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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2018-0011
Tatra Mt. Math. Publ. **71** (2018), 123–137

EXISTENCE OF POSITIVE BOUNDED SOLUTIONS OF SYSTEM OF THREE DYNAMIC EQUATIONS WITH NEUTRAL TERM ON TIME SCALES

Urszula Ostaszewska — Ewa Schmeidel — Małgorzata Zdanowicz

University of Bialystok, Białystok, POLAND

ABSTRACT. In this paper the system of three dynamic equations with neutral term in the following form

$$\begin{cases} \big(x(t) + p(t)\,x(u_1(t))\big)^{\Delta} = \; a(t)\,f\big(y(u_2(t))\big), \\ y^{\Delta}(t) = b(t)\,g\big(z(u_3(t))\big), \\ z^{\Delta}(t) = c(t)\,h\big(x(u_4(t))\big) \end{cases}$$

on time scales is considered. The aim of this paper is to present sufficient conditions for the existence of positive bounded solutions of the considered system for $0 < p(t) \le const < 1$. The main tool of the proof of presented here result is Krasnoselskii's fixed point theorem. Also, the useful generalization of the Arzelá-Ascoli theorem on time scales to the three-dimensional case is proved.

1. Introduction

Consider a nonlinear dynamic system of three equations of the form

$$\begin{cases} \big(x(t) + p(t)\,x(u_1(t))\big)^{\Delta} = \; a(t)\,f\big(y(u_2(t))\big), \\ y^{\Delta}(t) = b(t)\,g\big(z(u_3(t))\big), \\ z^{\Delta}(t) = c(t)\,h\big(x(u_4(t))\big) \end{cases} \tag{1}$$

on a time scale $\mathbb{T}$.

Throughout this paper $x, y, z\colon \mathbb{T} \to \mathbb{R}$ are unknown functions and $p, a, b, c\colon \mathbb{T} \to \mathbb{R}$, $f, g, h\colon \mathbb{R} \to \mathbb{R}$. Moreover, $u_i\colon \mathbb{T} \to \mathbb{T}$ is such that $\lim_{t\to\infty} u_i(t) = \infty$ for $i = 1, 2, 3, 4$. Here $\mathbb{R}$ is the set of real numbers and $\mathbb{T}$ is an arbitrary time scale.

The time scale theory was found promising because it demonstrates the interplay between the theories of continuous-time and discrete-time systems

2010 Mathematics Subject Classification: 34N05, 39A10, 39A22.
Keywords: system of difference equation, three-dimensional, neutral type, boundedness, nonoscillatory solution, time scales.

(see [2], [3]), and from twenty years attracting attention of many researchers.

Particularly, the system of dynamic equations on time scales was studied by many authors. For example, T a o u s s e r, D e f o o r t and D j e m a i in [10] deal with the stability analysis of a class of uncertain switched systems on non-uniform time domains. The classical results on stabilization of nonlinear continuous-time and discrete-time systems are extended to systems on arbitrary time scales with bounded graininess function in [1] by B a r t o s i e w i c z and P i o t r o w s k a. A necessary and sufficient condition for the exponential stability of time-invariant linear systems on time scales in terms of the eigenvalues of the system matrix is found by P ö t z s c h e, S i e g m u n d and W i r t h in [8]. In [12], Z h u and W a n g considered the existence of nonoscillatory solutions to neutral dynamic equations on time scales. In the discrete case, the existence of a bounded nonoscillatory solution of nonlinear neutral type difference systems has been studied in [9] and [11]. In this paper the authors improved and generalised for arbitrary time scales some results obtained by M i g d a, S c h m e i d e l and Z d a n o w i c z in [7] in discrete case.

Let us recall some basic definitions and facts related to time scales.

Definition 1 ([2])**.** A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.

The mapping $\sigma\colon \mathbb{T}\to\mathbb{T}$, defined by $\sigma(t)=\inf\{s\in\mathbb{T}\colon s>t\}$ with $\inf\emptyset=\sup\mathbb{T}$ is called the forward jump operator. Similarly, we define the backward jump operator $\rho\colon \mathbb{T}\to\mathbb{T}$, given by $\rho(t)=\sup\{s\in\mathbb{T}\colon s<t\}$ with $\sup\emptyset=\inf\mathbb{T}$. The following classification of points is used within the theory: a point $t\in\mathbb{T}$ is called right-dense, right-scattered, left-dense and left-scattered if $\sigma(t)=t$ (for $t<\sup\mathbb{T}$), $\sigma(t)>t$, $\rho(t)=t$ (for $t>\inf\mathbb{T}$) and $\rho(t)<t$, respectively. We say that t is isolated if $\rho(t)<t<\sigma(t)$, and that t is dense if $\rho(t)=t=\sigma(t)$. The function $\mu\colon \mathbb{T}\to[0,\infty)$ defined by $\mu(t)=\sigma(t)-t$ is called the graininess function. The delta (or Hilger) derivative of $f\colon \mathbb{T}\to\mathbb{R}$ at a point $t\in\mathbb{T}^\kappa$, where

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T}\setminus(\rho(\sup\mathbb{T}),\sup\mathbb{T}] & \text{if } \sup\mathbb{T}<\infty,\\ \mathbb{T} & \text{if } \sup\mathbb{T}=\infty \end{cases}$$

is defined in the following way.

Definition 2 ([2])**.** The delta derivative of function f at a point t, denoted by $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\varepsilon>0$, there is a neighborhood U of t (i.e., $U=(t-\delta;t+\delta)\cap\mathbb{T}$ for some $\delta>0$) such that

$$\Big|\Big(f(\sigma(t))-f(s)\Big)-f^\Delta(t)\big(\sigma(t)-s\big)\Big|\le\varepsilon|\sigma(t)-s| \qquad \text{for all} \quad s\in U.$$

We say that a function f is delta (or Hilger) differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta\colon \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^\kappa$.

Definition 3 ([2]). A function $f\colon \mathbb{T} \to \mathbb{R}$ is called regulated provided its right--sided limits exist (finite) at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f\colon \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 4 ([2]). Assume $f\colon \mathbb{T} \to \mathbb{R}$ is a regulated function. We define the indefinite integral of a regulated function f by $\int f(t)\Delta t = F(t) + C$, where C is an arbitrary constant and F is a pre-derivative of f. We define the Cauchy integral by $\int_a^b f(t)\Delta t = F(b) - F(a)$ for all $a, b \in \mathbb{T}$.

We are interested in the nonoscillatory behaviour of system (1), that is why the general assumption on the time scale $\mathbb{T}$ is the following $\inf \mathbb{T} = T_0$ and $\sup \mathbb{T} = \infty$.

By a solution of (1), we mean a sequence $\big(X(t)\big) = \big[x(t), y(t), z(t)\big]^T$ of delta differentiable functions which are defined on $\mathbb{T}$ and satisfy (1) for $t \geq T_1 \geq T_0$. A function φ is called eventually positive (or eventually negative) if there exists $T \in \mathbb{T}$ such that $\varphi(t) > 0$ (or $\varphi(t) < 0$) for all $t \geq T$ in $\mathbb{T}$. If the function φ is either eventually positive or eventually negative we call it nonoscillatory. A solution X of system (1) is called nonoscillatory if all its components, i.e., x, y, z are nonoscillatory.

2. Preliminaries

For $T_1, T_2 \in \mathbb{T}$, let

$$[T_1, \infty)_\mathbb{T} = \{t \in \mathbb{T}\colon t \geq T_1\} \quad \text{and} \quad [T_1, T_2]_\mathbb{T} = \{t \in \mathbb{T}\colon T_1 \leq t \leq T_2\}.$$

By $C(A, B)$, $C_{rd}(A, B)$ we denote the set of all continuous functions mapping A to B, the set of all rd-continuous functions mapping A to B, respectively.

For elements of $\mathbb{R}^3$ the symbol $|\cdot|$ stands for the maximum norm. By $\mathcal{B}(\mathbb{T})$ we denote the Banach space of all triples of bounded and continuous functions with the supremum norm defined on time scale $\mathbb{T}$, i.e.,

$$\mathcal{B}(\mathbb{T}) = \left\{X\colon X \in C\big([T_0, \infty)_\mathbb{T}, \mathbb{R}^3\big), \text{ and } \ ||X|| = \sup_{t\in\mathbb{T}} |X(t)| < \infty\right\}. \qquad (2)$$

Let Ω be a subset of $\mathcal{B}(\mathbb{T})$.

Definition 5. Ω is called uniformly Cauchy if for every $\varepsilon > 0$ exists $T_1 \in [T_0, \infty)_{\mathbb{T}}$ such that for any $X \in \Omega$

$$|X(t_1) - X(t_2)| < \varepsilon \qquad \text{for all} \quad t_1, t_2 \in [T_1, \infty)_{\mathbb{T}}.$$

Definition 6. Ω is called equi-continuous on $[a, b]_{\mathbb{T}}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $X \in \Omega$ and $t_1, t_2 \in [a, b]_{\mathbb{T}}$ with $|t_1 - t_2| < \delta$,

$$|X(t_1) - X(t_2)| < \varepsilon.$$

The analogue of the Arzelá-Ascoli theorem on time scales was proved by Zhu and Wang in [12]. We need the generalization of this theorem to the three-dimensional case.

Lemma 1. *Assume that $\Omega \subset \mathcal{B}(\mathbb{T})$ is bounded and uniformly Cauchy. Moreover, assume that Ω is equi-continuous on $[T_0, T_1]_{\mathbb{T}}$ for any $T_1 \in [T_0, \infty)_{\mathbb{T}}$. Then Ω is relatively compact.*

Proof. Since Ω is uniformly Cauchy we have that for any $\varepsilon > 0$ there exists $T_1 \in [T_0, \infty)_{\mathbb{T}}$ such that for any $X \in \Omega$

$$|X(t_1) - X(t_2)| < \frac{\varepsilon}{3}, \qquad t_1, t_2 \in [T_1, \infty)_{\mathbb{T}}. \tag{3}$$

By the assumption of boundness there exists $\alpha > 0$ such that $\|X\| \le \alpha$ for every $X \in \Omega$. We can choose the sequence of $N_1 + 1$ real numbers $\beta_0, \beta_1, \dots, \beta_{N_1}$ such that $-\alpha = \beta_0 < \beta_1 < \dots < \beta_{N_1} = \alpha$ and

$$|\beta_{i+1} - \beta_i| < \frac{\varepsilon}{3}, \qquad i = 0, 1, \dots, N_1 - 1. \tag{4}$$

The equi-continuity of Ω on $[T_0, T_1]_{\mathbb{T}}$ implies that for the chosen $\varepsilon > 0$ there exists $\delta > 0$ such that for any $X \in \Omega$

$$|X(t) - X(s)| < \frac{\varepsilon}{3} \qquad \text{if } |t - s| \le \delta, \quad s, t \in [T_0, T_1]_{\mathbb{T}}. \tag{5}$$

Obviously, we can choose N_2 real numbers from the interval $[T_0, T_1]$ so that $T_0 = t_1 < t_2 < \dots < t_{N_2} = T_1$ and

$$|t_{i+1} - t_i| \le \delta, \qquad i = 1, 2, \dots, N_2 - 1. \tag{6}$$

Now, we construct a continuous mapping class $\mathcal{U} \subset C\big([T_0, \infty)_{\mathbb{T}}, \mathbb{R}^3\big)$. For each $k \in \{1, 2, 3\}$, $i \in \{1, 2, \dots, N_2 - 1\}$ and $j \in \{1, 2, \dots, N_1 - 1\}$ we define a function u_{ij}^k on $[t_i, t_{i+1}]$ as follows

$$u_{ij}^{(k)}(t) = \beta_j + \frac{\beta_{j+1} - \beta_j}{t_{i+1} - t_i}(t - t_i), \qquad t \in [t_i, t_{i+1}]$$

or

$$u_{ij}^{(k)}(t) = \beta_{j+1} + \frac{\beta_j - \beta_{j+1}}{t_{i+1} - t_i}(t - t_i), \quad t \in [t_i, t_{i+1}].$$

Observe that function $u_{ij}^{(k)}$ connects with the line points (t_i, β_j) and (t_{i+1}, β_{j+1}) or (t_i, β_{j+1}) and (t_{i+1}, β_j) being opposite vertices of the rectangle domain: $t_i \le t \le t_{i+1}$ and $\beta_j \le \beta \le \beta_{j+1}$. Let $\mathcal{U}_k$ be the set of all continuous functions $u^{(k)}$ on $[T_0, T_1]$ connecting functions $u_{ij}^{(k)}$ as above from $[t_1, t_2]$ to $[t_{N_2-1}, t_{N_2}]$. Note that each $\mathcal{U}_k$ is a finite set for any fixed numbers N_1 and N_2. Every function $u^{(k)} \in \mathcal{U}_k$ we extend to the function $\bar{u}^{(k)}$ defined on the whole $[T_0, \infty)_{\mathbb{T}}$ in the following way

$$\bar{u}^{(k)}(t) = \begin{cases} u^{(k)}(t), & t \in [T_0, T_1]_{\mathbb{T}}, \\ u^{(k)}(T_1), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Let $\mathcal{U}$ be the set of all possible triples $U(t) = \left[\bar{u}^{(1)}(t), \bar{u}^{(2)}(t), \bar{u}^{(3)}(t)\right]^T$. Clearly, $\mathcal{U}$ is finite since $\mathcal{U}_k$ is a finite set for $k = 1, 2, 3$. We will show that $\mathcal{U}$ is a finite ε-net for Ω. Since inequalities (4) and (5) and the definition of functions $\bar{u}^{(k)}$ for any $X = [x, y, z]^T \in \Omega$ we can choose $U = \left[\bar{u}^{(1)}, \bar{u}^{(2)}, \bar{u}^{(3)}\right]^T \in \mathcal{U}$ such that

$$|\bar{u}^{(1)}(t) - x(t)| < \frac{\varepsilon}{3}, \quad |\bar{u}^{(2)}(t) - y(t)| < \frac{\varepsilon}{3}, \quad |\bar{u}^{(3)}(t) - z(t)| < \frac{\varepsilon}{3} \tag{7}$$

for any $t \in [T_0, T_1]_{\mathbb{T}}$, so on the interval $[T_0, T_1]_{\mathbb{T}}$ we have

$$|U(t) - X(t)| < \frac{\varepsilon}{3}. \tag{8}$$

In case when $t \in [T_1, \infty)_{\mathbb{T}}$, from (3) and (7), we obtain

$$\left|\bar{u}^{(1)}(t) - x(t)\right| = \left|u^{(1)}(T_1) - x(t)\right| \le |x(T_1) - x(t)| + \left|u^{(1)}(T_1) - x(T_1)\right| < \frac{2\varepsilon}{3},$$

and the same arguments give us

$$\left|\bar{u}^{(2)}(t) - y(t)\right| < \frac{2\varepsilon}{3} \quad \text{and} \quad \left|\bar{u}^{(3)}(t) - z(t)\right| < \frac{2\varepsilon}{3}.$$

This means that for $t \in [T_1, \infty)_{\mathbb{T}}$ we have

$$|U(t) - X(t)| < \frac{2\varepsilon}{3}. \tag{9}$$

Finally, since (8) and (9) we conclude that

$$\|U - X\| = \sup_{t \in [T_0, \infty)_{\mathbb{T}}} |U(t) - X(t)| \le \frac{2\varepsilon}{3}.$$

Thus $\mathcal{U}$ is a finite ε-net for Ω and this completes the proof of relative compactness of Ω. □

We also recall Krasnoselskii's fixed point theorem which will be crucial to establish the existence of nonoscillatory solutions for (1).

Theorem 1 ([5]). *Let B be a Banach space, let Ω be a bounded, convex and closed subset of B and let F, G be maps of Ω into B such that*

(i) *$FX + GY \in \Omega$ for any $X, Y \in \Omega$,*

(ii) *F is a contraction,*

(iii) *G is completely continuous.*

Then the equation $FX + GX = X$ has a solution in Ω.

3. Main results

We will assume in (1) that

(A1) $f, g, h \in C(\mathbb{R}, \mathbb{R})$,

(A2) $a, b, c \in C_{rd}(\mathbb{T}, \mathbb{R})$ and

$$\int_{T_0}^{\infty} |a(s)|\Delta s < \infty, \quad \int_{T_0}^{\infty} |b(s)|\Delta s < \infty, \quad \text{and} \quad \int_{T_0}^{\infty} |c(s)|\Delta s < \infty,$$

(A3) $u_i \in C_{rd}(\mathbb{T}, \mathbb{T})$ and $\lim_{t\to\infty} u_i(t) = \infty$ for $i = 1, 2, 3, 4$,

(A4) $p\colon \mathbb{T} \to \mathbb{R}$ is delta differentiable.

Using Krasnoselskii's fixed point theorem we will prove the following

Theorem 2. *Assume that conditions* (A1)–(A4) hold. If there exists a positive real number c_p such that

(A5) $$0 < p(t) \le c_p < 1 \qquad \text{for any} \quad t \in \mathbb{T},$$

then system (1) has a positive bounded solution.

P r o o f. For the fixed positive real number r we define set

$$\Omega = \big\{X \in \mathcal{B}(\mathbb{T})\colon x(t), y(t), z(t) \in I,\ t \in \mathbb{T}\big\},$$

where $I = \big[\frac{1}{3}(1 - c_p)r, r\big]$. Ω is bounded closed convex subset of the Banach space $\mathcal{B}(\mathbb{T})$. Since condition (A1) is satisfied, we can set

$$M = \max\big\{|f(x)|, |g(x)|, |h(x)|\colon x \in I\big\}.$$

From (A2), there exists $T_1 \in \mathbb{T}$ such that

$$\int_{T_1}^{\infty} |a(s)|\Delta s \le \frac{(1 - c_p)r}{3M}, \quad \int_{T_1}^{\infty} |b(s)|\Delta s \le \frac{(1 - c_p)r}{3M}, \quad \int_{T_1}^{\infty} |c(s)|\Delta s \le \frac{(1 - c_p)r}{3M}.$$

Next, we define the maps $F, G\colon \Omega \to B(\mathbb{T})$ where

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}, \qquad G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix},$$

$$(FX)(t) = \begin{bmatrix} -p(t)x(u_1(t)) + \frac{(2+c_p)r}{3} \\ \frac{2(1-c_p)r}{3} \\ \frac{2(1-c_p)r}{3} \end{bmatrix} \quad \text{for} \quad t \geq T_1,$$

$$(FX)(t) = (FX)(T_1) \qquad \text{for} \quad T_0 \leq t < T_1,$$

$$(GX)(t) = \begin{bmatrix} -\int_t^\infty a(s)\, f\big(y(u_2(s))\big)\Delta s \\ -\int_t^\infty b(s)\, g\big(z(u_3(s))\big)\Delta s \\ -\int_t^\infty c(s)\, h\big(x(u_4(s))\big)\Delta s \end{bmatrix} \quad \text{for} \quad t \geq T_1 \tag{10}$$

and

$$(GX)(t) = (GX)(T_1) \qquad \text{for} \quad T_0 \leq t < T_1.$$

We will show that F and G satisfy the conditions of the Theorem 1. First we show that for any $X, \bar{X} \in \Omega$ we have that $FX + G\bar{X} \in \Omega$. For $t \geq T_1$ we get the following upper and lower estimations

$$\begin{aligned} (F_1X)(t) + (G_1\bar{X})(t) &= -p(t)\, x\big(u_1(t)\big) + \frac{(2+c_p)r}{3} - \int_t^\infty a(s)\, f\Big(\bar{y}\big(u_2(s)\big)\Big)\Delta s \\ &\leq \frac{(2+c_p)r}{3} + \int_t^\infty |a(s)| \, \Big| f\Big(\bar{y}\big(u_2(s)\big)\Big)\Big| \,\Delta s \\ &\leq \frac{(2+c_p)r}{3} + M\int_t^\infty |a(s)|\Delta s \\ &\leq \frac{2}{3}r + \frac{1}{3}c_p r + M \cdot \frac{(1-c_p)r}{3M} = r, \end{aligned}$$

$$\begin{aligned}
(F_1X)(t)+(G_1\bar{X})(t) &= -p(t)\,x\big(u_1(t)\big)+\frac{(2+c_p)r}{3}-\int_t^\infty a(s)\,f\Big(\bar{y}\big(u_2(s)\big)\Big)\Delta s\\
&\ge \frac{(2+c_p)r}{3}-\int_t^\infty |a(s)|\,\Big|f\Big(\bar{y}\big(u_2(s)\big)\Big)\Big|\,\Delta s-p(t)\,x\big(u_1(t)\big)\\
&\ge \frac{2}{3}r+\frac{1}{3}c_pr-M\cdot\frac{(1-c_p)r}{3M}-c_pr\\
&= \frac{2}{3}r+\frac{1}{3}c_pr-\frac{1}{3}r+\frac{1}{3}c_pr-c_pr\\
&= \frac{1}{3}(1-c_p)r.
\end{aligned}$$

Therefore $(F_1X)(t)+(G_1\bar{X})(t)\in I$ for all $t\in\mathbb{T}$ and any $X,\bar{X}\in\Omega$.

Below we present reasoning for maps F_2 and G_2, but the same conclusions can be drawn for F_3 and G_3.

$$\begin{aligned}
(F_2X)(t)+(G_2\bar{X})(t) &= \frac{2(1-c_p)r}{3}-\int_t^\infty b(s)\;g\Big(\bar{z}\big(u_3(s)\big)\Big)\Delta s\\
&\le \frac{2(1-c_p)r}{3}+\int_t^\infty |b(s)|\,\Big|g\Big(\bar{z}\big(u_3(s)\big)\Big)\Big|\,\Delta s\\
&\le \frac{2}{3}r-\frac{2}{3}c_pr+M\cdot\frac{(1-c_p)r}{3M}=(1-c_p)r\le r,
\end{aligned}$$

$$\begin{aligned}
(F_2X)(n)+(T_2\bar{X})(n) &= \frac{2(1-c_p)r}{3}-\int_t^\infty b(s)\;g\Big(\bar{z}\big(u_3(s)\big)\Big)\Delta s\\
&\ge \frac{2(1-c_p)r}{3}-\int_t^\infty |b(s)|\,\Big|g\Big(\bar{z}\big(u_3(s)\big)\Big)\Big|\,\Delta s\\
&\ge \frac{2}{3}r-\frac{2}{3}c_pr-M\cdot\frac{(1-c_p)r}{3M}=\frac{1}{3}(1-c_p)r.
\end{aligned}$$

Hence for any $X,\bar{X}\in\Omega$ we have that $FX+G\bar{X}\in\Omega$.

The next step is to prove that F is a contraction mapping. It is easy to see that

$$\begin{aligned} \big|(F_1X)(t) - (F_1\bar{X})(t)\big| &\leq p(t)\ \big|x\big(u_1(t)\big) - \bar{x}\big(u_1(t)\big)\big| \\ &\leq c_p\ \big|x\big(u_1(t)\big) - \bar{x}\big(u_1(t)\big)\big| \\ &\leq c_p \sup_{t\in\mathbb{T}} \big|x\big(u_1(t)\big) - \bar{x}\big(u_1(t)\big)\big| \\ &\leq c_p \sup_{t\in\mathbb{T}} |x(t) - \bar{x}(t)|, \\ \big|(F_2X)(t) - (F_2\bar{X})(t)\big| &= 0, \\ \big|(F_3X)(t) - (F_3\bar{X})(t)\big| &= 0, \qquad \text{for } X, \bar{X} \in \Omega \quad \text{and} \quad t \geq T_1. \end{aligned}$$

Hence

$$\|FX - F\bar{X}\| \leq c_p\|X - \bar{X}\|,$$

where, by (A5), there is $0 < c_p < 1$.

It remains to show that G is a completely continuous mapping. We start it showing that G is continuous. Consider sequence $X_n = [x_n, y_n, z_n]^T \in \Omega$ for any $n \in \mathbb{N}$ such that $||X_n - X|| \to 0$ as $n \to \infty$, then $X \in \Omega$ and for any $t \in \mathbb{T}$ we have that $|x_n(t) - x(t)| \to 0$, $|y_n(t) - y(t)| \to 0$, $|z_n(t) - z(t)| \to 0$ as $n \to \infty$. Because f is continuous, then for any $t \in \mathbb{T}$ we have

$$|a(t)| \left| f\Big(y_n\big(u_2(t)\big)\Big) - f\Big(y\big(u_2(t)\big)\Big) \right| \to 0 \qquad \text{as } n \to \infty. \tag{11}$$

On the other hand, for $y_n(t), y(t) \in I$ we get that

$$|a(t)| \left| f\Big(y_n\big(u_2(t)\big)\Big) - \Big(y\big(u_2(t)\big)\Big) \right| \leq 2M|a(t)|. \tag{12}$$

From (10) we obtain

$$|(G_1X_n)(t) - (G_1X)(t)| \leq \int_t^\infty |a(s)| \left| f\Big(y_n\big(u_2(s)\big)\Big) - f\Big(y\big(u_2(s)\big)\Big) \right| \Delta s$$

for any $t \geq T_1$, and

$$|(G_1X_n)(t) - (G_1X)(t)| \leq \int_{T_1}^\infty |a(s)| \left| f\Big(y_n\big(u_2(s)\big)\Big) - f\Big(y\big(u_2(s)\big)\Big) \right| \Delta s$$

for $T_0 \leq t < T_1$. Therefore, we can conclude that

$$\sup_{t\in\mathbb{T}} |(G_1X_n)(t) - (G_1X)(t)| \leq \int_{T_1}^\infty |a(s)| \left| f\Big(y_n\big(u_2(s)\big)\Big) - f\Big(y\big(u_2(s)\big)\Big) \right| \Delta s. \tag{13}$$

Since (11), (12) and (13) applying the Lebesgue dominated convergence theorem on time scales for the integral on time scales [4], [6] we obtain

$$\sup_{t\in\mathbb{T}} |(G_1X_n)(t) - (G_1X)(t)| \to 0 \qquad \text{if } n \to \infty.$$

Analogously, if $n \to \infty$, then

$$\sup_{t\in\mathbb{T}} |(G_2X_n)(t) - (G_2X)(t)| \to 0 \quad \text{and} \quad \sup_{t\in\mathbb{T}} |(G_3X_n)(t) - (G_3X)(t)| \to 0.$$

Hence,

$$\|GX_n - GX\| \to 0 \qquad \text{if} \;\; n \to \infty.$$

It means that G is a continuous mapping on Ω.

To prove that $G\Omega$ is relatively compact it is sufficient to verify that $G\Omega$ satisfies all conditions in the Lemma 1. Obviously $G\Omega$ is a bounded set. Now we will show that it is uniformly Cauchy. Let $X \in \Omega$. Observe that for any given $\varepsilon > 0$ by assumptions (A1) and (A2) there exists $T_2 > T_1$ such that for all $t \geq T_2$ the following inequality holds

$$\int_t^\infty |a(s)| \left| f\Big(y\big(u_2(s)\big)\Big)\right| \Delta s < \frac{\varepsilon}{2}.$$

Hence by definition of G,

$$|(G_1X)(t_1) - (G_1X)(t_2)| = \left| \int_{t_1}^\infty a(s) f\Big(y\big(u_2(s)\big)\Big)\Delta s - \int_{t_2}^\infty a(s) f\Big(y\big(u_2(s)\big)\Big)\Delta s \right| < \varepsilon$$

for arbitrary $t_1, t_2 \in [T_2, \infty)_\mathbb{T}$. Since similar arguments can be apply to G_2 and G_3 we conclude that $G\Omega$ is uniformly Cauchy.

Finally, it remains to prove the equi-continuity on $[T_0, T_2]_\mathbb{T}$ for any $T_2 \in [T_0, \infty)_\mathbb{T}$. Observe that for any $X \in \Omega$ and $t_1, t_2 \in [T_0, T_1]_\mathbb{T}$

$$|(GX)(t_1) - (GX)(t_2)| \equiv 0,$$

that is why we can assume that $T_2 > T_1$. Taking now $t_1, t_2 \in [T_1, T_2]_\mathbb{T}$ we obtain the following estimation

$$\begin{aligned} |(G_1X)(t_1) - (G_1X)(t_2)| &= \left| \int_{t_1}^\infty a(s) f\Big(y\big(u_2(s)\big)\Big)\Delta s - \int_{t_2}^\infty a(s) f\Big(y\big(u_2(s)\big)\Big)\Delta s \right| \\ &\leq M \left| \int_{t_1}^{t_2} a(s)\Delta s \right|. \end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists

$$\delta_1 = \frac{\varepsilon}{M \cdot \max\limits_{t\in[T_1,T_2]_\mathbb{T}} |a(t)|}$$

such that, when $t_1, t_2 \in [T_1, T_2]_\mathbb{T}$ and $|t_1 - t_2| < \delta_1$, we get that

$$|(G_1X)(t_1) - (G_1X)(t_2)| < \varepsilon.$$

Since

$$|(GX)(t_1)-(GX)(t_2)| \le \max\left\{M\left|\int_{t_1}^{t_2} a(s)\Delta s\right|, M\left|\int_{t_1}^{t_2} b(s)\Delta s\right|, M\left|\int_{t_1}^{t_2} c(s)\Delta s\right|\right\},$$

we obtain the equi-continuity of $G\Omega$ with $\delta = \min\{\delta_1,\delta_2,\delta_3\}$, where $\delta_2 = \frac{\varepsilon}{M\cdot \max\limits_{t\in[T_1,T_2]_{\mathbb{T}}}|b(t)|}$ and $\delta_3 = \frac{\varepsilon}{M\cdot \max\limits_{t\in[T_1,T_2]_{\mathbb{T}}}|c(t)|}$. Lemma 1 implies that $G\Omega$ is relatively Cauchy. From the above we obtain that G is a completely continuous mapping.

By the Theorem 1, there exists X^* such that $FX^*+GX^*=X^*$. We will verify that $X^*(t)$ satisfies system (1) for $t\ge T_1$. Since $(F_1X^*)(t)+(G_1X^*)(t)=x^*(t)$ we have

$$-p(t)x^*\big(u_1(t)\big)+\frac{(2+c_p)r}{3}-\int_t^{\infty} a(s)f\Big(y^*\big(u_2(s)\big)\Big)\Delta s = x^*(t). \tag{14}$$

After moving the term $-p(t)x^*\big(u_1(t)\big)$ to the right-hand side of the equation and then applying to its both sides delta differentiation we get

$$\Big(x^*(t)+p(t)x^*\big(u_1(t)\big)\Big)^{\Delta} = \left[-\int_t^{\infty} a(s)f\Big(y^*\big(u_2(s)\big)\Big)\Delta s,\right]^{\Delta}.$$

Thus

$$\Big(x^*(t)+p(t)x^*\big(u_1(t)\big)\Big)^{\Delta} = a(t)f\Big(y^*\big(u_2(t)\big)\Big), \tag{15}$$

since $a\big(t\big)f(y^*(u_2(t)))$, by assumption (A3), as rd-continuous function has its antiderivative.

Let us notice that the Theorem 1 guaranties the equality (14) and by the assumptions (A1)–(A5) we arrive to (15). Hence, we see that $x^*(t)$ is not only rd-continuous but, moreover, rd-continuously delta differentiable. Similarly, from equation $(F_2X^*)(t)+(G_2X^*)(t)=y^*(t)$ we get that

$$(y^*)^{\Delta}(t) = \left[\int_t^{\infty} b(s)g\Big(z^*\big(u_3(s)\big)\Big)\Delta s\right]^{\Delta}.$$

Again, by assumption (A3), we know that $b(t)g\big(z^*(u_3(t))\big)$ is rd-continuous function and in consequence

$$(y^*)^{\Delta}(t) = b(t)g\Big(z^*\big(u_3(t)\big)\Big).$$

In the same manner we verify that $(F_3X^*)(t)+(G_3X^*)(t)=z^*(t)$ implies the third equation of (1). Hence X^* is the solution of system (1). The proof is complete. □

Finally, the above theorem is illustrated with examples in which four different time scales are employed.

Example 1. Let $\mathbb{T} = [5, \infty)$. Consider the following system

$$\begin{cases} \big(x(t) + \frac{1}{2t}\, x(t-1)\big)^{\Delta} = a(t)\, y(t-2), \\ y^{\Delta}(t) = b(t)\, \big(z(t-1)\big)^2, \\ z^{\Delta}(t) = c(t)\, x(t) \end{cases}$$

with

$$\begin{aligned} a(t) &= -\frac{(4t^2 - 6t + 3)(t-2)}{2t^2(t-1)^2(3t-5)}, \\ b(t) &= -\frac{(t-1)^4}{t^2(3t^2 - 6t + 4)^2}, \\ c(t) &= -\frac{2}{t^2(2t+1)}. \end{aligned}$$

It easy to check that all the conditions (A1)–(A5) are satisfied. Here $c_p = \frac{1}{10}$. One of the bounded solutions of the above system is

$$X(t) = \left[2 + \frac{1}{t}, 3 + \frac{1}{t}, 3 + \frac{1}{t^2}\right]^T.$$

Example 2. Let $\mathbb{T} = \{n \colon n \geq 3,\ n \in \mathbb{N}\}$. Consider the following system

$$\begin{cases} \big(\, x(t) \,+\, \frac{1}{3t}\, x(t-2)\big)^{\Delta} = a(t)\, \big[(y(t))^2 + 2\big], \\ y^{\Delta}(t) = b(t)\, \big(z(t-1)\big)^3, \\ z^{\Delta}(t) = c(t)\, \big(x(t-2)\big)^2 \end{cases}$$

with

$$\begin{aligned} a(t) &= -\frac{(7t^2 - 17t + 12)t}{3(t+1)(t-1)(t-2)(6t^2 + 4t + 1)}, \\ b(t) &= -\frac{(t-1)^6}{t^4(t+1)(t-2)^3}, \\ c(t) &= \frac{(2t+1)(t-2)^2}{t^4(t+1)^2}. \end{aligned}$$

Again it is easy to see that all the conditions (A1)–(A4) are satisfied. Also the condition (A5) of the Theorem 2 is satisfied with $c_p = \frac{1}{9}$. One of the bounded solutions of the above system is

$$X(t) = \left[1 + \frac{2}{t}, 2 + \frac{1}{t}, 1 - \frac{1}{t^2}\right]^T.$$

Example 3. Let $\mathbb{T} = \{2^n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ is the set of nonnegative integers. Consider the following system

$$\begin{cases} \big(x(t) + \frac{1}{2}\, x(\rho(t))\big)^{\Delta} = a(t)\, y\big(\rho(t)\big), \\ y^{\Delta}(t) = b(t)\, z(t), \\ z^{\Delta}(t) = c(t)\, \big(x(t)\big)^3 \end{cases}$$

with

$$\begin{aligned} \rho(t) &= \frac{t}{2}, \\ a(t) &= -\frac{1}{2t(t+1)}, \\ b(t) &= -\frac{1}{2t(t+1)}, \\ c(t) &= -\frac{t}{2(t+1)^3}. \end{aligned}$$

One can verify that all the conditions (A1)–(A5) are satisfied (condition (A5) with constant $c_p = \frac{1}{2}$). One of the bounded solutions of this system is

$$X(t) = \left[1 + \frac{1}{t}, 2 + \frac{1}{t}, 1 + \frac{1}{t}\right]^T.$$

Example 4. Let $\mathbb{T} = P_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$. Consider the following system

$$\begin{cases} \big(x(t) + \frac{1}{3}\, x(t-2)\big)^{\Delta} = a(t)\, y(t-2), \\ y^{\Delta}(t) = b(t)\, \big(z(t-2)\big)^2, \\ z^{\Delta}(t) = c(t)\, (x(t))^2 \end{cases}$$

with

$$\begin{aligned} a(t) &= -\frac{4(t^2+7t+13)}{3(2t+7)(t+3)(t+5)^2}, \\ b(t) &= -\frac{(t+1)^2}{(t+5)^2(2t+3)^2}, \\ c(t) &= -\frac{(t+5)^2}{(t+3)^2(4t+21)^2}, \end{aligned}$$

for $t \in \bigcup_{k=0}^{\infty} [2k, 2k+1)$ and

$$a(t) = -\frac{2(2t^2+16t+33)}{3(2t+7)(t+6)(t+5)(t+4)},$$

$$b(t) = -\frac{(t+1)^2}{(t+5)(t+6)(2t+3)^2},$$

$$c(t) = -\frac{(t+5)^2}{(t+3)(t+4)(4t+21)^2}$$

for $t \in \bigcup_{k=0}^{\infty} \{2k+1\}$. One of the bounded solutions of this system is

$$X(t) = \left[4 + \frac{1}{t+5}, 2 + \frac{1}{t+5}, 2 + \frac{1}{t+3}\right]^T.$$

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Received October 19, 2017

Department of Mathematics
Faculty of Mathematics and Informatics
University of Bialystok
K. Ciołkowskiego 1M
PL–15-245 Białystok
POLAND

E-mail: uostasze@math.uwb.edu.pl
eschmeidel@math.uwb.edu.pl
mzdan@math.uwb.edu.pl