sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2018-0009
Tatra Mt. Math. Publ. **71** (2018), 99–107

STOCHASTIC DIFFERENTIAL EQUATIONS DESCRIBING SYSTEMS WITH COLOURED NOISE

Edita Kolářová* — Lubomír Brančík**

Brno University of Technology, Brno, CZECH REPUBLIC

ABSTRACT. In this paper we deal with stochastic differential equations, that describe systems effected by coloured noise. In electrical systems this can be the case, when e.g. transmission line is modelled by means of proper higher-order ladder network. We define the mathematical representation of the coloured noise as a solution of the Langevin equation and formulate the corresponding Itô type stochastic differential equation. Applying this theory we derive the stochastic model of the network and find sets of individual stochastic trajectories numerically via a stochastic version of the backward Euler scheme. Afterwards respective confidence intervals are computed statistically while utilizing Student's t distribution. The theoretical results are illustrated by an example of a higher-order ladder network. Numerical simulations in the example are carried out using Matlab.

1. Introduction

Ordinary differential equations describe systems without the influence of randomness. By incorporating noise effects into these equations we get systems of stochastic differential equations (SDEs). The mathematical representation of the noise is a stochastic process called the Wiener process or Brownian motion, which can be considered as the integral of the so-called Gaussian white noise.

Definition 1.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space. A real-valued, continuous stochastic process $W(t) = \{W(t,\omega),\ t \geq 0,\ \omega \in \Omega\}$ on $(\Omega, \mathcal{A}, P)$ is called the Wiener process if

(i) $W(0) = 0$,

(ii) $W(t) - W(s)$ is $N(0, t-s)$ for all $t \geq s \geq 0$,

2010 Mathematics Subject Classification: 34F05, 60H10, 65C30.
Keywords: Stochastic integral equations, Wiener process, Itô formula, white noise, coloured noise, higher-order ladder network, transmission line model.

(iii) for all $0 < t_1 < t_2 < \cdots < t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W\big(t_{n-1}\big)$ are independent.

For the moments of the Wiener process we have

$$E[W(t)] = 0 \quad \text{and} \quad E[W^2(t)] = t \quad \text{for } t \geq 0.$$

Definition 1.2. $\mathbf{W}(t) = \big(W_1(t), \ldots, W_M(t)\big)$ is called an M-dimensional Wiener process if $W_i(t)$ are independent Wiener processes for all $i = 1 \ldots M$.

2. Stochastic differential equations

Definition 2.1. An N-dimensional stochastic differential equation (SDE) can be written as

$$\mathrm{d}\mathbf{Y}(t) = \mathbf{F}\big(t, \mathbf{Y}(t)\big)\ \mathrm{d}t + \mathbf{G}\big(t, \mathbf{Y}(t)\big)\ \mathrm{d}\mathbf{W}(t), \mathbf{Y}(0) = \mathbf{Y}_0, \tag{1}$$

where $\mathbf{F} : \langle 0, T\rangle \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathbf{G} : \langle 0, T\rangle \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^M$ are functions and $\mathbf{W}(t)$ is an M-dimesional Wiener processes representing the noise.
The solution is a stochastic vector process $\mathbf{Y}(t) = \big(Y_1(t), \ldots, Y_N(t)\big)$.

We denote the stochastic solution by capital letters, to distinguish it from the deterministic one. By an SDE we understand in fact an integral equation

$$\mathbf{Y}(t) = \mathbf{Y}_0 + \int_0^t \mathbf{F}\big(s, \mathbf{Y}(s)\big)\ \mathrm{d}s + \int_0^t \mathbf{G}\big(s, \mathbf{Y}(s)\big)\ \mathrm{d}\mathbf{W}(s), \tag{2}$$

where the integral with respect to $\mathrm{d}s$ is the Lebesgue integral and the integrals with respect to $\mathrm{d}W_i(s)$ are stochastic integrals, that we call Itô integrals (see [1] and [2]).

2.1. The Langevin equation

In SDEs the randomness of the system is considered as the so-called "white noise process", which is represented by the Wiener process. In some physical problems the noise has to be considered more generally, as a coloured noise, that can be represented by the solution of the **Langevin equation**:

$$\mathrm{d}X(t) = -\beta X(t)\ \mathrm{d}t + \sigma\ \mathrm{d}W(t), \quad X(0) = X_0, \tag{3}$$

for some initial distribution X_0, independent of the Wiener process, some coefficients β, σ and $\beta > 0$. Then the solution $X(t)$ is the process:

$$X(t) = e^{-\beta\, t} X_0 + \sigma \int_0^t e^{-\beta(t-s)}\ \mathrm{d}W(s), \tag{4}$$

where the integral with respect to the Wiener process is the Itô integral.

Definition 2.2. Let X_0 be constant or normally distributed, then the solution of the Langevine equation (3) is called the coloured noise process.

Let us assume that the moment $E[X_0] < \infty$. As the expectation of the Itô integral is zero, for the expectation of the solution of the Langevin equation we have

$$E[X(t)] = e^{-\beta\, t} \cdot\, E[X_0]. \tag{5}$$

We can compute the second moment of $X(t)$ as well

$$E[X^2(t)] = e^{-2\beta\, t} \cdot\, E[X_0^2] + \frac{\sigma^2}{2\beta}\big(1 - e^{-2\beta t}\big). \tag{6}$$

Thus the variance $V[X(t)] = E[X^2(t)] - \big(E[X(t)]\big)^2$ is

$$V[X(t)] = e^{-2\beta\, t} \cdot\, V[X_0] + \frac{\sigma^2}{2\beta}\big(1 - e^{-2\beta t}\big). \tag{7}$$

We can see that $E[X(t)] \to 0$, $V[X(t)] \to \frac{\sigma^2}{2\beta}$ as $t \to \infty$ and the distribution of $X(t)$ approaches $N\left(0, \frac{\sigma^2}{2\beta}\right)$ for large t. If X_0 is constant or normally distributed, then the solution is a Gaussian process, that represents a more general noise process as the white noise.

Remark 1. If we extend the equation (3) for $\beta = 0$, we get $X(t) = \sigma W(t)$. In that special case, the coloured noise process is a multiple of the Brownian motion (or the Wiener process in mathematical notation). See the top left picture in Fig. 1, where we also pictured 3 examples of the coloured noise with the same coefficients β and σ but with different initial conditions.

2.2. Stochastic differential equations with coloured noise

Definition 2.3. A coloured noise influenced system can be described by an SDE with coloured noise as follows

$$\mathrm{d}\mathbf{Y}(t) = \mathbf{F}\big(t, \mathbf{Y}(t)\big)\, \mathrm{d}t + \mathbf{G}\big(t, \mathbf{Y}(t)\big)\, \mathrm{d}\mathbf{X}(t), \quad \mathbf{Y}(0) = \mathbf{Y}_0, \tag{8}$$

where

$$\mathbf{F} : \langle 0, T\rangle \times \mathbb{R}^N \to \mathbb{R}^N, \quad \mathbf{G} : \langle 0, T\rangle \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^M$$

and all components $X_i(t)$, $i = 1, \ldots, M$ of the process $\mathbf{X}(t) = \big(X_1(t), \ldots, X_M(t)\big)$ satisfy the Langevin equation (3) for some constants $\beta > 0$, $\sigma \neq 0$ and for X_0 constant or normally distributed.

While examining the expectation of the stochastic solutions of the SDE with white noise we can use the fact that the expectation of the Wiener process as well the expectation of the Itô integral with respect to the Wiener process are 0. While examining the SDE with coloured noise, we can have initial value X_0 with nonzero expectation that influences the expectation of the stochastic solutions.

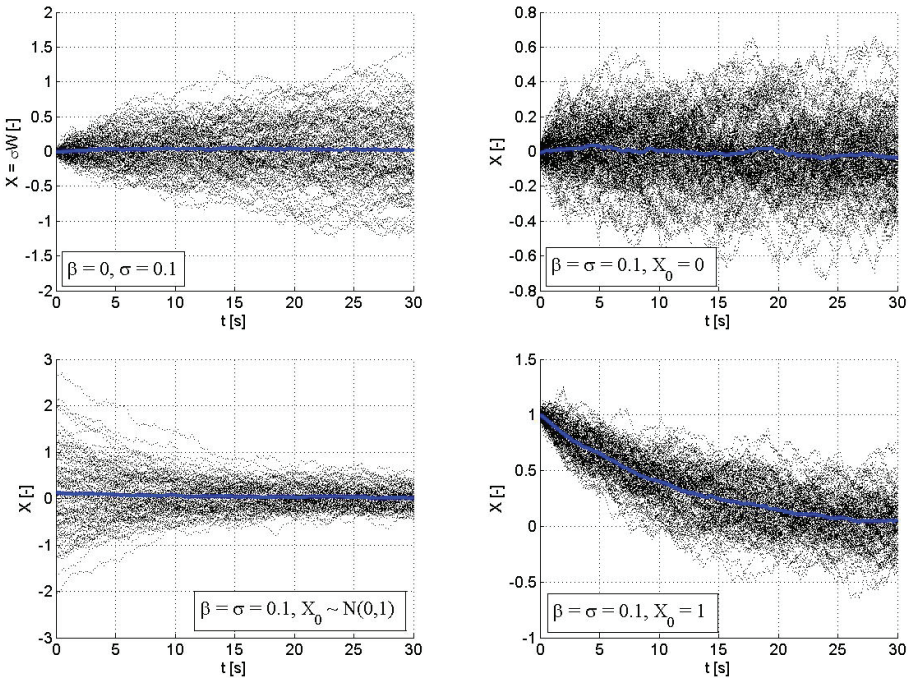

FIGURE 1. Trajectories of the white noise (top left picture) and the coloured noise process with different initial conditions.

3. Application to higher-order ladder network

A higher-order ladder network composed of RLGC passive elements is shown in Fig. 2. Such a type of the circuit structure plays often role of a lumped-parameter model of a lossy transmission line (TL). Here the asterisk at the voltage source exciting the network identifies the presence of the noise.

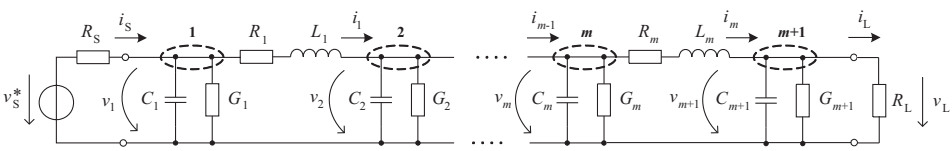

FIGURE 2. Higher-order ladder network with coloured-noise excitation source.

3.1. Deterministic model formulation

First, we consider a mathematical description for a deterministic source, $v_S(t)$, based on a state-variable method. Then we can write a first-order ordinary differential equation (ODE) in a matrix form (see [3])

$$\mathbf{M}\,\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = -(\mathbf{H}+\mathbf{P})\,\mathbf{y}(t) + \mathbf{P}\,\mathbf{u}(t), \tag{9}$$

with individual terms as follows. The column vector

$$\mathbf{y}(t) = \begin{pmatrix} \mathbf{v}_C(t) \\ \mathbf{i}_L(t) \end{pmatrix}$$

consists of the unknown state variables, namely subvectors $\mathbf{v}_C(t)$ and $\mathbf{i}_L(t)$ holding $m+1$ capacitor voltages and m inductor currents, respectively. The matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{pmatrix}$$

is composed of two diagonal submatrices, $\mathbf{C}$ with $m+1$ capacitances, and $\mathbf{L}$ with m inductances, and

$$\mathbf{H} = \begin{pmatrix} \mathbf{G} & \mathbf{E} \\ -\mathbf{E}^T & \mathbf{R} \end{pmatrix}$$

is similarly formed by diagonal submatrices $\mathbf{G}$ and $\mathbf{R}$, with $m+1$ conductances and m resistances, respectively, but $\mathbf{E}$ as $(m+1)\times m$ bi-diagonal matrix containing items ± 1, more details can be found in [8]. The matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{G}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

depends on external elements and contains an $m+1$ order diagonal matrix

$$\mathbf{G}_e = \mathrm{diag}(G_S, 0, \ldots, 0, G_L), \quad \text{where} \quad G_S = 1/R_S \quad \text{and} \quad G_L = 1/R_L.$$

Finally, the column vector

$$\mathbf{u}(t) = \begin{pmatrix} \mathbf{v}_e(t) \\ \mathbf{0} \end{pmatrix}$$

contains an $m+1$ order subvector $\mathbf{v}_e(t) = \big(v_S(t), 0, \ldots, 0\big)^T$. If we introduce markings

$$\mathbf{A} = -\mathbf{M}^{-1}(\mathbf{H}+\mathbf{P}) \quad \text{and} \quad \mathbf{B} = \mathbf{M}^{-1}\,\mathbf{P},$$

then (9) leads to a simplified notation

$$\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = \mathbf{A}\,\mathbf{y}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{10}$$

which is a basis for the formulation of stochastic differential equation with a coloured noise.

3.2. Stochastic model formulation

In this section we formulate the stochastic differential equations for the circuit with a source influenced by random effects. Instead of $v_S(t)$ a non-deterministic version is considered as

$$v_S^*(t) = v_S(t) + \text{“}noise\text{”}. \tag{11}$$

By considering (10) and (11) a respective SDE can be formulated

$$\mathrm{d}\mathbf{Y}(t) = \Big(\mathbf{A} \cdot \mathbf{Y}(t) + \mathbf{B}\ \mathbf{u}(t)\Big)\ \mathrm{d}t + \mathbf{b}\ \mathrm{d}X(t), \tag{12}$$

where $\mathbf{b} = \mathbf{B}^{(1)}$ designates the first column of the matrix $\mathbf{B}$, corresponding to a fact that just the first item in the $\mathbf{u}(t)$ is influenced by a noise as in (11). The term $\mathrm{d}X(t)$ is a stochastic process described by the Langevin equation (3), whose coefficients β and σ define a type of a coloured noise. Now combining (12) and (3) we can reformulate the resultant SDE with a new variable

$$\mathrm{d}\tilde{\mathbf{Y}}(t) = \big(\ \tilde{\mathbf{A}}\,\tilde{\mathbf{Y}}(t) +\ \tilde{\mathbf{B}}\ \tilde{\mathbf{u}}(t)\big)\ \mathrm{d}t + \tilde{\mathbf{b}}\ \mathrm{d}W(t), \tag{13}$$

where

$$\tilde{\mathbf{Y}}(t) = \begin{pmatrix} \mathbf{Y}(t) \\ X(t) \end{pmatrix}, \quad \tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & -\beta\mathbf{b} \\ \mathbf{0}^T & -\beta \end{pmatrix}, \quad \tilde{\mathbf{B}}(t) = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix},$$

$$\mathbf{b} = \sigma \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{u}}(t) = \begin{pmatrix} \mathbf{u}(t) \\ 0 \end{pmatrix}.$$

with $\mathbf{0}$ as $(2m+1) \times 1$ zero column vector.

4. Examples of Matlab simulations

Let the network in Fig. 2 be excited from a voltage source influenced by coloured noises of various parameters. We are interested in voltage and/or current responses at the end of the network. A deterministic part of this source is a unit step, $v_S(t) = 1(t)$. Its internal resistance $R_S = 1\,\Omega$, the output port of the circuit is open, i.e., $R_L \to \infty$. We consider $m = 10$, it means, the electrical circuit is of the 21st order. All longitudinal resistances and shunt conductances have values 10 mΩ and 10 mS, respectively, while inductances and capacitances are 1 H and 1 F, respectively. We substituted this values to the equation (9) and created the corresponding stochastic model (13), while we considered $\beta = \sigma = 0.1$. We found the numerical solutions of this equation by the stochastic backward Euler scheme and we computed 99 % confidence intervals via statistical processing of sets of stochastic trajectories, see [4]. The results of simulations are shown in Fig. 3.

From the results of simulation in Fig. 3, we can identify responses with time delays around 10 seconds corresponding to right behavior of the system with

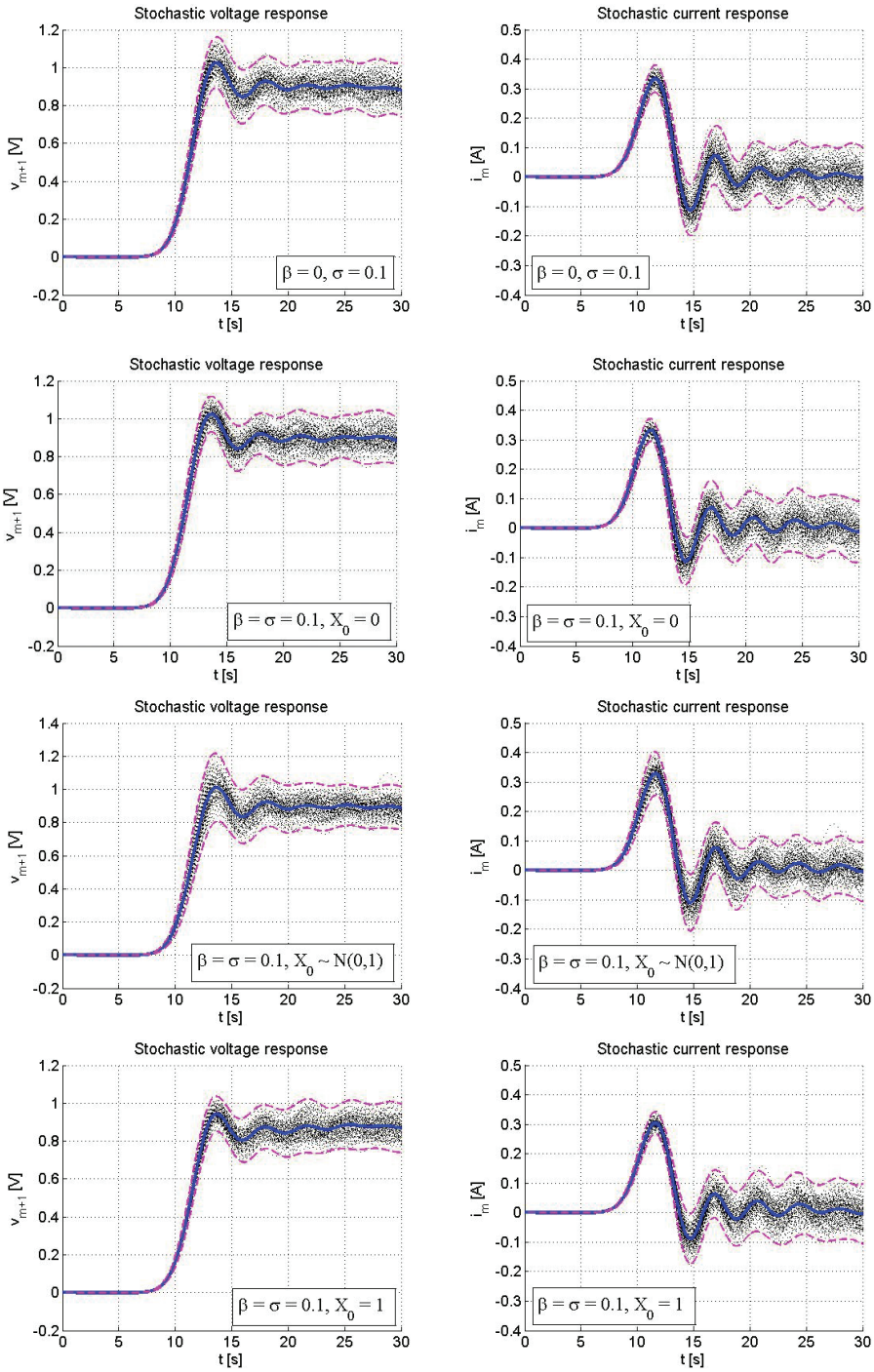

FIGURE 3. Stochastic voltage and current responses of the network.

distributed parameters (a transmission line). The ringings of mean values are given by the finite number of sections in the model and depend on concrete values of respective elements. Presented confidence intervals show clear practical usability of the model to cover stochastic variances in the systems with distributed parameters, while voltage excitation affected by a coloured noise is considered for illustration. For increasing t the confidence intervals are very similar, no matter, what type of the noise is considered. The effect of the coefficient β and the nonzero initial condition in equation (3) is significant near the expected delay of the system (in our case about 10 seconds), for large t the filtering effect of the system settles the confidence intervals. For sufficient long time, the expectations are approaching their steady-state values given by clearly resistive model in Fig. 2, while neglicting effects of reactive elements (all models's inductors and capacitors), namely they are roughly 0.896 V for the voltage responses and 8.96 mA for the current responses.

5. Conclusion

It is very natural to involve randomness into physical systems while solving problems in economics and finance, see for example [5]. In last 50 years, systems with random effects are considered in many fields as biology, chemistry, physics and engineering.

Stochastic differential equations approach in electrical engineering can cover numerous random processes arising in electrical circuits. The case of first-order (RL, RC) circuits is still being studied (e.g., [6] and [7]). Higher order white noise effected systems can be found in [8]. RLCG circuits with coloured noise effected parameters are studied in [9]. These are the basic building blocks of higher order circuits discussed in this paper.

In this paper we present a new, original approach to coloured noise effected high order electrical systems. We stated SDEs for such systems, numerically solved them and computed confidence intervals statistically from the obtained stochastic trajectories. In this way we can see complete stochastic responses in detail. However, it is also possible to find confidence intervals by more straightforward manner, see, e.g., [9] and [10] for the case of second-order circuits, when one is interested primarily in the expectations of solutions but not in individual stochastic trajectories. Generalization of the method from [9] towards higher-order systems will be done in near future. Besides, from practical point of view, it is also important to consider multiple excitation sources along the transmission line model which will enable to incorporate colour noises produced by electronic devices, like, e.g., varicaps often used for modelling nonlinear transmission lines.

Acknowledgements. This work was supported by Czech Science Foundation under the grant 15–18288S. For research, infrastructure of the SIX Center was used.

REFERENCES

[1] ARNOLD, L.: *Stochastic Differential Equations: Theory and Applications*. John Wiley & Sons, New York, 1974.

[2] ØKSENDAL, B.: *Stochastic Differential Equations, An Introduction with Applications*. Springer-Verlag, Berlin, 2000.

[3] BRANČÍK, L.: *Time and Laplace-domain methods for MTL transient and sensitivity analysis*, COMPEL—The International Journal for Computation and Mathematics in Electrical and Electronic Engineering **30**, 2011, no. 4, 1205–1223.

[4] KLOEDEN, P.—PLATEN, E.—SCHURZ, H.: *Numerical Solution of SDE Through Computer Experiments*. Springer-Verlag, Berlin, 1997.

[5] DZHALLADOVA, I.—MICHALKOVÁ, M.—RŮŽIČKOVÁ, M. : *Model of stabilizing of the interest rate on deposits banking system using by moment equatioons* Tatra Mt. Math. Publ. **54** (2013), 4–59.

[6] KOLÁŘOVÁ, E.: *Statistical estimates of stochastic solutions of RL electrical circuits*. In: International Conference on Industrial Technology ICIT '06, December 15–17, 2006, Mumbai, India. Proc. of the IEEE, 2007, pp. 2546–2550.

[7] RAHMAN, F.—PARISA, N.: *A stochastic perspective of RL electrical circuit using different noise terms*, COMPEL—The International Journal for Computation and Mathematics in Electrical and Electronic Engineering **30** (2011), no. 2, pp. 812–822.

[8] BRANČÍK, L.—KOLÁŘOVÁ, E.: *Simulation of higher-order electrical circuits with stochastic parameters via SDEs*, Advances in Electrical and Computer Engineering **13** (2013), no. 1, 17–22.

[9] KOLÁŘOVÁ, E.—BRANČÍK, L.: *Confidence intervals for RLCG cell influenced by coloured noise*, COMPEL—The International Journal for Computation and Mathematics in Electrical and Electronic Engineering **36** (2017), no. 4, 838–848.

[10] KOLÁŘOVÁ, E.: *Applications of second order stochastic integral equations to electrical networks*, Tatra Mt. Math. Publ. **63** (2015). 163–173.

Received December 11, 2017

*Department of Mathematics**
*Department of Radio Electronic***
Brno University of Technology
Technická 8
CZ-616 00 Brno
CZECH REPUBLIC

E-mail: kolara@feec.vutbr.cz*
brancik@feec.vutbr.cz**