sciendo

Tatra Mountains
Mathematical Publications
DOI: 10.2478/tmmp-2018-0015
Tatra Mt. Math. Publ. **71** (2018), 175–193

OSCILLATION CRITERIA FOR FORCED FIRST ORDER NONLINEAR NEUTRAL IMPULSIVE DIFFERENCE SYSTEM

Arun Kumar Tripathy — Gokula Nanda Chhatria

Sambalpur University, Sambalpur, INDIA

ABSTRACT. In this work, we have established sufficient conditions for oscillation and nonoscillation of a class of forced first order neutral impulsive difference equations with deviating arguments and fixed moments of impulsive effect.

1. Introduction

Consider a class of forced first order nonlinear neutral difference equations of the form:

$$(E_1)\quad \begin{cases} \Delta\big[y(n)+p(n)y(n-\tau)\big]+q(n)G\big(y(n-\sigma)\big)=f(n), \quad n\neq m_j,\ j\in\mathbb{N} \\ \underline{\Delta}\big[y(m_j-1)+p(m_j-1)y(m_j-\tau-1)\big] \\ \quad +r(m_j-1)G\big(y(m_j-\sigma-1)\big)=h(m_j-1), \end{cases}$$

where p, q, f are real valued functions with discrete arguments such that $q(n)>0$, $|p(n)|<\infty$ for $n\in\mathbb{N}(n_0)=\{n_0,n_0+1,\dots\}$, $G\in C(\mathbb{R},\mathbb{R})$ satisfying the properties $xG(x)>0$ for $x\neq 0$ and Δ is the forward difference operator defined by $\Delta u(n)=u(n+1)-u(n)$, and $m_1,m_2,m_3,\dots$ are the moments of impulsive effect satisfying the property

(A_0) $0<m_1<m_2<\cdots$, $\lim_{j\to\infty} m_j=+\infty$. For (E_1), $\underline{\Delta}$ is the difference operator defined by

$$\begin{aligned}&\underline{\Delta}\big[y(m_j-1)+p(m_j-1)y(m_j-\tau-1)\big]\\ &\quad = y(m_j)+p(m_j)y(m_j-\tau)-\big[y(m_j-1)+p(m_j-1)y(m_j-\tau-1)\big].\end{aligned}$$

2010 Mathematics Subject Classification: 39A10, 39A12.

Keywords: oscillation, nonoscillation, impulsive difference equation, nonlinear, delay.

This work is supported by Rajiv Gandhi National fellowship(UGC), New Delhi, India, through the Letter No. F1-17.1/2017-18/RGNF-2017-18-SC-ORI-35849, dated. 11th july, 2017.

In this work, our objective is to study the oscillatory behaviour of solutions of the system (E_1) when $|p(n)| < \infty$. About the impulsive difference equations we refer the monograph [2] by Lakshmikantham et al.

In [11], the authors Tripathy and Chhatria have discussed the oscillatory behaviour of solutions of homogeneous counterpart of (E_1)

$$(E_h)\ \begin{cases} \Delta\big[y(n)+p(n)y(n-\tau)\big]+q(n)G\big(y(n-\sigma)\big)=0, & n\neq m_j,\ j\in\mathbb{N} \\ \underline{\Delta}\big[y(m_j-1)+p(m_j-1)y(m_j-\tau-1)\big] & \\ \quad +r(m_j-1)G\big(y(m_j-\sigma-1)\big)=0. & \end{cases}$$

And it is learnt that the work based on the restriction upon G as either G is superlinear or sublinear. However, this work is free from any restrictions upon G. An attempt is made here to study (E_1) with a suitable choice on the forcing terms $f(n)$ and $h(m_j - 1)$.

In literature, the theory of impulsive differential/difference equations is richer than the corresponding theory of differential/difference equations without impulse. In addition, it provides a more adequate mathematical model for numerous process and phenomena studied in physics, biology, engineering and to mention a few. But, the development of the theory of impulsive difference equations especially of neutral type is comparatively very slow due to theoretical and technical difficulties caused by their peculiarities.

In [5], Li et al. have established the oscillation criteria for third order impulsive difference equations of the form:

$$(E_2)\ \begin{cases} \Delta^3 y(n)+p(n)y(n-\tau)=0, & n\neq n_k \\ y(n_k)=a_k y(n_k-1), & k\in\mathbb{N}, \\ \Delta y(n_k)=b_k\Delta y(n_k-1), & k\in\mathbb{N}, \\ \Delta^2 y(n_k)=c_k\Delta^2 y(n_k-1), & k\in\mathbb{N}, \end{cases}$$

and the work [6] is extended to nonlinear third order impulsive difference equations of the form:

$$(E_{2N})\ \begin{cases} \Delta^3 y(n)+p(n)f\big(y(n-\tau)\big)=0, & n\neq n_k \\ \Delta^i y(n_k)=g_{i,k}\Delta^i y(n_k-1), & i=0,1,2,\ k\in\mathbb{N}. \end{cases}$$

Unlike the methods for $(E_2)/(E_{2N})$, our impulsive effect satisfies another neutral equation corresponding to its difference equation. The present work for the impulsive difference system (E_1) is a different approach as compared to the existing works in the literature. We may note that in present years much effort has been given to the study of functional difference equations of neutral type. However, the impulsive difference equations of neutral type especially $(E_1)/(E_h)$ is not well studied. In this direction, we refer the reader to some works [3], [4], [7]–[13] and the references cited there in.

Definition 1.1. By a solution of (E_1) we mean a real valued function $y(n)$ defined on $\mathbb{N}(n_0-\rho)$ which satisfy (E_1) for $n \geq n_0$ with the initial conditions

$$y(r)=\phi(r), \qquad r=n_0-\rho,\dots,n_0,$$

where $\phi(r)$, $r=n_0-\rho,\dots,n_0$ are given and $\rho=\max\{\tau,\sigma\}$.

Definition 1.2. A nontrivial solution $y(n)$ of (E_1) is said to be nonoscillatory, if it is either eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory.

Definition 1.3. A solution $y(n)$ of (E_1) is said to be regular, if it is defined on $\mathbb{N}(0)$ and

$$\sup\bigl\{|y(n)|\colon n \geq N > 0\bigr\} > 0.$$

A regular solution $y(n)$ of (E_1) is said to be eventually positive (eventually negative), if there exists $n_0>0$ such that $y(n)>0$ $\bigl(y(n)<0\bigr)$ for $n \geq n_0$.

2. Oscillation criteria

In this section, we discuss the oscillatory behavior of solutions of the impulsive system (E_1). For our discussion, we need the following assumptions:

There exists a sequence of reals $F(n)$ such that

$$\Delta F(n)=f(n) \quad \text{and} \quad \underline{\Delta} F(m_j-1)=h(m_j-1)$$

satisfying the properties

(A_1) $F(n)$ changes sign such that
$-\infty < \liminf_{n\to\infty} F(n) < 0 < \limsup_{n\to\infty} F(n) < \infty$,

(A_2) $F(n)$ changes sign and $F^+(n)=\max\bigl\{F(n),0\bigr\}$, $F^-(n)=\max\bigl\{-F(n),0\bigr\}$.

Throughout our discussion we use the following notations:

$$\begin{cases} z(n)=y(n)+p(n)y(n-\tau), \\ z(m_j-1)=y(m_j-1)+p(m_j-1)y(m_j-\tau-1), \quad j\in\mathbb{N} \end{cases} \tag{2.1}$$

and

$$\begin{cases} w(n)=z(n)-F(n), \\ w(m_j-1)=z(m_j-1)-H(m_j-1), \qquad j\in\mathbb{N}, \end{cases} \tag{2.2}$$

where $m_j-1>0$ for $j\in\mathbb{N}$.

Theorem 2.1. *Let $-1 \leq p(n) \leq 0$. Assume that (A_0)–(A_2) and*

(A_3) $G(uv)=G(u)G(v)$, $G(-u)=-G(u)$, $u,v\in\mathbb{R}$

hold. If one of following conditions:

(A_4) $\sum_{n=n^*}^{\infty} q(n)G\big(F^+(n-\sigma)\big) + \sum_{j=1}^{\infty} r(m_j-1)G\big(F^+(m_j-\sigma-1)\big) = \infty$, $n^* > \sigma$,

(A_5) $\sum_{n=n^*}^{\infty} q(n)G\big(F^-(n-\sigma)\big) + \sum_{j=1}^{\infty} r(m_j-1)G\big(F^-(m_j-\sigma-1)\big) = \infty$, $n^* > \sigma$,

(A_6) $\sum_{n=n^*}^{\infty} q(n)G\big(F^+(n+\tau-\sigma)\big)+\sum_{j=1}^{\infty} r(m_j-1)G\big(F^+(m_j+\tau-\sigma-1)\big) = \infty$, $n^* > \sigma-\tau$

and

(A_7) $\sum_{n=n^*}^{\infty} q(n)G\big(F^-(n+\tau-\sigma)\big)+\sum_{j=1}^{\infty} r(m_j-1)G\big(F^-(m_j+\tau-\sigma-1)\big) = \infty$, $n^* > \sigma-\tau$

hold, then every regular solution of (E_1) *oscillates.*

P r o o f. On the contrary, let $y(n)$ be a regular solution of (E_1) such that

$$y(n) > 0, \quad y(n-\tau) > 0, \quad y(n-\sigma) > 0 \quad \text{for } n \ge n_1 = n_0 + \rho.$$

Using (2.1) and (2.2) in (E_1), we obtain

$$\Delta w(n) = -q(n)G\big(y(n-\sigma)\big) \le 0, \qquad n \ne m_j \tag{2.3}$$

$$\underline{\Delta} w(m_j-1) = -r(m_j-1)G\big(y(m_j-\sigma-1)\big) \le 0, \qquad j \in \mathbb{N}. \tag{2.4}$$

Now, $w(n)$ is monotonic decreasing for $n \ge n_2 > n_1+\sigma$. Due to (A_0), $w(m_j-1)$ is monotonic decreasing when we assume that $m_j > n_2 > n_1+\sigma+1$, $j \in \mathbb{N}$. Hence, $w(n)$ is monotonic decreasing for all $n \ge n_2$. Therefore, $w(n) > 0$ or < 0 for $n \ge n_2$. If $w(n) > 0$ for $n \ge n_2$, then $y(n) > -p(n)y(n-\tau) + F(n)$ implies that $y(n) > F(n)$ and hence $y(n) > F^+(n)$ so also $y(m_j-1) > F^+(m_j-1)$. Ultimately, (E_1) reduces to

$$(E_3) \quad \begin{cases} \Delta w(n) + q(n)G\big(F^+(n-\sigma)\big) \le 0, & n \ne m_j, \\ \underline{\Delta} w(m_j-1) + r(m_j-1)G\big(F^+(m_j-\sigma-1)\big) \le 0, & j \in \mathbb{N} \end{cases}$$

for $n \ge n_3 > n_2$. Summing (E_3) from n_3 to $n-1$, we obtain

$$w(n) - w(n_3) - \sum_{n_3 \le m_j-1 \le n-1} \underline{\Delta} w(m_j-1) + \sum_{s=n_3}^{n-1} q(s)G(F^+\big(s-\sigma)\big) \le 0,$$

that is,

$$w(n) - w(n_3) + \sum_{s=n_3}^{n-1} q(s)G\big(F^+(s-\sigma)\big) + \sum_{n_3 \le m_j-1 \le n-1} r(m_j-1)G\big(F^+(m_j-\sigma-1)\big) \le 0.$$

Consequently,

$$\sum_{s=n_3}^{n-1} q(s)G\big(F^+(s-\sigma)\big) + \sum_{n_3 \le m_j-1 \le n-1} r(m_j-1)G\big(F^+(m_j-\sigma-1)\big) \le w(n_3)-w(n)$$

implies that

$$\sum_{s=n_3}^{n-1} q(s)G\big(F^+(s-\sigma)\big) + \sum_{n_3\le m_j-1\le n-1} r(m_j-1)G\big(F^+(m_j-\sigma-1)\big) < \infty, \text{ as } n\to\infty,$$

a contradiction to (A_4). Hence, $w(n)<0$ for $n\ge n_1$, that is, $y(n)+p(n)y(n-\tau)-F(n)<0$ implies that $-p(n+\tau)y(n)>y(n+\tau)-F(n+\tau)$. Consequently, $y(n)\ge -p(n+\tau)y(n)>y(n+\tau)-F(n+\tau)\ge -F(n+\tau)$ for $n\ge n_1$. Therefore, $y(n)>\max\{0,-F(n+\tau)\}$ and hence $y(n)>F^-(n+\tau)$ so also $y(m_j-1)>F^-(m_j+\tau-1)$ for $n\ge n_2>n_1+1$. Let $\lim_{n\to\infty} w(n)=l, -\infty\le l<0$. Suppose that $l=-\infty$. Then there exist $n\ge n_3>n_2$ and $\beta>\alpha>0$ such that $F(n)\le\alpha+\epsilon$, $0<\epsilon<\beta-\alpha$ and $w(n)\le-\beta$, that is,

$$y(n)\le -p(n)y(n-\tau)-\beta+F(n)\le y(n-\tau)+\alpha+\epsilon-\beta$$

for $n\ge n_3$. Proceeding inductively we get $y(n)\le y(n_3)+\frac{(n-n_3)}{\tau}(\alpha+\epsilon-\beta)$, that is, $y(n)<0$ as $n\to\infty$, a contradiction. Hence, $\lim_{n\to\infty} w(n)$ exists. Also, it is true that $\lim_{j\to\infty} w(m_j-1)$ exist due to the non-impulsive points $m_j-1, j\in\mathbb{N}$. Using the above argument, (E_1) can be viewed as

$$\begin{aligned} &\Delta w(n)+q(n)G(F^-\big(n+\tau-\sigma)\big)\le 0, && n\ne m_j\\ &\underline{\Delta} w(m_j-1)+r(m_j-1)G\big(F^-(m_j+\tau-\sigma-1)\big)\le 0, && j\in\mathbb{N}\end{aligned}$$

for $n\ge n_4>n_3$. Summing the above impulsive system from n_4 to $n-1$, we obtain

$$w(n)-w(n_4) - \sum_{n_4\le m_j-1\le n-1}\underline{\Delta} w(m_j-1)+\sum_{s=n_4}^{n-1} q(s)G\big(F^-(s+\tau-\sigma)\big)\le 0,$$

that is,

$$\begin{aligned} w(n)-w(n_4) &+\sum_{s=n_4}^{n-1} q(s)G\big(F^-(s+\tau-\sigma)\big)\\ &+\sum_{n_4\le m_j-1\le n-1} r(m_j-1)G\big(F^-(m_j+\tau-\sigma-1)\big)\le 0.\end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{s=n_4}^{n-1} q(s)G\big(F^-(s+\tau-\sigma)\big)\\ &+\sum_{n_4\le m_j-1\le n-1} r(m_j-1)G\big(F^-(m_j+\tau-\sigma-1)\big)\le w(n_4)-w(n)<\infty\\ &\hspace{20em}\text{as } n\to\infty,\end{aligned}$$

a contradiction to (A_7).

If $y(n)<0$ for $n\geq n_0$, then we put $x(n)=-y(n)$ in (E_1) and because of (A_3), (E_1) becomes

$$(E_4)\begin{cases}\Delta\big[x(n)+p(n)x(n-\tau)\big]+q(n)G\big(x(n-\sigma)\big)=\hat{f}(n), \quad n\neq m_j,\ j\in\mathbb{N},\\ \underline{\Delta}\big[x(m_j-1)+p(m_j-1)x(m_j-\tau-1)\big]\\ \quad +r(m_j-1)G\big(x(m_j-\sigma-1)\big)=\hat{h}(m_j-1),\end{cases}$$

where $\hat{f}(n)=-f(n)$, $\hat{h}(m_j-1)=-h(m_j-1)$. If we choose $-F(n)=\hat{F}(n)$ and $-H(m_j-1)=\hat{H}(m_j-1)$, then it follows that $-\infty<\liminf_{n\to\infty}\hat{F}(n)<0<\limsup_{n\to\infty}\hat{F}(n)<\infty$. Proceeding as above for (E_4), we obtain contradictions to (A_5) and (A_6). This completes the proof of the theorem. □

Theorem 2.2. *Let $0\leq p(n)\leq p_4<\infty$. Assume that (A_0)–(A_3) and (A_8) there exists $\lambda>0$ such that $G(u)+G(v)\geq\lambda G(u+v)$, $u,v\in\mathbb{R}_+$ hold. Furthermore, assume that one of following conditions:*

(A_9) $\sum_{n=n^*}^{\infty}Q(n)G\big(F^+(n-\sigma)\big)+\sum_{j=1}^{\infty}R(m_j-1)G\big(F^+(m_j-\sigma-1)\big)=\infty$, $n^*>\sigma$

and

(A_{10}) $\sum_{n=n^*}^{\infty}Q(n)G\big(F^-(n-\sigma)\big)+\sum_{j=1}^{\infty}R(m_j-1)G\big(F^-(m_j-\sigma-1)\big)=\infty$, $n^*>\sigma$

hold, where $Q(n)=\min\{q(n),q(n-\tau)\}$, $R(m_j-1)=\min\big\{r(m_j-1), r(m_j-\tau-1)\big\}$, $n\geq\tau$, $m_j\geq\tau+1$, $j\in\mathbb{N}$. Then every regular solution of (E_1) is oscillatory.

P r o o f. Let $y(n)$ be a regular nonoscillatory solution of (E_1). Proceeding as in Theorem 2.1, it follows that $w(n)$ is monotonic decreasing for $n\geq n_1$. Therefore, either $w(n)<0$ or $w(n)>0$ for $n\geq n_2>n_1$. Suppose that $w(n)<0$ for $n\geq n_2$. Then $z(n)-F(n)<0$ implies that $0<z(n)<F(n)$ for $n\geq n_2$, which contradicts (A_1). Ultimately, $w(n)>0$ for $n\geq n_2$ and hence $z(n)>\max\{0,F(n)\}=F^+(n)$ for $n\geq n_2$. Clearly, $z(m_j-1)>H^+(m_j-1)$ for $m_j\geq n_2+1$, $j\in\mathbb{N}$. Using (E_1) we have

$$\Delta w(n)+q(n)G\big(y(n-\sigma)\big) + G(p_4)\Big[\Delta w(n-\tau)+q(n-\tau)G\big(y(n-\tau-\sigma)\big)\Big]=0,$$

$$\underline{\Delta}w(m_j-1)+q(m_j-1)G\big(y(m_j-\sigma-1)\big) + G(p_4)\Big[\underline{\Delta}w(m_j-\tau-1) + q(m_j-\tau-1)G\big(y(m_j-\tau-\sigma-1)\big)\Big]=0,$$

that is,

$$\Delta\big[w(n)+G(p_4)w(n-\tau)\big]+Q(n)\Big[G\big(y(n-\sigma)\big)+G\big(y(n-\tau-\sigma)p_4\big)\Big]\leq 0,$$

$$\underline{\Delta} w(m_j-1)+G(p_4)\underline{\Delta} w(m_j-\tau-1) \\ +R(m_j-1)\Big[G\big(y(m_j-\sigma-1)\big)+G\big(y(m_j-\tau-\sigma-1)p_4\big)\Big]\leq 0.$$

Upon using (A_8) in the preceding inequalities, we obtain

$$(E_5)\ \begin{cases}\Delta\big[w(n)+G(p_4)w(n-\tau)\big]+\lambda Q(n)G\big(z(n-\sigma)\big)\leq 0,\\ \underline{\Delta}\big[w(m_j-1)+G(p_4)w(m_j-\tau-1)\big]\\ \quad +\lambda R(m_j-1)G\big(z(m_j-\sigma-1)\big)\leq 0\end{cases}$$

due to $z(n-\sigma)\leq y(n-\sigma)+p_4y(n-\tau-\sigma)$ for $n\geq n_3>n_2+\sigma$, and $z(m_j-\sigma-1)\leq y(m_j-\sigma-1)+p_4y(m_j-\tau-\sigma-1)$ for $m_j\geq n_3+1, j\in\mathbb{N}$. Summing the system (E_5) from n_3 to $n-1$, we get

$$w(n)-w(n_3)+G(p_4)w(n-\tau)-G(p_4)w(n_3-\tau) \\ -\sum_{n_3\leq m_j-1\leq n-1}\underline{\Delta}\big(w(m_j-1)+G(p_4)w(m_j-\tau-1)\big) \\ +\sum_{s=n_3}^{n-1}\lambda Q(s)G\big(z(s-\sigma)\big)\leq 0,$$

that is,

$$w(n)-w(n_3)+G(p_4)w(n-\tau)-G(p_4)w(n_3-\tau) \\ +\lambda\sum_{n_3\leq m_j-1\leq n-1}R(m_j-1)G\big(z(m_j-\sigma-1)\big) \\ +\lambda\sum_{s=n_3}^{n-1}Q(s)G\big(z(s-\sigma)\big)\leq 0.$$

As a result,

$$\lambda\left[\sum_{s=n_3}^{n-1}Q(s)G\big(z(s-\sigma)\big)\ +\sum_{n_3\leq m_j-1\leq n-1}R(m_j-1)G\big(z(m_j-\sigma-1)\big)\right] \\ \leq w(n_3)-w(n)+G(p_4)w(n_3-\tau)-G(p_4)w(n-\tau)$$

implies that

$$\lambda\left[\sum_{s=n_3}^{n-1}Q(s)G\big(F^+(n-\sigma)\big)+\sum_{n_3\leq m_j-1\leq n-1}R(m_j-1)G\big(F^+(m_j-\sigma-1)\big)\right] \\ \leq w(n_3)-w(n)+G(p_4)w(n_3-\tau)-G(p_4)w(n-\tau).$$

Because $\lim_{n\to\infty} w(n)$ exists, the last inequality becomes

$$\lambda\left[\sum_{s=n_3}^{n-1} Q(s)G\bigl(F^+(n-\sigma)\bigr) + \sum_{n_3\le m_j-1\le n-1} R(m_j-1)G\bigl(F^+(m_j-\sigma-1)\bigr)\right] < \infty$$

as $n\to\infty$, a contradiction to (A_9).

The case $y(n) < 0$ is similar. Hence, the theorem is proved. □

Theorem 2.3. *Let $-\infty < -p_6 \le p(n) \le -1$, $p_6 > 0$. If all conditions of Theorem 2.1 are satisfied, then every bounded regular solution of (E_1) oscillates.*

Proof. On contrary, let $y(n)$ be a regular nonoscillatory solution of (E_1). Proceeding as in the proof of Theorem 2.1, we get that $\Delta w(n)$ is decreasing for $n \ge n_1 > n_0+\sigma$. Therefore, either $w(n) > 0$ or $w(n) < 0$ for $n \ge n_1$. Suppose that $w(n) > 0$. Then $y(n) > -p(n)y(n-\tau) + F(n)$. Consequently, $y(n) > F^+(n-\sigma)$ for $n \ge n_2 > n_1$. Hence, we obtain a contradiction as in the proof of Theorem 2.1. Thus, $w(n) < 0$ for $n \ge n_1$. Since $y(n)$ is bounded, then $\lim_{n\to\infty} w(n)$ exists and hence $\lim_{j\to\infty} w(m_j-1)$ exists due to nonimpulsive points m_j-1, $m_j-2,\dots$ and so on. Clearly, $w(n) < 0$ implies that $-p(n+\tau)y(n) > y(n+\tau) - F(n+\tau)$, that is, $y(n)p_6 \ge -p(n+\tau)y(n) > y(n+\tau) - F(n+\tau) \ge -F(n+\tau)$ for $n \ge n_1$. Therefore, $y(n) > \max\{0, \frac{-F(n+\tau)}{p_6}\}$ implies that $y(n-\sigma) > \frac{F^-(n+\tau-\sigma)}{p_6}$ for $n \ge n_2 > n_1$. Therefore, (E_1) reduces to

$$\begin{aligned}
&\Delta w(n) + G(p_6^{-1})q(n)G\bigl(F^-(n+\tau-\sigma)\bigr) \le 0, && n \ne m_j\\
&\underline{\Delta} w(m_j-1) + G(p_6^{-1})r(m_j-1)G\bigl(F^-(m_j+\tau-\sigma-1)\bigr) \le 0, && j\in\mathbb{N}.
\end{aligned}$$

Summing the above impulsive system from n_2 to $n-1$, we get

$$G(p_6^{-1})\left[\sum_{s=n_2}^{n-1} q(s)G\bigl(F^-(s+\tau-\sigma)\bigr) + \sum_{n_2\le m_j-1\le n-1} r(m_j-1)G\bigl(F^-(m_j+\tau-\sigma-1)\bigr)\right] < \infty$$

as $n\to\infty$, a contradiction to (A_7). The rest of the proof follows from the proof of Theorem 2.1. This completes the proof of the theorem. □

3. Nonoscillation criteria

In this section, we discuss the nonoscillation properties of solutions of the impulsive system (E_1). We assume the following hypothesis for use in the sequel:

(A_{11}) $\lim_{n\to\infty} F(n) = M$, $|M| < \infty$.

Theorem 3.1. *Let* $-1 < p_1 \le p(n) \le 0$. *Assume that* (A_{11}) *and*
(A_{12}) $\sum_{n=N}^{\infty} q(n) + \sum_{j=1}^{\infty} r(m_j - 1) < \infty$, $N > 0$
hold. Then (E_1) *admits a positive bounded nonoscillatory solution.*

P r o o f. Let $X = l_{\infty}^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \ge n_0$ with sup norm defined by

$$\|y\| = \sup\{|y(n)| \colon n \ge n_0\}.$$

Let $K = \{y(n) \in X \colon y(n) \ge 0 \; for \; n \ge n_0\}$. For $y_1, y_2 \in X$, we define $y_1 \le y_2$ if and only if $y_2 - y_1 \in K$. Thus, X is a partially ordered Banach space. Set

$$S = \left\{ y \in X \colon \frac{1+p_1}{12} \le y(n) \le 1, \, n \ge n_0 \right\}.$$

Let $x_0(n) = \frac{1+p_1}{12}$ for $n \ge n_0$. Then $x_0(n) \in S$ and $x_0(n) = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \left\{ y \in X \colon l_1 \le y(n) \le l_2, \frac{1+p_1}{12} \le l_1, l_2 \le 1, \, n \ge n_0 \right\}.$$

Let

$$x_1(n) = l_2' = \sup \left\{ l_2 \colon \frac{1+p_1}{12} \le l_2 \le 1 \right\}.$$

Then $x_1(n) \in S$ and $x_1(n) = \sup S^*$. From (A_{11}) and (A_{12}), it is possible to choose $n_1 > n_0$ such that

$$\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \le m_j - 1 \le n} r(m_j - 1) < \frac{1+p_1}{6G(1)}, \qquad n \ge n_1 \tag{3.1}$$

and

$$|F(n) - M| < \frac{1+p_1}{12}, \qquad n \ge n_1. \tag{3.2}$$

Since S is a closed subset of X, S is a complete metric space. For $y \in S$ define a map

$$Ty(n) = \begin{cases} Ty(n_1 + \rho), & n_1 \le n \le n_1 + \rho, \\ \frac{1+p_1}{6} - p(n)y(n-\tau) + \sum_{s=n}^{\infty} q(s) G\big(y(s-\sigma)\big) & \\ \quad + \sum_{n_1 \le m_j - 1 \le n} r(m_j - 1) G\big(y(m_j - \sigma - 1)\big) & \\ \quad + (F(n) - M), & n \ge n_1 + \rho. \end{cases}$$

For $y \in S$ and using (3.1), (3.2), we have

$$\begin{aligned} Ty(n) &\le \frac{1+p_1}{6} - p(n)y(n-\tau) \\ &\quad + G(1)\left[\sum_{s=n}^{\infty} q(s) \ + \sum_{n_1 \le m_j - 1 \le n} r(m_j - 1)\right] + (F(n) - M) \\ &\le \frac{1+p_1}{6} - p_1 + \frac{1+p_1}{6} + \frac{1+p_1}{12} \\ &= \frac{5-7p_1}{12} < 1 \end{aligned}$$

and

$$Ty(n) \ge \frac{1+p_1}{6} - (F(n) - M) \ge \frac{1+p_1}{6} - \frac{1+p_1}{12} = \frac{1+p_1}{12}$$

implies that $Ty(n) \in S$ for every $n \ge n_1$. Let $y_1, y_2 \in S$ such that $y_1 \le y_2$. It is easy to verify that $Ty_1 \le Ty_2$. Hence, by Knaster-Tarski fixed point theorem [1], T has a unique $y \in S$ such that $Ty = y$. Therefore,

$$y(n) = \begin{cases} y(n_1+\rho), & n_1 \le n \le n_1 + \rho, \\ \frac{1+p_1}{6} - p(n)y(n-\tau) + \sum_{s=n}^{\infty} q(s) G\big(y(s-\sigma)\big) & \\ \quad + \sum_{j=1}^{\infty} r(m_j - 1) G\big(y(m_j - \sigma - 1)\big) & \\ \quad +(F(n) - M), & n \ge n_1 + \rho \end{cases}$$

and it is easy to see that $y(n)$ is a nonoscillatory solution of (E_1). This completes the proof of the theorem. □

Theorem 3.2. *Let $0 \le p(n) \le p_2 < 1$. Assume that (A_{11}) and (A_{12}) hold. Let G be Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$, then (E_1) has a bounded nonoscillatory solution.*

Proof. Let $X = l_\infty^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \ge n_0$ with sup norm defined by

$$\|y\| = \sup\{|y(n)| \colon n \ge n_0\}.$$

Set

$$S = \left\{ y \in X \colon \frac{1-p_2}{6} \le y(n) \le 1,\, n \ge n_0 \right\}.$$

From (A_{12}), it is possible to choose $n_1 > n_0$ such that

$$\sum_{s=n}^{\infty} q(s) \ + \sum_{n_1 \le m_j - 1 \le n} r(m_j - 1) < \frac{1-p_2}{3L}, \qquad n \ge n_1, \tag{3.3}$$

where $L = \max\{L_1, G(1)\}$, L_1 is the Lipschitz constant of G on $\left[\frac{1-p_2}{6}, 1\right]$. Also, from (A_{11}) it is possible to choose $n_2 > n_1$ such that

$$|F(n) - M| < \frac{1-p_2}{6}, \qquad n \ge n_2. \tag{3.4}$$

Since S is a closed subset of X, S is a complete metric space. For $y \in S$ define a map

$$Ty(n) = \begin{cases} Ty(n_2+\rho), & n_2 \le n \le n_2+\rho, \\ \frac{1+2p_2}{6} - p(n)y(n-\tau) + \sum_{s=n}^{\infty} q(s)G\big(y(s-\sigma)\big) \\ \quad + \sum_{n_2 \le m_j - 1 \le n} r(m_j-1)G\big(y(m_j-\sigma-1)\big) \\ \quad + \big(F(n) - M\big), & n \ge n_2+\rho. \end{cases}$$

Using (3.3) and (3.4) and for $y \in S$, we have

$$\begin{aligned} Ty(n) &\le \frac{1+2p_2}{6} + G(1)\left[\sum_{s=n}^{\infty} q(s) + \sum_{n_2 \le m_j-1 \le n} r(m_j-1)\right] + (F(n) - M) \\ &\le \frac{1+2p_2}{6} + \frac{1-p_2}{3} + \frac{1-p_2}{6} \\ &= \frac{5+p_2}{6} < 1 \end{aligned}$$

and

$$\begin{aligned} Ty(n) &\ge \frac{1+2p_2}{6} - p(n)y(n-\tau) - \big(F(n) - M\big) \\ &\ge \frac{1+2p_2}{6} - p_2 - \frac{1-p_2}{6} \\ &= \frac{1-p_2}{6} \end{aligned}$$

implies that $Ty(n) \in S$ for every $n \ge n_2$. For $y_1, y_2 \in S$, we have

$$\begin{aligned} &|Ty_1(n) - Ty_2(n)| \\ &\le |p(n)|\,|y_1(n-\tau) - y_2(n-\tau)| \\ &\quad + L_1 \sum_{s=n}^{\infty} q(s)|y_1(s-\sigma) - y_2(s-\sigma)| \\ &\quad + L_1 \sum_{n_2 \le m_j-1 \le n} r(m_j-1)|y_1(m_j-\sigma-1) - y_2(m_j-\sigma-1)|, \end{aligned}$$

that is,

$$|Ty_1(n) - Ty_2(n)| \leq p_2\|y_1 - y_2\| + L_1\|y_1 - y_2\| \left[\sum_{s=n}^{\infty} q(s) \ + \sum_{n_2 \leq m_j - 1 \leq n} r(m_j - 1)\right]$$
$$\leq \left[p_2 + \frac{1 - p_2}{3}\right] \|y_1 - y_2\|$$

implies that

$$\|Ty_1 - Ty_2\| \leq \lambda\|y_1 - y_2\|.$$

Therefore, T is a contraction with $\lambda = \frac{1+2p_2}{2} < 1$. Since S is complete and T is a contraction on S, then by Banach's fixed point theorem, T has a unique solution $y \in S$ such that $Ty = y$. Clearly, $y(n)$ is a nonoscillatory solution of (E_1). This completes the proof of the theorem. □

Theorem 3.3. *Let* $1 < p_3 \leq p(n) \leq p_4 < \infty, p_3^2 > p_4$. *Assume that* (A_{11}), (A_{12}) *hold. Let* G *be Lipschitzian on the intervals of the form* $[a, b]$, $0 < a < b < \infty$, *then* (E_1) *admits a bounded nonoscillatory solution.*

Proof. Let $X = l_\infty^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \geq n_0$ with sup norm defined by

$$\|y\| = \sup\{|y(n)| \colon n \geq n_0\}.$$

Let's choose

$$\alpha = \frac{2\gamma(p_3^2 - p_4) - p_4(p_3^2 + p_3 - 2)}{2p_3^2 p_4}, \quad \beta = \frac{p_3 + \gamma - 1}{p_3},$$

for

$$\gamma = \frac{p_4(p_3^2 + p_4 - 2)}{2(p_3^2 - p_4)} > 0.$$

Set

$$S = \{y \in X \colon \alpha \leq y(n) \leq \beta,\ n \geq n_0\}.$$

From (A_{12}), it is possible to choose $n_1 > n_0$ such that

$$\sum_{s=n}^{\infty} q(s) \ + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1) < \frac{p_3 - 1}{2L}, \qquad n \geq n_1, \tag{3.5}$$

where $L = \max\{L_1, G(\beta)\}$, L_1 is the Lipschitz constant of F on $[\alpha, \beta]$. Also, from (A_{11}) it is possible to choose $n_2 > n_1$ such that

$$|F(n) - M| < \frac{p_3 - 1}{2}, \qquad n \geq n_2. \tag{3.6}$$

Since S is a closed subset of X, then S is a complete metric space.

For $y \in S$ define a map

$$Ty(n) = \begin{cases} Ty(n_2+\rho), & n_2 \le n \le n_2+\rho, \\ \frac{\gamma}{p(n+\tau)} - \frac{y(n+\tau)}{p(n+\tau)} + \frac{1}{p(n+\tau)}\Big[\sum_{s=n+\tau}^{\infty} q(s)G\big(y(s-\sigma)\big) & \\ \quad + \sum_{n_2 \le m_j-1 \le n+\tau} r(m_j-1)G\big(y(m_j-\sigma-1)\big)\Big] & \\ \quad + \frac{1}{p(n+\tau)}(F(n)-M), & n \ge n_2+\rho. \end{cases}$$

For $y \in S$ and using (3.5), (3.6) we have

$$\begin{aligned} Ty(n) &\le \frac{\gamma}{p(n+\tau)} \\ &\quad + \frac{G(\beta)}{p(n+\tau)}\left[\sum_{s=n+\tau}^{\infty} q(s) + \sum_{n_2 \le m_j-1 \le n+\tau} r(m_j-1)\right] \\ &\quad + \frac{1}{p(n+\tau)}\big(F(n)-M\big) \\ &\le \frac{1}{p_3}\left[\gamma + \frac{p_3-1}{2} + \frac{p_3-1}{2}\right] \\ &= \frac{\gamma+p_3-1}{p_3} = \beta \end{aligned}$$

and

$$\begin{aligned} Ty(n) &\ge \frac{\gamma}{p(n+\tau)} - \frac{y(n+\tau)}{p(n+\tau)} - \frac{1}{p(n+\tau)}\big(F(n)-M\big) \\ &\ge \frac{-1}{p_3}\big[y(n+\tau) + (F(n)-M)\big] + \frac{\gamma}{p_4} \\ &\ge \frac{-1}{p_3}\left[\beta + \frac{p_3-1}{2}\right] + \frac{\gamma}{p_4} = \frac{2\gamma p_3 - p_4(p_3+2\beta-1)}{2p_3p_4} = \alpha \end{aligned}$$

implies that $Ty(n) \in S$ for every $n \ge n_2$. For $y_1, y_2 \in S$, we have

$$\begin{aligned} &|Ty_1(n) - Ty_2(n)| \\ &\le \frac{1}{|P(n+\tau)|}\,|y_1(n+\tau) - y_2(n+\tau)| \\ &\quad + \frac{L_1}{|P(n+\tau)|}\sum_{s=n+\tau}^{\infty} q(s)|y_1(s-\sigma) - y_2(s-\sigma)| \\ &\quad + \frac{L_1}{|P(n+\tau)|}\sum_{n_2 \le m_j-1 \le n+\tau} r(m_j-1)|y_1(m_j-\sigma-1) - y_2(m_j-\sigma-1)|, \end{aligned}$$

that is,

$$\begin{aligned}
&|Ty_1(n) - Ty_2(n)| \\
&\le \frac{1}{p_3}\|y_1 - y_2\| + \frac{L_1}{p_3}\|y_1 - y_2\| \Bigg[\sum_{s=n+\tau}^{\infty} q(s) \ + \sum_{n_2 \le m_j - 1 \le n+\tau} r(m_j - 1) \Bigg] \\
&\le \frac{1}{p_3}\|y_1 - y_2\| + \frac{L_1}{p_3} M \|y_1 - y_2\| \\
&\le \frac{1}{p_3}\left[1 + \frac{p_3 - 1}{2}\right] \|y_1 - y_2\|
\end{aligned}$$

implies that

$$\|Ty_1 - Ty_2\| \le \lambda \|y_1 - y_2\|.$$

Therefore, T is a contraction with $\lambda = \frac{1+p_3}{2p_3} < 1$. Hence, by Banach's fixed point theorem, T has a unique solution $y \in S$ such that $Ty = y$. Indeed, $y(n)$ is a nonoscillatory solution of (E_1). This completes the proof of the theorem. □

Theorem 3.4. *Let $-\infty < p_6 \le p(n) \le p_5 < -1$. Assume that (A_{11}) and (A_{12}) hold. Let G be Lipschitzian on the intervals of the form $[a, b]$, $0 < a < b < \infty$, then (E_1) admits a bounded nonoscillatory solution.*

Proof. Let $X = l_\infty^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \ge n_0$ with sup norm defined by

$$\|y\| = \sup\bigl\{|y(n)| : n \ge n_0\bigr\}.$$

Choose

$$\mu = \frac{-p_5}{\xi - p_5}, \qquad \nu = \frac{2\xi - p_5(\xi + 1)}{(p_5 - \xi)(p_5 + 1)},$$

for

$$\xi > \max\left\{-p_6, p_5 + \frac{p_5}{1 + p_5}\right\}.$$

Set

$$S = \bigl\{y \in X : \mu \le y(n) \le \nu, n \ge n_0\bigr\}.$$

From (A_{12}), it is possible to choose $n_1 > n_0$ such that

$$\sum_{s=n}^{\infty} q(s) \ + \sum_{n_1 \le m_j - 1 \le n} r(m_j - 1) < \frac{-p_5}{(\xi - p_5)L}, \qquad n \ge n_1, \tag{3.7}$$

where $L = \max\{L_1, G(\nu)\}$, L_1 is the Lipschitz constant of G on $[\mu, \nu]$. Also, from (A_{11}) it is possible to choose $n_2 > n_1$ such that

$$|F(n) - M| < \frac{-p_5}{(\xi - p_5)}, \qquad n \ge n_2. \tag{3.8}$$

Since S is a closed subset of X, S is a complete metric space. For $y \in S$ define a map

$$Ty(n) = \begin{cases} Ty(n_2+\rho), & n_2 \le n \le n_2+\rho, \\ -\frac{\xi f(2-p_5)}{p(n+\tau)(\xi-p_5)} - \frac{y(n+\tau)}{p(n+\tau)} + \frac{1}{p(n+\tau)}\big[\sum_{s=n+\tau}^{\infty} q(s)G(y(s-\sigma)) \\ \quad + \sum_{n_2 \le m_j-1 \le n+\tau} r(m_j-1)G(y(m_j-\sigma-1))\big] \\ \quad + \frac{1}{p(n+\tau)}(F(n)-M), & n \ge n_2+\rho. \end{cases}$$

For $y \in S$ and using (3.7), (3.8), we have

$$\begin{aligned} Ty(n) &\le -\frac{\xi(2-p_5)}{p(n+\tau)(\xi-p_5)} - \frac{y(n+\tau)}{p(n+\tau)} + \frac{1}{p(n+\tau)}\big(F(n)-M\big) \\ &\le -\frac{\xi(2-p_5)}{p_5(\xi-p_5)} - \frac{\nu}{p_5} + \frac{p_5}{p_5(\xi-p_5)} \\ &= \frac{-\nu(\xi-p_5) - 2\xi + \xi p_5 + p_5}{p_5(\xi-p_5)} \le \nu \end{aligned}$$

and

$$\begin{aligned} Ty(n) \ge &-\frac{\xi(2-p_5)}{p(n+\tau)(\xi-p_5)} + \frac{G(\nu)}{p(n+\tau)}\left[\sum_{s=n+\tau}^{\infty} q(s) + \sum_{n_2 \le m_j-1 \le n+\tau} r(m_j-1)\right] \\ &- \frac{1}{p(n+\tau)}\big(F(n)-M\big), \end{aligned}$$

that is,

$$\begin{aligned} Ty(n) &\ge -\frac{\xi(2-p_5)}{p_6(\xi-p_5)} - \frac{p_5}{p_5(\xi-p_5)} - \frac{p_5}{p_5(\xi-p_5)} \\ &\ge -\frac{\xi(2-p_5)}{p_6(\xi-p_5)} - \frac{2}{(\xi-p_5)} \ge \mu \end{aligned}$$

implies that $Ty(n) \in S$ for every $n \ge n_2$. For $y_1, y_2 \in S$, we have

$$\begin{aligned} &|Ty_1(n) - Ty_2(n)| \\ &\le -\frac{1}{|p(n+\tau)|}\, |y_1(n+\tau) - y_2(n+\tau)| \\ &\quad - \frac{L_1}{|p(n+\tau)|} \sum_{s=n+\tau}^{\infty} q(s)|y_1(s-\sigma) - y_2(s-\sigma)| \\ &\quad - \frac{L_1}{|p(n+\tau)|} \sum_{n_2 \le m_j-1 \le n+\tau} r(m_j-1)|y_1(m_j-\sigma-1) - y_2(m_j-\sigma-1)|, \end{aligned}$$

that is,

$$
\begin{aligned}
&|Ty_1(n) - Ty_2(n)| \\
&\leq -\frac{1}{p_5}\|y_1 - y_2\| - \frac{L_1}{p_5}\|y_1 - y_2\| \left[\sum_{s=n+\tau}^{\infty} q(s) \ + \sum_{n_2 \leq m_j - 1 \leq n+\tau} r(m_j - 1)\right] \\
&\leq -\frac{1}{p_5}\|y_1 - y_2\| - \frac{1}{p_5}\frac{-p_5}{(\xi - p_5)}\|y_1 - y_2\| \\
&\leq \left(-\frac{1}{p_5} + \frac{1}{\xi - p_5}\right)\|y_1 - y_2\|
\end{aligned}
$$

implies that

$$\|Ty_1 - Ty_2\| \leq \lambda \|y_1 - y_2\|.$$

Therefore, T is a contraction with $\lambda = -\frac{1}{p_5} + \frac{1}{\xi - p_5} < 1$. Hence, by Banach's fixed point theorem, T has a unique solution $y \in S$ such that $Ty = y$. It is easy to see that $y(n)$ is a nonoscillatory solution of (E_1). This completes the proof of the theorem. □

4. Discussion and examples

In this work, we have discussed sufficient conditions for oscillation and nonoscillation of the impulsive difference system (E_1). However, we failed to obtain the necessary and sufficient conditions for oscillation of the impulsive system (E_1). It would be interesting to establish the necessary and sufficient conditions for oscillation of (E_1) by using (A_{11}). We may note the following facts about our work:

(i) If $F(n) \equiv 0$, then Theorems 3.1–3.4 are hold true for the homogeneous counterpart of (E_1).

(ii) If $\lim_{n\to\infty} |F(n)| = 0$, then Theorems 3.1–3.4 are hold true for (E_1).

(iii) Theorems 2.1–2.3 are hold true when G is sublinear, superlinear or linear.

Remark 4.1. In Theorems 2.1–2.3, (E_1) is oscillatory due to (A_1) and (A_2). But, the problem is still incomplete when we assume that

(A_{13}) $F(n)$ changes sign such that

$$\liminf_{n\to\infty} F(n) = -\infty \quad \text{and} \quad \limsup_{n\to\infty} F(n) = \infty.$$

However, we state the following result without proof:

Theorem 4.2. *Assume that (A_{13}) holds. If $0 \leq p(n) < \infty$, then (E_1) is oscillatory. If $-\infty < p(n) \leq 0$, then every bounded solution of (E_1) oscillates.*

We conclude this section with the following illustrative examples:

EXAMPLE 4.3. Consider the impulsive difference system:

$$(E_4)\begin{cases}\Delta\big[y(n)-((-1)^nB(n-1)+B(n))y(n-2)\big] \\ \quad +\frac{6}{n^{\frac{1}{3}}}y^{\frac{1}{3}}(n-2)=2(-1)^n, & n\neq m_j,\ n>2, \\ \underline{\Delta}\big[y(m_j-1)-((-1)^{m_j-1}B(m_j-2)+B(m_j-1))y(m_j-3)\big] \\ \quad +\frac{6}{(m_j-1)^{\frac{1}{3}}}y^{\frac{1}{3}}(n-2)=2(-1)^{m_j+1} \end{cases}$$

where, $\tau=2,\sigma=2$, $p(n)=-\big((-1)^nB(n-1)+B(n)\big)$, $q(n)=\frac{6}{n^{\frac{1}{3}}}$, $r(m_j-1)=\frac{6}{(m_j-1)^{\frac{1}{3}}}$, $f(n)=2(-1)^n$, $g(m_j-1)=2(-1)^{m_j+1}$, $G(u)=u^{\frac{1}{3}}$, $m_j=3j$, $j\in\mathbb{N}$, $F(n)=(-1)^{n+1}$ and

$$B(n)=\begin{cases}1, & n\ odd,\\ 0, & n\ even.\end{cases}$$

Indeed,

$$F^+(n)=\begin{cases}1, & n\ odd,\\ 0, & n\ even\end{cases}$$

and

$$F^-(n)=\begin{cases}0, & n\ odd,\\ 1, & n\ even.\end{cases}$$

Since,

$$\sum_{n=2}^{\infty}q(n)=\sum_{n=2}^{\infty}\frac{6}{n^{\frac{1}{3}}}=6\sum_{n=2}^{\infty}\frac{1}{n^{\frac{1}{3}}}=\infty,$$

then (A_4)–(A_7) hold. By Theorem 2.1, every solution of (E_4) oscillates. In particular, $y(n)=(n+m_j+2)(-1)^{n+m_j}$ is an oscillatory solution of (E_4).

EXAMPLE 4.4. Consider the impulsive difference system of the form:

$$(E_5)\begin{cases}\Delta\big[y(n)+\frac{1}{e^n}y(n-2)\big]+(e^2+e)e^{\frac{2n-1}{3}}y^{\frac{1}{3}}(n-2) \\ =-2e^{-2}(-1)^n, & n\neq m_j,\ n>2, \\ \underline{\Delta}\big[y(m_j-1)+\frac{1}{e^{m_j-1}}y(m_j-3)\big] \\ \quad +(e^2+e)e^{\frac{2m_j-3}{3}}+(\frac{1-e^2}{e^2})e^{\frac{-m_j-3}{3}}y^{\frac{1}{3}}(m_j-3)=2e^{-2}(-1)^{m_j}, & j\in\mathbb{N},\end{cases}$$

where $\tau=2=\sigma$, $p(n)=\frac{1}{e^n}$, $q(n)=(e^2+e)e^{\frac{2n-1}{3}}$, $r(m_j-1)=(e^2+e)e^{\frac{2m_j-3}{3}}+(\frac{1-e^2}{e^2})e^{\frac{-m_j-3}{3}}$, $G(u)=u^{\frac{1}{3}}$, $m_j=3j$ for $j\in\mathbb{N}$, $F(n)=e^{-2}(-1)^n$. Hence

$$F^+(n)=\begin{cases}0, & n\ odd,\\ e^{-2}, & n\ even\end{cases}$$

and

$$F^-(n) = \begin{cases} e^{-2}, & n \ odd, \\ 0, & n \ even. \end{cases}$$

Also

$$Q(n) = (e^2+e)e^{\frac{2n-5}{3}}, \quad R(m_j-1) = (e^2+e)e^{\frac{2m_j-7}{3}} + \left(\frac{1-e^2}{e^2}\right)e^{\frac{-m_j-5}{3}}.$$

Since

$$\sum_{n=2}^{\infty} Q(n) = \sum_{n=1}^{\infty}(e^2+e)e^{\frac{2n-5}{3}} = (e^2+e)\sum_{n=1}^{\infty} e^{\frac{2n-5}{3}} = \infty,$$

then (A_9) and (A_{10}) hold true. By Theorem 2.2, all solutions of (E_5) are oscillatory. In particular, $y(n) = (-1)^{(n+m_j)}e^{(n+m_j)}$ is an oscillatory of (E_5).

Example 4.5. Consider the impulsive difference equations of the form:

$$(E_6)\begin{cases} \Delta\big[y(n) - (2+(-1)^n)y(n-2)\big] & \\ \quad +3y^3(n-1) = (-1)^n, & n \neq m_j,\ n > 2, \\ \underline{\Delta}\big[y(m_j-1) - (2+(-1)^{m_j-1})y(m_j-3)\big] & \\ \quad +3y^3(m_j-2) = (-1)^{m_j+1}, & j \in \mathbb{N}, \end{cases}$$

where $\tau = 2 = \sigma = 1$, $p(n) = -\big(2+(-1)^n\big)$, $q(n) = 3 = r(n)$, $G(u) = u^3$, $m_j = 3j$ for $j \in \mathbb{N}$, $F(n) = \frac{1}{2}(-1)^{n+1}$. Clearly,

$$F^+(n) = \begin{cases} \frac{1}{2}, & n \ odd, \\ 0, & n \ even \end{cases}$$

and

$$F^-(n) = \begin{cases} 0, & n \ odd, \\ \frac{1}{2}, & n \ even. \end{cases}$$

Since,

$$\sum_{n=1}^{\infty} q(n)G\big(F^+(n-\sigma)\big) = 3\sum_{n=1}^{\infty}\big(F^+(n-2)\big)^3 = 3\sum_{i=0}^{\infty}\big(F^+(2i)\big)^3 = \infty,$$

and

$$\sum_{n=1}^{\infty} q(n)G\big(F^-(n+\tau-\sigma)\big) = 3\sum_{n=1}^{\infty}\big(F^-(n-1)\big)^3 = 3\sum_{i=0}^{\infty}\big(F^-(2i)\big)^3 = \infty,$$

then (A_4)–(A_7) hold. By Theorem 2.3, every bounded solution of (E_6) oscillates. In particular, $y(n) = (-1)^{(n+m_j)}$ is an oscillatory of (E_6).

REFERENCES

[1] AGARWAL, R. P.—BOHNER, M.—GRACE, S. R.—O'REGAN, D.: *Discrete Oscillation Theory*. Hindawi Publ. Corp., New York, 2005.
[2] LAKSHMIKANTHAM, V.—BAINOV, D. D.—SIMIEONOV, P. S.: *Oscillation Theory of Impulsive Differential Equations*. World Scientific, Singapore, 1989.
[3] LALLI, B. S.—ZHANG, B. G.: *On existance of positive solutions and bounded oscillations for neutral difference equations*, J. Math. Anal. Appl. **166** (1992), 272–287.
[4] LI, W. T.—FAN, X. L.—ZHANG, C. K.: *On unbounded solutions of second-order difference equations with a singular nonlinear term*, J. Math. Anal. Appl. **246** (2000), 80–88.
[5] LI, Q. L.—ZHANG, Z. G.—GOU, F.—LIANG, H. Y.: *Oscillation criteria for third-order difference equations with impulses,* J. Comput. Appl. Math. **225** (2009), 80–86.
[6] LU, W.—GE, W. G.—ZHAO, Z. H.: *Oscillatory criteria for third-order nonlinear difference equations with impulses,* J. Comput. Appl. Math. **234** (2010), 3366–3372.
[7] PARHI, N.—TRIPATHY, A. K.: *Oscillation criteria for forced nonlinear neutral delay difference equations of first order,* Differ. Equ. Dyn. Syst. **8** (2000), 81–97.
[8] ______ *Oscillation of forced nonlinear neutral delay difference equations of first order,* Czechoslovak Math. J. **53** (2003), 83–101.
[9] ______ *Oscillation of a class of neutral difference equations of first order,* J. Difference Equ. Appl. **9** (2003), 933–946.
[10] TRIPATHY, A. K.: *Oscillation criteria for a class of first order neutral impulsive differential-difference equations,* J. Appl. Anal. Comput. **4** (2014), 89–101.
[11] TRIPATHY, A. K.—CHHATRIA, G. N.: *On oscillatory first order neutral impulsive difference equations* (communicated).
[12] WEI, G. P.: *The persistance of nonoscillatory solutions of difference equation under impulsive perturbations,* Computers Math. Appl. **50** (2005), 1579–1586.
[13] ZHANG, H.—CHEN, L.: *Oscillations criteria for a class of second-order impulsive delay difference equations,* Adv. Complex Syst. **9** (2006), 69–76.

Received April 4, 2018

Department of Mathematics
Sambalpur University
Sambalpur – 768019
INDIA
E-mail: arun_tripathy70@rediffmail.com
c.gokulananda@gmail.com