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STABILITY OF THE EQUILIBRIUM OF NONLINEAR DYNAMICAL SYSTEMS

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ABSTRACT. The algorithm for estimating the stability domain of zero equilibrium to the system of nonlinear differential equations with a quadratic part and a fractional part is proposed in the article. The second Lyapunov method with quadratic Lyapunov functions is used as a method for studying such systems.

1. Introduction

The theory and applications of nonlinear differential equations form an important part of modern nonlinear dynamics. These equations are natural mathematical models of various real-life phenomena, such as population dynamics and ecology, physiology and medicine, economics and other natural sciences. For this reason, the study of the stability of such models is extremely important.

One of the first methods to study the stability of zero solution to nonlinear systems is the method of linearization and stability analysis, based on the stability of the linear approximation system. Such types of work were done in the second half of the last century, for example [6], [8], [9]. The basic idea is: "If zero solution of the linear approximation is asymptotically stable, then in a sufficiently small neighbourhood of the equilibrium, the trivial solution of the original nonlinear system will be also stable."

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If the system is asymptotically stable, then it comes to zero equilibrium position in an infinite time interval. An important characteristic of stability is the time for which the solution of the system goes into an ε neighbourhood of the origin of the system and will not leave it, this is, $\|x(t)\| \le \varepsilon$. Using the obtained convergence estimate, it is possible to find the time for which the solution of the system from a position $x(t_0)$ falls into the ε neighbourhood of the zero position.

For systems of linear stationary differential equations

$$x'(t) = Ax(t), \quad t \ge t_0\,, \tag{1}$$

where A is an $n \times n$ constant matrix, the estimates of the above type were obtained in [3]. In order to formulate the results obtained there, we introduce the notations associated with the Lyapunov matrix equation

$$A^T H + HA = -C. \tag{2}$$

If the matrix A is asymptotically stable, then for any positive definite $n \times n$ matrix C there exists a unique solution to (2). Such a solution is a positive definite $n \times n$ matrix H, see in [4]. Matrices C and H from the Lyapunov matrix equation (2) play an important role in estimating the convergence of solutions to an equilibrium. Namely, the following estimate of the exponential convergence of solutions to linear system (1) is derived in [4]

$$\|x(t)\| \le \big[\varphi(H)\,\|x(t_0)\|\big] e^{-\frac{1}{2}\gamma(H)(t-t_0)}, \tag{3}$$

where

$$\varphi(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \gamma(H) = \frac{\lambda_{\min}(C)}{\lambda_{\max}(H)}, \tag{4}$$

$\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the largest and smallest eigenvalues of the corresponding symmetric matrix, and

$$\|x(t)\| = \sqrt{\sum_{i=1}^{n} x_i^2(t)} \tag{5}$$

denotes the vector norm.

In this paper we deal with nonlinear systems in order to estimate the convergence of their solutions to stable singular points. The stability domain of the zero equilibrium of the systems of nonlinear differential equations with quadratic part and a fractional part is estimated. As a method of investigation such systems, the second Lyapunov method with quadratic Lyapunov functions is used [2]. The second Lyapunov method using the Lyapunov function of Lurje-Postnikov type was used in [5] to obtain an estimation of a solution to a control equation with nonlinearity of a sector form.

It should be noted that the Lyapunov function in quadratic form is determined by a symmetric, positive definite matrix H, which is the solution to the Lyapunov matrix equation (2) in the case, when the matrix A in the linear part of nonlinear system is a constant matrix.

2. Systems with the quadratic right-hand side

In this section we consider systems with special form of nonlinearity, namely, systems with the quadratic right-hand side, written in a vector-matrix form [1, 4, 7],

$$x'(t) = Ax(t) + X^T(t)Bx(t), \tag{6}$$

where A is an $n \times n$ constant matrix, $B = (B_1, B_2, \ldots, B_n)^T$, $B_i, i = 1, 2, \ldots, n$ are $n \times n$ constant matrices,

$$B_i = \begin{pmatrix} b^i_{11} & b^i_{12} & \ldots & b^i_{1n} \\ b^i_{21} & b^i_{22} & \ldots & b^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b^i_{n1} & b^i_{n2} & \ldots & b^i_{nn}, \end{pmatrix},$$

and $X^T = \big(X_1(t), X_2(t), \ldots, X_n(t)\big)$, $X_i(t), i = 1, 2, \ldots, n$ are $n \times n$ matrices in which only the i-th row is nonzero,

$$X_1(t) = \begin{pmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \ \ldots, \ X_n(t) = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1(t) & x_2(t) & \ldots & x_n(t) \end{pmatrix}.$$

Recall that if the matrix A of the linear part in (6) be asymptotically stable, that is, all its eigenvalues have negative real parts, $\mathrm{Re}\lambda_i(A) < 0, i = 1, \ldots, n$, then, as follows from the stability theory of linear approximation, the zero solution of the corresponding nonlinear system is also asymptotically stable.

The following form of matrix norm will be used in our considerations

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)}. \tag{7}$$

Theorem 2.1. *Suppose that the matrix A in* (6) *is asymptotically stable. Then the trivial solution to* (6) *is asymptotically stable. Moreover, the domain*

$$G_{r_0} = \max_{r>0} \{G_r \colon G_r \subset G_0\}, \tag{8}$$

where

$$G_r = \big\{x \in R^n \colon x^T H x < r^2\big\}, \quad G_0 = \left\{x \in \mathbb{R}^n \colon \|x\| < \frac{\lambda_{\min}(C)}{2\|H\|\,\|B\|}\right\},$$

is the domain of stability.

P r o o f. We calculate the total derivative of the Lyapunov function in the quadratic form, $V(x) = x^T Hx$, along the trajectories of system (6)

$$\frac{\mathrm{d}\,\mathrm{V(x(t))}}{\mathrm{d}\,t} = x^T(t)\big[\big(A^T H + HA\big) + \big(B^T X(t)H + HX^T B\big)\big]x(t). \tag{9}$$

Taking into account (2), equation (9) can be rewritten into the form

$$\frac{\mathrm{d}\,\mathrm{V(x(t))}}{\mathrm{d}\,t} = -x^T(t)\big[C - \big(B^T X(t)H + HX^T(t)B\big)\big]x(t).$$

Therefore, the stability domain is the interior of the level surface of the Lyapunov function, which lies within the domain

$$G_0 = \big\{x \in \mathbb{R}^n \colon C - B^T XH - HX^T B > \Theta\big\},$$

where Θ denotes the zero matrix, and the expression inside denotes that the relevant matrix is positive definite. Since, in view of the vector and matrix norms defined by (5) and (7), we have $\|X(t)\| = \|x(t)\|$, thus the total derivative of the Lyapunov function can be estimated as

$$\frac{\mathrm{d}\,\mathrm{V(x(t))}}{\mathrm{d}\,t} < -\big[\lambda_{\min}(C) - 2\|H\|\,\|B\|\,\|x(t)\|\,\big]\,\|x(t)\|^2. \tag{10}$$

Therefore, if the inequality

$$\|x(t)\| < \frac{\lambda_{\min}(C)}{2\|H\|\,\|B\|}$$

is satisfied, then the total derivative of the Lyapunov function is negative. □

Remark 2.2. Obviously, to obtain the "maximum" domain of stability, the sphere G_0 should have the radius

$$R = \frac{\lambda_{\min}(C)}{2\|H\|\,\|B\|},$$

and the r should "stretch" as long as the ellipse $x^T Hx = r^2$ touches the sphere.

Theorem 2.3. *Suppose that the matrix A in* (6) *is asymptotically stable. Then for any solution to* (6) *satisfying the initial condition*

$$\|x(0)\| < \frac{\gamma(H)}{2\|B\|\varphi(H)}, \tag{11}$$

the following estimate

$$\|x(t)\| \le \frac{\gamma(H)\|x(0)\|}{\big[\gamma(H) - 2\|B\|\,\varphi(H)\,\|x(0)\|\,\big]\,e^{\frac{1}{2}\gamma(H)t} + 2\|B\|\varphi(H)\|x(0)\|} \tag{12}$$

for the convergence of solutions to the zero singular point holds.

P r o o f. The total derivative of the Lyapunov function $V(x) = x^T Hx$ along trajectories of system (6) is given by (9). Since for $V(x)$ two-sided inequality,

$$\lambda_{\min}(H)\|x\|^2 \le V(x) \le \lambda_{\max}(H)\|x\|^2, \tag{13}$$

is satisfied, then the estimate (10) of the Lyapunov function can be rewritten as

$$\frac{\mathrm{d\,V(x(t))}}{\mathrm{d}\,t} \le -\frac{\lambda_{\min}(C)}{\lambda_{\max}(H)}V(x(t)) + 2\lambda_{\max}(H)\|B\|\frac{V^{\frac{3}{2}}(x(t))}{\lambda_{\min}(H)},$$

or, taking into account (4), we get

$$\frac{\mathrm{d\,V(x(t))}}{\mathrm{d}\,t} \le -\gamma(H)V(x(t)) + 2\|B\|\frac{V^{\frac{3}{2}}(x(t))\varphi(H)}{\sqrt{\lambda_{\min}(H)}}.$$

Dividing by $V^{\frac{3}{2}}(x)$ and denoting

$$V^{-\frac{1}{2}}(x(t)) = z(t), \tag{14}$$

we obtain

$$-2\frac{\mathrm{d}\,z(t)}{\mathrm{d}\,t} \le -\gamma(H)z(t) + 2\|B\|\frac{\varphi(H)}{\sqrt{\lambda_{\min}(H)}},$$

and from here

$$\frac{\mathrm{d}\,z(t)}{\mathrm{d}\,t} \ge \frac{1}{2}\gamma(H)z(t) - \frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}.$$

Solving the inequality by analogy with a linear nonhomogeneous equation, we have

$$\|z(t)\| \ \ge \ \left[z(0) - 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right]e^{\frac{1}{2}\gamma(H)t} + 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}.$$

Since (14), we get

$$V^{-\frac{1}{2}}(x(t)) \ \ge \ \left[V^{-\frac{1}{2}}(x(0)) - 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right]e^{\frac{1}{2}\gamma(H)t} + 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}},$$

or

$$V^{\frac{1}{2}}(x(t)) \ge \left(\left[V^{-\frac{1}{2}}(x(0)) - 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right]e^{\frac{1}{2}\gamma(H)t} + 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right)^{-1}.$$

Consequently, using two-sided inequality (13), we obtain

$$\sqrt{\lambda_{\min}(H)}\|x(t)\| \le \frac{\gamma(H)\sqrt{\lambda_{\min}(H)}\|x(0)\|}{\big(\gamma(H) - 2\|B\|\varphi(H)\|x(0)\|\big)e^{\frac{1}{2}\gamma(H)t} + 2\|B\|\varphi(H)\|x(0)\|}.$$

Therefore, any solution $x(t)$ to (6) satisfying the initial condition (11) under the assumption $x(0) \in G_0$, is estimated by (12). □

Remark 2.4. In applications one can find nonlinear systems with a quadratic part in the form

$$x_i'(t) = \left[-a_i + \sum_{j=1}^{n} b_{ij} x_j(t) \right] x_i(t), \quad i = 1, 2, \dots, n, \tag{15}$$

where $a_i, b_{ij} \in \mathbb{R}^+,\ i, j = 1, 2, \dots, n$.
If we denote $A = \operatorname{diag}(a_1, a_2, \dots, a_n)$, $B = (B_1, B_2, \dots, B_n)^T$, $B_i, i = 1, 2, \dots, n$ are $n \times n$ constant matrices in which only the i-th column is nonzero,

$$B_i = \begin{pmatrix} 0 & \dots & b_{i2} & \dots & 0 \\ 0 & \dots & b_{i2} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & b_{in} & \dots & 0 \end{pmatrix},$$

and X^T as in system (6), then system (15) can be written in the form

$$x'(t) = -Ax(t) + X^T(t)Bx(t). \tag{16}$$

Since $a_i > 0, i = 1, 2, \dots, n$, all eigenvalues are negative, which means that the trivial solution to (15) is asymptotically stable, and the result of Theorem 2.3 can be applied to estimate solutions to system (16) as well as (15).

The second singular point $x_0 = (x_1^0, x_2^0, \dots, x_n^0)^T$ to system (16) is solution to algebraic system

$$B_0 x = a, \tag{17}$$

where

$$B_0 = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}, \quad a = \big(a_1, a_2, ..., a_n\big)^T,$$

under assumption that $\det B_0 \neq 0$.
Then, using substitution $y(t) = x(t) - x_0$, we obtain the transformed system with the zero equilibrium in the form

$$y'(t) = \bar{A}y(t) + Y^T(t)By(t), \tag{18}$$

where

$$\bar{A} = \begin{pmatrix} b_{11}x_1^0 & b_{12}x_1^0 & \dots & b_{1n}x_1^0 \\ b_{21}x_2^0 & b_{22}x_2^0 & \dots & b_{2n}x_2^0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}x_n^0 & b_{n2}x_n^0 & \dots & b_{nn}x_n^0 \end{pmatrix}.$$

If the matrix $\bar{A}$ is asymptotically stable, then the result of Theorem 2.3 can be applied to estimate solutions to system (18) as well as (15).

Example 2.1. We illustrate the result obtained on a scalar equation

$$x'(t) = -ax(t) + bx^2(t), \quad a, b > 0. \tag{19}$$

Since $\lambda = -1$, the trivial solution to equation (19) is stable. Any solution to (19), satisfying the initial condition $x(0) = x_0$, can be determined by the formula

$$x(t) = \frac{ax(0)e^{-at}}{a - bx(0)\left[1 - e^{-at}\right]}.$$

To estimate these solutions in a neighbourhood of the trivial solution, we take the Lyapunov function in the form $V(x) = x^2$. So, $H = 1$, $\lambda_{\min}(H) = \lambda_{\max}(H) = 1$, $C = 1$, $\varphi(H) = 1$, $\gamma(H) = 2a$. In accordance with the result of Theorem 2.3, convergence to the zero singular point of any solution to (19), satisfying initial condition $x(0) < \frac{a}{b}$, is estimated as follows

$$x(t) \leq \frac{ax(0)}{\big[a - bx(0)\big]e^{at} + bx(0)}.$$

As a result, the exact solution to equation (19) coincides with the obtained estimate, using by quadratic Lyapunov function.

It should be noted, the second equilibrium $x = \frac{a}{b}$ of equation (19) is unstable.

Remark 2.5. Interesting results of estimating the convergence of solutions to the zero singular point using the Lyapunov function can be obtained for the planar system with the quadratic right-hand side, this is, for system

$$\begin{aligned} x_1'(t) &= a_{11}x_2(t) + a_{12}x_2(t) + b_{11}^1 x_1^2(t) + 2b_{12}^1 x_1 x_2 + b_{22}^1 x_2^2(t), \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + b_{11}^2 x_1^2(t) + 2b_{12}^2 x_1 x_2 + b_{22}^2 x_2^2(t). \end{aligned} \tag{20}$$

Using notations

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B_1 = \begin{pmatrix} b_{11}^1 & b_{12}^1 \\ b_{12}^1 & b_{22}^1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_{11}^2 & b_{12}^2 \\ b_{12}^2 & b_{22}^2 \end{pmatrix}, \quad B = \big(B_1, B_2\big)^T$$

system (20) can be rewritten into the vector-matrix form (6). Under the assumption that $\operatorname{Re}\lambda_{1,2}(A) < 0$, Theorem 2.3 can be applied to estimate solutions to (20). Thus, the total derivative of the Lyapunov function along trajectories of system (20) can be estimated as in (10), where

$$\|H\| = \lambda_{\max}(H) = \frac{1}{2}\left(h_{11} + h_{22} + \sqrt{(h_{11} - h_{22})^2 + 4h_{12}^2}\right),$$

$$\lambda_{\min}(C) = \frac{1}{2}\left(c_{11} + c_{22} - \sqrt{(c_{11} - c_{22})^2 + 4c_{12}^2}\right),$$

$$\|B\| = \lambda_{\max}(B^T B).$$

Therefore, in view of (8), the interior of the ellipse

$$h_{11}x^2 + 2h_{12}xy + h_{22}y^2 < r_0^2, \quad r_0 = \frac{\lambda_{\min}(C)}{2\|H\|\,\|B\|}$$

is the guaranteed domain of stability.

3. Systems with a fractional part

Let us consider a nonlinear system of two differential equations with fractional parts depending on four parameters $\alpha, \varepsilon, \gamma, \mu \in \mathbb{R}$ in the form

$$\begin{aligned} x'(t) &= \quad x(t) - \frac{x(t)y(t)}{1+\alpha x(t)} - \varepsilon x^2(t), \\ y'(t) &= -\gamma y(t) + \frac{x(t)y(t)}{1+\alpha x(t)} + \mu y^2(t). \end{aligned} \tag{21}$$

By using the linearization method, we determine the behavior of trajectories in a neighbourhood of singular points and the phase portrait of the system.

Solving the system of algebraic equations,

$$\begin{aligned} \left[1 - \frac{y(t)}{1+\alpha x(t)} - \varepsilon x(t)\right] x(t) &= 0, \\ \left[-\gamma + \frac{x(t)}{1+\alpha x(t)} + \mu y(t)\right] y(t) &= 0 \end{aligned}$$

we obtain singular points which we will discussed about.

1. In a neighbourhood of the singular point $O_1(x_1, y_1)$, $x_1 = y_1 = 0$, the associated linear system to system (21) is

$$\begin{aligned} x'(t) &= \quad x(t), \\ y'(t) &= -\gamma\, y(t), \end{aligned}$$

with eigenvalues $\lambda_1 = 1, \lambda_2 = -\gamma$. Therefore, if $\gamma > 0$, then the zero equilibrium is a saddle.

2. In a neighbourhood of the singular point $O_2(x_2, y_2)$, $x_2 = 0$, $y_2 = \frac{\gamma}{\mu}$, the associated linear system to system (21) is the system

$$\begin{aligned} x'(t) &= \left(1 - \frac{\gamma}{\mu}\right) x(t), \\ y'(t) &= \frac{\gamma}{\mu} x(t) + \gamma\left(y(t) - \frac{\gamma}{\mu}\right), \end{aligned}$$

with eigenvalues $\lambda_1 = \gamma$, $\lambda_2 = 1 - \frac{\gamma}{\mu}$. Therefore, if $\gamma > 0$ and $\frac{\gamma}{\mu} > 1$, then the equilibrium O_2 is a saddle. However, if $\gamma > 0$, but $\frac{\gamma}{\mu} < 1$, then the equilibrium O_2 is an unstable knot. Given the considerations in the previous point, we do not consider the case $\gamma < 0$.

3. In a neighbourhood of the singular point $O_3(x_3, y_3)$, $x_3 = \frac{1}{\varepsilon}$, $y_3 = 0$, the associated linear system to system (21) is the system

$$x'(t) = -\left(x(t) - \frac{1}{\varepsilon}\right) - \frac{1}{\varepsilon + \alpha}\, y(t),$$

$$y'(t) = \left(-\gamma + \frac{1}{\varepsilon + \alpha}\right) y(t),$$

with eigenvalues $\lambda_1 = -1, \lambda_2 = -\gamma + \frac{1}{\varepsilon+\alpha}$. Therefore, if $\gamma > \frac{1}{\varepsilon+\alpha}$, then the equilibrium O_3 is a stable knot. However, if $\gamma < \frac{1}{\varepsilon+\alpha}$, then the equilibrium O_3 is a saddle.

4. Solving system of algebraic equations

$$1 - \frac{y}{1 + \alpha x} - \varepsilon x = 0,$$

$$-\gamma + \frac{x}{1 + \alpha x} + \mu y = 0,$$

the last singular points can be obtained. However, the system implies third order algebraic equation of the form

$$\mu(1 - \varepsilon\, x)(1 + \alpha\, x)^2 - \gamma(1 + \alpha\, x) = 0,$$

that has at least one real root. It seems that this equilibrium is a saddle.

Consequently, from our considerations follows the statement.

Corollary 3.1. *If $\gamma > \frac{1}{\varepsilon+\alpha}$ and $\gamma > \mu$, then the trajectories arise from the point $O_2(x_2, y_2)$, $x_2 = 0$, $y_2 = \frac{\gamma}{\mu}$, and passing $O_1(x_1, y_1)$, $x_1 = y_1 = 0$ they converge to the stable position $O_3(x_3, y_3)$, $x_3 = \frac{1}{\varepsilon}$, $y_3 = 0$.*

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