,\mathrm{d}\xi_m\,.$$

In other words, we obtain a system of linear algebraic equations

$$\sum_{j=0}^{n} t_{kj}(\xi')\tilde{b}_j(\xi') = f_k(\xi'), \qquad k = 0, 1, \dots, n\,,$$

where

$$t_{kj}(\xi') = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^k \frac{K_j(\xi', \xi_m)}{A_{d,-}(\xi', \xi_m)}\,\mathrm{d}\xi_m\,,$$

$$f_k(\xi') = \frac{1}{2\pi}\int_{-\hbar\pi}^{\hbar\pi}\left(\frac{e^{ih\xi_m}-1}{h}\right)^k A_{d,-}^{-1}(\xi', \xi_m)\widetilde{(\ell v_d)}(\xi', \xi_m)\,\mathrm{d}\xi_m\,.$$

After some additional arguments like in [5], [9] we can obtain the following assertion.

Theorem 4. *Let* $æ - s = -n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$. *Then the equation* (14) *has unique solution*

$$u_d \in H^s(h\mathbb{Z}^m), \quad c_j \in H^{s_j}(h\mathbb{Z}^{m-1}), \quad s_j = s - \alpha + \alpha_j + 1/2, \quad j = 0, 1, \dots, n\,,$$

if and only if

$$ess \inf_{\xi' \in h\mathbb{T}^{m-1}} |\det\big(t_{kj}(\xi')\big)_{k,j=0}^{n} > 0\,. \tag{15}$$

A priori estimates

$$||u_d||_s \le a||v_d||^+_{s-\alpha}, \quad ||b_j||_{s_j} \le a_j||v_d||^+_{s-\alpha}, \qquad j = 0, 1, \dots, n,$$

hold with constants $a, a_1, \dots, a_n$, non-depending on h.

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