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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2018-0002
Tatra Mt. Math. Publ. **71** (2018), 9–25

ON THE DENSE CONTROLLABILITY FOR THE PARABOLIC PROBLEM WITH TIME-DISTRIBUTED FUNCTIONAL

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ABSTRACT. We consider a control problem for one-dimensional heat equation with quadratic cost functional. We prove the existence and uniqueness of a control function from a prescribed set, and study the structure of the set of accessible temperature functions. We also prove the dense controllability of the problem for some set of control functions.

1. Introduction

When growing plants in industrial hothouses, some temperature conditions are needed at some fixed height corresponding to the growth point of the plants. These conditions should be maintained according to a circadian schedule with small deviations admitted. One can make the temperature to rise by heating the floor of the hothouse and to fall by opening ventilator windows at the ceil. A hothouse can be treated as an elongated parallelepiped. Consider its cross-sections that are perpendicular to its longer side. We suppose that temperature distribution does not depend on the section, so we can use the model based on the one-dimensional heat equation.

Let us consider in the semi-infinite stripe $Q = (0, \pi) \times (0, +\infty)$ the mixed problem for the equation

$$u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0, \tag{1}$$

with the boundary conditions

2010 Mathematics Subject Classification: 35K20, 35Q93, 35Q79, 49J20.
Keywords: heat equation, quadratic cost functional, extremal problem, exact controllability, dense controllability.

$$u(0,t) = \varphi(t), \quad u_x(\pi,t) = \psi(t), \quad t > 0, \tag{2}$$

and the initial condition

$$u(x,0) = 0, \quad 0 < x < \pi, \tag{3}$$

with the functions

$$\varphi \in W_2^1(0,T), \quad \psi \in W_2^1(0,T) \quad \text{for any } T > 0.$$

We have to maintain the temperature $z(t)$ at some given height $c \in (0,\pi]$ during the whole time interval $0 \le t \le T$. We mean that the function $\psi(t)$ is fixed and $\varphi(t)$ is a control function to be found. The problem consists in finding the control function $\varphi_0(t)$ making the temperature at the point c maximally close to the given one $z(t)$. The quality of the control is estimated by the quadratic cost functional. We prove the existence and uniqueness of the control function $\varphi_0(t)$ from a prescribed set (the minimizer) giving the minimum to this functional, and study the structure of the set of accessible temperature functions. We also prove the "dense controllability" of the problem for some set of control functions.

Let us note that extremum problems for partial differential equations with integral functionals were considered by different authors (see [1]–[4]). The problem of minimization of functional with final observation and the problem of optimal time of control were considered in [2]–[6]. The review of early results on this problem is contained in [5], survey of later works is contained in [6], see also [7], [8]. Note that our formulation of the extremal problem with time-distributed functional is quite different from those formulated in the papers listed.

2. Mathematical model and preliminary results

Propose a mathematical model to solve the problem.

Denote $Q_T = (0,\pi) \times (0,T)$. Just as in [13, p. 26], by $V_2^{1,0}(Q_T)$ we denote the Banach space of functions $u \in W_2^{1,0}(Q_T)$ with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0 \le t \le T} \|u(x,t)\|_{L_2(0,\pi)} + \|u_x\|_{L_2(Q_T)} \tag{4}$$

and such that $t \mapsto u(\cdot,t)$ is a continuous mapping $[0,T] \to L_2(0,\pi)$. The derivatives in (4) are weak derivatives. The formula to the norm in the space $V_2^{1,0}(Q_T)$ introduced in the book [13, p. 26]. This norm naturally corresponds to the energy balance equation for the mixed problem to the heat equation [13, Ch. 3, formula (2.22)].

By $\widetilde{W}_2^1(Q_T)$ we denote the space of all functions $\eta \in W_2^1(Q_T)$ such that $\eta(x,T) = 0$, $\eta(0,t) = 0$. The values of the functions $\eta(x,T)$ and $\eta(0,t)$ are considered in the trace sense (see [13, Ch. 1, Th. 6.3, p. 71]).

We will consider the energy class of weak solutions to problem (1)–(3), i.e., the set of functions $u \in V_2^{1,0}(Q_T)$ satisfying the boundary condition $u(0,t) = \varphi(t)$ and the integral identity

$$\int\limits_{Q_T} (u_x \eta_x - u\eta_t)\ \mathrm{d}x\ \mathrm{d}t = \int\limits_0^T \psi(t)\,\eta(\pi,t)\,\mathrm{d}t \tag{5}$$

for any function $\eta(x,t) \in \widetilde{W}_2^1(Q_T)$. See [13, Ch. 3, § 2, p. 161]. Under the conditions $\varphi, \psi \in W_2^1(0,T)$ the weak solution from the set $W_2^{1,0}(Q_T)$ automatically belongs to $V_2^{1,0}(Q_T)$ [13, Ch. 3, § 3], so that the following statement holds.

Lemma 1 ([9]). *There exists a unique weak solution to problem* (1)–(3) *belonging to* $V_2^{1,0}(Q_T)$.

Hereafter we denote by u_φ the unique solution to the problem (1)–(3) with $\varphi(t) \in W_2^1(0,T)$, $\psi(t) \in W_2^1(0,T)$ for any $T > 0$, existing according to Lemma 1.

Suppose $T>0$, $z \in L_2(0,T)$. By Φ_M with $M>0$ we denote the set of functions

$$\Phi_M = \big\{\varphi \in W_2^1(0,T)\colon \|\varphi\|_{W_2^1(0,T)} \le M\big\}.$$

For some $c \in (0,\pi]$ we define the functional

$$J[\varphi] = \int\limits_0^T \big(u_\varphi(c,t) - z(t)\big)^2\ \mathrm{d}t.$$

The value of the function $u_\varphi(c,t) \in L_2(0,T)$ is also considered in the trace sense.

Consider the minimization problem for this functional and put

$$m = \inf_{\varphi \in \Phi_M} J[\varphi].$$

Theorem 1. *There exists a unique function* $\varphi_0(t) \in \Phi_M$ *such that* $m = J[\varphi_0]$.

P r o o f. The proofs of results on the existence and uniqueness are based on the following lemma concerning the best approximation in the Hilbert space.

Lemma 2 ([7]). *Let A be a convex closed set in a Hilbert space H. Then for any $x \in H$ there exists a unique element $y \in A$ such that*

$$\|x - y\| = \inf_{z \in A} \|x - z\|.$$

Denote

$$B_M = \big\{y = u_\varphi(c,\cdot)\colon \varphi \in \Phi_M\big\} \subset L_2(0,T).$$

Let us prove that the set B_M is a convex closed subset in $L_2(0,T)$. Suppose $y_1, y_2 \in B_M$ with $y_j = u_{\varphi_j}(c,\cdot)$. Then $\|\varphi_j\|_{W_2^1(0,T)} \le M$, $j = 1,2$, and for any $\alpha \in (0,1)$ we have

$$\big\|\alpha\varphi_1 + (1-\alpha)\varphi_2\big\|_{W_2^1(0,T)} \le \alpha\big\|\varphi_1\big\|_{W_2^1(0,T)} + (1-\alpha)\big\|\varphi_2\big\|_{W_2^1(0,T)} \le M,$$

whence $\alpha y_1 + (1-\alpha) y_2 \in B_M$ and the set B_M is convex.

Now we prove that B_M is a closed subset in $L_2(0,T)$. Let

$$\{y_k(t)\}_{k=1}^{\infty} \subset B_M$$

be a fundamental sequence in $L_2(0,T)$ having the limit

$$y \in L_2(0,T), \quad \|y - y_k\|_{L_2(0,T)} \to 0, \quad k \to \infty.$$

The corresponding sequence $\{\varphi_k\} \subset \Phi_M$ is a weakly precompact set in $W_2^1(0,T)$. Hence, some subsequence φ_{k_j} tends weakly, as $j \to \infty$, to a function

$$\varphi \in W_2^1(0,T).$$

By the properties of weakly convergent sequences in Hilbert spaces [13, Ch. 1, Sec. 1, Th. 1.1] we obtain

$$\|\varphi\|_{W_2^1(0,T)} \le \limsup_{j\to\infty} \|\varphi_{k_j}\|_{W_2^1(0,T)} \le M, \tag{6}$$

whence $\varphi \in \Phi_M$.

Next, by the Banach-Saks theorem [14, Ch. 2, Sec. 3] there exists a subsequence k_{j_n} such that

$$\lim_{n\to\infty} \|\tilde{\varphi}_n - \varphi\|_{W_2^1(0,T)} = 0\,, \tag{7}$$

where

$$\tilde{\varphi}_n = \frac{1}{n}\sum_{l=1}^{n} \varphi_{k_{j_l}}.$$

Therefore,

$$\|\tilde{\varphi}_n\|_{W_2^1(0,T)} \le \frac{1}{n}\sum_{l=1}^{n} \|\varphi_{k_{j_l}}\|_{W_2^1(0,T)} \le M$$

and by (6) we obtain

$$\tilde{y}_n = \frac{1}{n}\sum_{l=1}^{n} y_{k_{j_l}} \in B_M.$$

By standard technique (see [12], [13]) we can obtain the following estimate for the solution to problem (1)–(3):

$$\begin{aligned}\|u_\varphi\|_{V_2^{1,0}(Q_T)} &\le C_1\big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)}\big)\\ &\le C_2\|(\varphi,\psi)\|_{W_2^1(0,T)\times W_2^1(0,T)},\end{aligned} \tag{8}$$

where the constants C_1 and C_2 are independent of φ and ψ. So, the operator $(\varphi,\psi) \mapsto u$ is bounded in $W_2^1(0,T) \times W_2^1(0,T) \to V_2^{1,0}(Q_T)$.

Therefore, for the corresponding sequence of solutions

$$u_{\tilde{\varphi}_n} = \frac{1}{n}\sum_{l=1}^{n} u_{\varphi_{k_{j_l}}},$$

we obtain the inequalities

$$\|u_{\tilde{\varphi}_m} - u_{\tilde{\varphi}_n}\|_{V_2^{1,0}(Q_T)} \le C_1\|\tilde{\varphi}_m - \tilde{\varphi}_n\|_{W_2^1(0,T)} \to 0, \qquad m,n\to\infty. \tag{9}$$

This means that $u_{\tilde{\varphi}_n}(0,t) = \tilde{\varphi}_n(t)$ and the integral identity

$$\int\limits_{Q_T} \big((u_{\tilde{\varphi}_n})_x \eta_x - u_{\tilde{\varphi}_n} \eta_t\big)\,\mathrm{d}x\,\mathrm{d}t = \int\limits_0^T \psi(t)\,\eta(\pi,t)\,\mathrm{d}t \tag{10}$$

holds for any function $\eta(x,t) \in \widetilde{W}_2^1(Q_T)$. Taking into account relations (7), (9), and (10), we see that there exists the limit function $u \in V_2^{1,0}(Q_T)$, which is a weak solution to the problem (1)–(3) with the boundary function φ and

$$\|u - u_{\tilde{\varphi}_n}\|_{V_2^{1,0}(Q_T)} \le C_1 \|\varphi - \tilde{\varphi}_n\|_{W_2^1(0,T)}.$$

So, by the embedding estimate (see [13, Ch. 1, Sec. 6, formula 6.15]) we obtain

$$\|u(c,\cdot) - u_{\tilde{\varphi}_n}(c,\cdot)\|_{L_2(0,T)} \le C_2 \|u - u_{\tilde{\varphi}_n}\|_{V_2^{1,0}(Q_T)} \le C_1 C_2 \|\varphi - \tilde{\varphi}_n\|_{W_2^1(0,T)},$$

whence $y = u(c,\cdot) \in B_M$ and B_M is a closed subset in $L_2(0,T)$.

Therefore, by Lemma 2, there exists a unique function $y = u(c,\cdot)$, where $u \in V_2^{1,0}(Q_T)$ is a solution to the problem (1)–(3) with some $\varphi_0 \in \Phi_M$ such that

$$\inf_{\varphi \in \Phi_M} J[\varphi] = J[\varphi_0].$$

Let us prove that such $\varphi_0 \in \Phi_M$ is unique. If not, consider a pair of such functions φ_1, φ_2 and the corresponding pair of solutions u_{φ_1}, u_{φ_2}. The function $\tilde{u} = u_{\varphi_1} - u_{\varphi_2}$ is a solution to the problem

$$\tilde{u}_t = \tilde{u}_{xx}, \qquad 0 < t < T, \quad 0 < x < \pi, \tag{11}$$
$$\tilde{u}(0,t) = \tilde{\varphi}(t), \qquad 0 < t < T, \quad \tilde{\varphi}(t) = \varphi_1(t) - \varphi_2(t), \tag{12}$$
$$\tilde{u}(c,t) = 0, \qquad 0 < t < T, \tag{13}$$
$$\tilde{u}_x(\pi,t) = 0, \qquad 0 < t < T, \tag{14}$$
$$\tilde{u}(x,0) = 0, \qquad 0 < x < \pi. \tag{15}$$

First we consider the case $c = \pi$. By (11)–(15) we have the following problem for the function $\tilde{u}$

$$\tilde{u}_t = \tilde{u}_{xx}, \qquad 0 < t < T, \quad 0 < x < \pi, \tag{16}$$
$$\tilde{u}(0,t) = \tilde{\varphi}(t), \qquad 0 < t < T, \tag{17}$$
$$\tilde{u}(\pi,t) = 0, \qquad 0 < t < T, \tag{18}$$
$$\tilde{u}_x(\pi,t) = 0, \qquad 0 < t < T, \tag{19}$$
$$\tilde{u}(x,0) = 0, \qquad 0 < x < \pi. \tag{20}$$

It follows from (16)–(20) that $\tilde{u}$ is a solution of the non-characteristic Cauchy problem

$$\tilde{u}_t = \tilde{u}_{xx}, \qquad 0 < t < T, \quad 0 < x < \pi, \tag{21}$$

$$\tilde{u}(\pi, t) = 0, \qquad 0 < t < T, \tag{22}$$

$$\tilde{u}_x(\pi, t) = 0, \qquad 0 < t < T. \tag{23}$$

The problem (21)–(23) has a zero solution. By the Holmgren local uniqueness theorem [16, Part 1, Ch. 3, § 3.4] and the analyticity in x of the function $\tilde{u}$ we have that $\tilde{u} = 0$ in Q_T. Therefore, $\tilde{\varphi}(t) = \tilde{u}(0,t) = 0$. Uniqueness of control function is proved.

Now consider the case $0 < c < \pi$. This case is considered separately because in this situation we have no Cauchy problem on the line $x = c$. So, we cannot apply the Holmgren uniqueness theorem. Taking into account integral identity (5) with the function $\eta(x,t)$ equal to 0 on $[0,c] \times [0,T]$, we obtain that the function $\tilde{u}$ on the rectangle $Q_T^{(c)} = (c,\pi) \times (0,T)$ is equal to the solution of the problem

$$\hat{u}_t = \hat{u}_{xx}, \qquad 0 < t < T, \quad c < x < \pi, \tag{24}$$

$$\hat{u}(c, t) = 0, \qquad 0 < t < T, \tag{25}$$

$$\hat{u}_x(\pi, t) = 0, \qquad 0 < t < T, \tag{26}$$

$$\hat{u}(x, 0) = 0, \qquad c < x < \pi. \tag{27}$$

But the solution to problem (24)–(27) vanishes on $[c,\pi] \times [0,T]$, whence we have

$$\tilde{u}(x,t) = 0, \qquad c < x < \pi, \quad 0 < t < T. \tag{28}$$

Now we prove that

$$\tilde{u}(x,t) = 0, \qquad 0 < x < \pi, \quad 0 < t < T. \tag{29}$$

Note that by Theorem 2 [17, Sec. 11], the weak solution $\tilde{u}$ is a classical solution to the equation (11) in Q_T. Now we use Theorem 5 [18, Sec. 3]. It establishes the following.

Consider a function $u(x,t) \in C^{2,1}(\Omega)$, $\Omega \subset R^2$, such that $u_t = u_{xx}$ on Ω. Suppose G_0 is a connected component of the set $\Omega \cap \{t = t_0\}$, and $\widetilde{G}$ is a connected open subset of G_0. If $u|_{\widetilde{G}} = 0$, then $u|_{G_0} = 0$.

Applying this theorem to the solution $\tilde{u}$ of the problem (11)–(15) for any $t_0 \in (0,T)$ with $G_0 = (0,\pi) \times \{t_0\}$ and $\widetilde{G} = (c,\pi) \times \{t_0\}$, we obtain that (29) follows from (28). Therefore, $\tilde{u}(x,t) = 0$ for any $x \in (0,\pi)$ and $t \in (0,T)$. This means that

$$\tilde{\varphi}(t) = \tilde{u}(0,t) = 0.$$

The proof of Theorem 1 is complete. □

By similar considerations we can obtain the existence and uniqueness theorems for other practically important classes of control functions (see [7]).

3. On the exact controllability

Besides the question of existence and uniqueness of the solution to the extremum problem, another important question concerns the *exact controllability* on some set $Z \subset L_2(0,T)$, which means the ability to obtain, at some point $x = c$, the restriction $u(c,t)$ equal almost everywhere on $[0,T]$ to a given function $z(t) \in Z$. Respectively, by the exact control we mean the function $\varphi_0(t) \in W_2^1(0,T)$ making the functional $J[\varphi]$ to vanish:

$$J[\varphi_0] = \int_0^T \big(u_{\varphi_0}(c,t) - z(t)\big)^2 \mathrm{d}t = 0.$$

The next theorem shows that the set Z of functions $z \in L_2(0,T)$ admitting exact controllability is sufficiently “small” subset of $L_2(0,T)$.

Theorem 2. *The set Z of all functions $z \in L_2(0,T)$ admitting exact control, i.e., such that $J[\varphi] = 0$ for some $\varphi(t) \in W_2^1(0,T)$ is a first Baire category subset in $L_2(0,T)$.*

P r o o f. Consider the solutions $u_{\varphi_j}(x,t) \in V_2^{1,0}(Q_T)$, $j = 1,2$. Denote $\tilde u = u_{\varphi_1} - u_{\varphi_2}$. The function $\tilde u$ is a solution to the equation (1) with the boundary conditions

$$\tilde u(0,t) = \tilde\varphi(t) = \varphi_1(t) - \varphi_2(t), \tag{30}$$

$$\tilde u_x(\pi,t) = 0, \tag{31}$$

and the initial condition

$$\tilde u(x,0) = 0. \tag{32}$$

Now, in the domain $Q_T^{(2\pi)} = (0,2\pi) \times (0,T)$ consider the problem

$$\bar u_t = \bar u_{xx}, \qquad 0 < x < 2\pi, \quad 0 < t < T, \tag{33}$$

$$\bar u(0,t) = \tilde\varphi(t), \tag{34}$$

$$\bar u(2\pi,t) = \tilde\varphi(t), \tag{35}$$

$$\bar u(x,0) = 0. \tag{36}$$

The weak solution of the problem (33)–(36) is a function $\bar u(x,t) \in V_2^{1,0}\big(Q_T^{(2\pi)}\big)$ satisfying the boundary condition $\bar u(0,t) = \bar u(2\pi,t) = \tilde\varphi(t)$ and the integral identity

$$\int_{Q_T^{(2\pi)}} (\bar u_x \eta_x - \bar u \eta_t)\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{37}$$

for any function $\eta(x,t) \in W_2^1(Q_T^{(2\pi)})$ such that $\eta(x,T) = 0$, $\eta(0,t) = 0$, $\eta(2\pi,t) = 0$. It follows from the equality (37) that

$$\bar u(x,t) = \tilde u(x,t), \qquad 0 < x < \pi, \quad 0 < t < T. \tag{38}$$

By the maximum principle for weak solutions [12, Ch. 3, Sec. 7, Th. 7.2], the solution $\bar{u}(x,t)$ satisfies the inequalities

$$\min\Big\{0, \operatorname{ess}\inf_{t\in[0,T]}\tilde{\varphi}(t)\Big\} \le \bar{u}(x,t) \le \max\Big\{0, \operatorname{ess}\sup_{t\in[0,T]}\tilde{\varphi}(t)\Big\}. \tag{39}$$

From (39) therefore

$$\|\tilde{u}\|_{L_\infty(Q_T^{(2\pi)})} \le \|\varphi_1 - \varphi_2\|_{L_\infty(0,T)}, \tag{40}$$

and, consequently by the continuity of solution of equation (33)

$$\sup_{(0,T)} |\tilde{u}(c,t)| \le \|\varphi_1 - \varphi_2\|_{L_\infty(0,T)}. \tag{41}$$

Integrating inequality (41), we obtain

$$\|\tilde{u}(c,t)\|_{L_2(0,T)} \le \sqrt{T}\|\varphi_1 - \varphi_2\|_{L_\infty(0,T)}. \tag{42}$$

Suppose the functions $\varphi_1(t)$ and $\varphi_2(t)$ are the exact control functions for given $z_1(t)$ and $z_2(t)$. This means that

$$J[\varphi_j] = \int_0^T \big(u_{\varphi_j}(c,t) - z_j(t)\big)^2 \mathrm{d}t = 0, \qquad j = 1,2.$$

In this situation, inequality (42) invokes the inequality

$$\|z_1 - z_2\|_{L_2(0,T)} \le \sqrt{T}\|\varphi_1 - \varphi_2\|_{L_\infty(0,T)} \tag{43}$$

for arbitrary functions $z_1(t)$ and $z_2(t)$ admitting exact controllability.

Let $Z \subset L_2(0,T)$ be a set of exactly controllable functions. We have

$$Z = \cup_{M=1}^{\infty} Z_M, \quad \text{where } \ Z_M \subset L_2(0,T)$$

is the set of functions exactly controllable with $\varphi(t) \in \Phi_M$. For any $M = 1,2,\dots$ consider an arbitrary sequence of control functions $\{\varphi_k(t)\} \subset \Phi_M$ and the corresponding sequence $\{z_k(t)\} = \{u_{\varphi_k}(c,t)\} \subset Z_M$. The set Φ_M is a bounded set in $W_2^1(0,T)$. By the embedding theorem for $W_2^1(0,T)$, we have

$$\|\varphi_{k_l} - \varphi_{k_j}\|_{L_\infty(0,T)} \to 0, \qquad l,j \to \infty, \quad \text{for some subsequence } \ \varphi_{k_j}. \tag{44}$$

Therefore, by (43), (44) we get for the sequence $\{z_{k_j}(t)\} \subset Z_M$ the relation

$$\|z_{k_l} - z_{k_j}\|_{L_2(0,T)} \le \sqrt{T}\|\varphi_{k_l} - \varphi_{k_j}\|_{L_\infty(0,T)} \to 0, \qquad j,l \to \infty. \tag{45}$$

It follows from (45) that Z_M is a pre-compact set in $L_2(0,T)$. So, Z_M is nowhere dense in $L_2(0,T)$. Thus, since $Z = \cup_{M=1}^{\infty} Z_M$, we conclude that Z is a first Baire category set in $L_2(0,T)$. Theorem 2 is proved. □

4. On the dense controllability

The Theorem 2 shows that the set of functions $z(t) \in L_2(0,T)$ admitting exact controllability is sufficiently "small". So, another important question concerns the *dense controllability* in some set $Z \subset L_2(0,T)$ of functions $z(t)$ which means that for some set $\Phi \subset W_2^1(0,T)$ of control functions $\varphi(t)$ for all $z \in Z$ we have

$$\inf_{\varphi\in\Phi} J[\varphi] = \inf_{\varphi\in\Phi} \int_0^T \bigl(u_\varphi(c,t) - z(t)\bigr)^2 \,\mathrm{d}t \, = 0\,.$$

The following result proves the dense controllability when

$$Z = L_2(0,T) \quad \text{and} \quad \Phi = W_2^1(0,T).$$

Theorem 3. *For any $z \in L_2(0,T)$ the following equality holds*

$$\inf_{\varphi\in W_2^1(0,T)} J[\varphi] = 0. \tag{46}$$

P r o o f. Let us represent the solution of the problem (1)–(3) in the form

$$u_\varphi = v + w,$$

where v and w are solutions of the following boundary value problems

$$\begin{aligned}
v_t - v_{xx} &= 0, && 0<x<\pi, \quad 0<t<T, && (47)\\
v(0,t) &= \varphi(t), && 0<t<T, && (48)\\
v_x(\pi,t) &= 0, && 0<t<T, && (49)\\
v(x,0) &= 0, && 0<x<\pi, && (50)
\end{aligned}$$

and

$$\begin{aligned}
w_t - w_{xx} &= 0, && 0<x<\pi, \quad 0<t<T, && (51)\\
w(0,t) &= 0, && 0<t<T, && (52)\\
w_x(\pi,t) &= \psi(t), && 0<t<T, && (53)\\
w(x,0) &= 0, && 0<x<\pi. && (54)
\end{aligned}$$

Therefore, denoting $v = v_\varphi$ we have

$$J[\varphi] = \int_0^T \bigl(v_\varphi(c,t) - z_1(t)\bigr)^2 \,\mathrm{d}t, \qquad c \in (0,\pi], \tag{55}$$

where $z_1(t) = z(t) - w(c,t) \in L_2(0,T)$. It follows from the inequality

$$\inf_{\varphi\in W_2^1(0,T)} J[\varphi] \;\le \inf_{\substack{\varphi\in W_2^1(0,T),\\ \varphi(0)=0}} J[\varphi] \;= \inf_{\substack{\varphi\in W_2^1(0,T),\\ \varphi(0)=0}} \int_0^T \bigl(v_\varphi(c,t) - z_1(t)\bigr)^2 \,\mathrm{d}t \tag{56}$$

that to establish (46) it is sufficient to prove that

$$\inf_{\substack{\varphi\in W_2^1(0,T),\\ \varphi(0)=0}} \int_0^T \big(v_\varphi(c,t)-z_1(t)\big)^2\,\mathrm{d}t = 0\,. \tag{57}$$

Let us construct the weak solution $v_\varphi \in W_2^{1,0}(Q_T)$ of the problem (47)–(50) for

$$\varphi \in W_2^1(0,T), \quad \varphi(0)=0\,.$$

Consider the function

$$y(x,t) = v_\varphi(x,t) - \varphi(t)$$

which is the solution of the following problem

$$y_t - y_{xx} = -\varphi'(t), \qquad 0<x<\pi, \quad 0<t<T, \tag{58}$$

$$y(0,t)=0, \qquad 0<t<T, \tag{59}$$

$$y_x(\pi,t)=0, \qquad 0<t<T, \tag{60}$$

$$y(x,0)=0, \qquad 0<x<\pi. \tag{61}$$

So,

$$y = -\frac{2}{\pi}\sum_{k=0}^{\infty}\left(\frac{\sin\left(\left(k+\frac12\right)x\right)}{k+\frac12}\int_0^t e^{-\left(k+\frac12\right)^2(t-\tau)}\varphi'(\tau)\,\mathrm{d}\tau\right).$$

Therefore,

$$\begin{aligned} v_\varphi(x,t) &= \varphi(t) - \frac{2}{\pi}\sum_{k=0}^{\infty}\left(\frac{\sin\left(\left(k+\frac12\right)x\right)}{k+\frac12}\int_0^t e^{-\left(k+\frac12\right)^2(t-\tau)}\varphi'(\tau)\,\mathrm{d}\tau\right) \\ &= \int_0^t \varphi'(\tau)d\tau - \frac{2}{\pi}\sum_{k=0}^{\infty}\left(\frac{\sin\left(\left(k+\frac12\right)x\right)}{k+\frac12}\int_0^t e^{-\left(k+\frac12\right)^2(t-\tau)}\varphi'(\tau)\,\mathrm{d}\tau\right) \\ &= \int_0^t \varphi'(\tau)\left(1-\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin\left(\left(k+\frac12\right)x\right)}{k+\frac12}e^{-\left(k+\frac12\right)^2(t-\tau)}\right)\mathrm{d}\tau \\ &= \int_0^t \varphi'(\tau)P(x,t-\tau)\,\mathrm{d}\tau, \end{aligned} \tag{62}$$

where

$$P(x,t) = 1-\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin\left(\left(k+\frac12\right)x\right)}{k+\frac12}e^{-\left(k+\frac12\right)^2 t}. \tag{63}$$

Let us prove that the function $P(x,t) \in V_2^{1,0}(Q_T)$ is a weak solution of the mixed problem

$$
\begin{aligned}
P_t - P_{xx} &= 0, \qquad 0 < x < \pi, \qquad 0 < t < T, && (64)\\
P(0,t) &= 1, \qquad 0 < t < T, && (65)\\
P_x(\pi,t) &= 0, \qquad 0 < t < T, && (66)\\
P(x,0) &= 0, \qquad 0 < x < \pi, && (67)
\end{aligned}
$$

and satisfies the integral identity

$$
\int_{Q_T} (P_x \eta_x - P\eta_t)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{68}
$$

for any function $\eta(x,t) \in \widetilde{W}_2^1(Q_T)$.

At first we show that $P(x,t) \in W_2^{1,0}(Q_T)$. By the equality

$$
P(x,t) = 1 - P_1(x,t), \quad P_1(x,t) = \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\sin\left(\left(k+\frac{1}{2}\right)x\right)}{k+\frac{1}{2}} e^{-\left(k+\frac{1}{2}\right)^2 t} \tag{69}
$$

it is sufficient to prove that $P_1(x,t) \in W_2^{1,0}(Q_T)$. We have the following estimates:

$$
\begin{aligned}
\int_0^\pi P_1^2(x,t)\,\mathrm{d}x &= \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{e^{-2\left(k+\frac{1}{2}\right)^2 t}}{\left(k+\frac{1}{2}\right)^2} \int_0^\pi \sin^2\left(\left(k+\frac{1}{2}\right)x\right)\mathrm{d}x\\
&= \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{e^{-2\left(k+\frac{1}{2}\right)^2 t}}{\left(k+\frac{1}{2}\right)^2} \le \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{2}\right)^2} \le C_1, \quad 0 < t < T,
\end{aligned}
$$

$$
\int_{Q_T} P_1^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le C_1 T. \tag{70}
$$

$$
\begin{aligned}
\int_0^\pi \big(P_{1x}(x,t)\big)^2 \mathrm{d}x &= \frac{4}{\pi^2}\sum_{k=0}^{\infty} e^{-2\left(k+\frac{1}{2}\right)^2 t} \int_0^\pi \cos^2\left(\left(k+\frac{1}{2}\right)x\right)\mathrm{d}x\\
&\le \frac{2}{\pi}\int_0^\infty e^{-2\left(s+\frac{1}{2}\right)^2 t}\mathrm{d}s \le \frac{2}{\pi}\int_0^\infty e^{-2s^2 t}\,\mathrm{d}s\\
&= \frac{\sqrt{2}}{\pi\sqrt{t}}\int_0^\infty e^{-z^2}\,\mathrm{d}z = \frac{1}{\sqrt{2\pi t}}, \qquad 0 < t < T,
\end{aligned}
$$

$$
\int_{Q_T} \big(P_{1x}(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \le C_2\sqrt{T}. \tag{71}
$$

It follows from (70), (71) that

$$\|P\|^2_{W_2^{1,0}(Q_T)} = \|P\|^2_{L_2(Q_T)} + \|P_x\|^2_{L_2(Q_T)} \le C_3(T + \sqrt{T}), \tag{72}$$

and we can define the trace $P(c,\cdot) \in L_2(0,T)$, $c \in (0,\pi]$. From the structure of series (63) we obtain that P is the Green function for problem (64)–(67) and satisfies the integral identity (68), and, by Lemma 1, we have $P \in V_2^{1,0}(Q_T)$.

We use the following property of linear manifolds in the Hilbert space [15, Ch. 2, § 4, Lemma 2]:

Lemma 3. *The linear manifold G is dense in the Hilbert space H if and only if there is no non-zero element which is orthogonal to any element of G.*

We apply this lemma to $H = L_2(0,T)$ and the linear manifold

$$G = \{v_\varphi(c,t), \varphi(t) \in D(0,T) = \mathring{C}^\infty(0,T)\}.$$

To prove (46) it is sufficient to prove that if for any $\varphi(t) \in D(0,T)$ we have

$$\begin{aligned}\int_0^T & z_1(t) v_\varphi(c,t)\,\mathrm{d}t \\ &= \int_0^T z_1(t) \left(\int_0^t P(c,t-\tau)\varphi'(\tau)\,\mathrm{d}\tau\right) \mathrm{d}t = 0,\end{aligned} \tag{73}$$

then $z_1(t) = 0$. We can transform (73) as

$$\begin{aligned}\int_0^T & z_1(t) \int_0^t P(c,t-\tau)\varphi'(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \int_0^T \varphi'(\tau) \int_\tau^T z_1(t) P(c,t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau = 0.\end{aligned} \tag{74}$$

By (74)

$$\int_\tau^T z_1(t) P(c,t-\tau)\,\mathrm{d}t = \mathrm{const}, \quad \tau \in [0,T],$$

but

$$\int_T^T z_1(t) P(c,t-T)\,\mathrm{d}t = 0,$$

so

$$\int_\tau^T z_1(t) P(c,t-\tau)\,\mathrm{d}t = 0, \qquad \tau \in [0,T].$$

After the transformation of variables we have

$$\begin{aligned}
(t \to \tau, \tau \to t) &= \int_{\tau}^{T} z_1(t) P(c, t-\tau)\,\mathrm{d}t = \int_{t}^{T} z_1(\tau) P(c, \tau - t)\,\mathrm{d}\tau \\
(s = T - \tau) &= \int_{0}^{T-t} z_1(T-s) P(c, T-s-t)\,\mathrm{d}s \\
(q = T - t) &= \int_{0}^{q} z_1(T-s) P(c, q-s)\,\mathrm{d}s \\
\big(z_2(s) = z_1(T-s)\big) &= \int_{0}^{q} z_2(s) P(c, q-s)\,\mathrm{d}s = 0
\end{aligned} \tag{75}$$

for almost all $q \in (0, T)$, here

$$z_2(t) = z_1(T-t) \in L_2(0,T) \subset L_1(0,T).$$

Now we apply the Titchmarsh convolution theorem [19, Theorem 7].

Theorem 4. *Let $\xi(t) \in L_1(0,T)$, $\zeta(t) \in L_1(0,T)$ be functions such that*

$$\int_0^t \xi(\tau)\zeta(t-\tau)\,\mathrm{d}\tau = 0 \quad \textit{a.e. in the interval } 0 < t < T, \tag{76}$$

then

$$\xi(t) = 0 \quad \textit{a.e. in } (0,\alpha)$$

and

$$\zeta(t) = 0 \quad \textit{a.e. in } (0,\beta),$$

where

$$\alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta \geq T.$$

We use Theorem 4 to the functions $P(c,\cdot)$ and $z_2(\cdot)$. By the equality (75) we obtain that there exist $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \geq T$ such that

$$z_2(s) = 0 \quad \text{a.e. in } (0,\alpha)$$

and

$$P(c,s) = 0 \quad \text{a.e. in } (0,\beta). \tag{77}$$

Now we prove that $\beta = 0$. Let us suppose on the contrary that $\beta > 0$. We define the function

$$\widetilde{P}(x,t) = \begin{cases} P(x,t), & 0 < x < \pi, \quad 0 < t < T, \\ P(2\pi - x, t), & \pi < x < 2\pi, \quad 0 < t < T, \end{cases} \tag{78}$$

which belongs to

$$V_2^{1,0}\left(Q_T^{(2\pi)}\right), \qquad Q_T^{(2\pi)} = (0,2\pi)\times(0,T).$$

Let

$$\eta(x,t) \in W_2^1\left(Q_T^{(2\pi)}\right)$$

such that

$$\eta(0,t) = \eta(2\pi,t) = 0, \quad t\in(0,T), \quad \eta(x,T) = 0, \quad x\in(0,2\pi).$$

By identity (68) and (78) we have

$$\begin{aligned}
&\int_{\pi}^{2\pi}\int_{0}^{T}\big(\widetilde{P}_x(x,t)\eta_x(x,t) - \widetilde{P}(x,t)\eta_t(x,t)\big)\mathrm{d}x\,\mathrm{d}t\\
&= \int_{0}^{\pi}\int_{0}^{T}\big(P_y(y,t)\eta_y(2\pi-y,t) - P(y,t)\eta_t(2\pi-y,t)\big)dy\,\mathrm{d}t\\
&= \int_{0}^{\pi}\int_{0}^{T}\big(P_y(y,t)\tilde{\eta}_y(y,t) - P(y,t)\tilde{\eta}_t(y,t)\big)dy\,\mathrm{d}t = 0, \qquad (79)\,(80)
\end{aligned}$$

where

$$\tilde{\eta}(y,t) = \eta(2\pi-y,t).$$

So, by equations (68) and (79) we have

$$\begin{aligned}
&\int_{Q_T^{(2\pi)}}\big(\widetilde{P}_x\eta_x - \widetilde{P}\eta_t\big)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{Q_T}\left(P_x\eta_x - P\eta_t\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{Q_T^{(2\pi)}\setminus Q_T}\big(\widetilde{P}_x\eta_x - \widetilde{P}\eta_t\big)\,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned} \tag{81}$$

for any function $\eta(x,t) \in W_2^1\big(Q_T^{(2\pi)}\big)$ such that

$$\eta(0,t) = \eta(2\pi,t) = 0, \quad t\in(0,T), \quad \eta(x,T) = 0, \quad x\in(0,2\pi).$$

So, the function $\widetilde{P}$ is the solution of the following mixed problem

$$\widetilde{P}_t - \widetilde{P}_{xx} = 0, \qquad 0 < x < 2\pi, \quad 0 < t < T, \tag{82}$$

$$\widetilde{P}(0,t) = 1, \qquad 0 < t < T, \tag{83}$$

$$\widetilde{P}(2\pi,t) = 1, \qquad 0 < t < T, \tag{84}$$

$$\widetilde{P}(x,0) = 0, \qquad 0 < x < 2\pi. \tag{85}$$

Applying the maximum principle [12, Ch. 3, § 7, Theorem 7.2] to the problem (82)–(85) we obtain

$$0 \leq \widetilde{P}(x,t) \leq 1 \quad \text{a.e. in } \ Q_T^{(2\pi)}.$$

It follows from equalities (69) that the function

$$P(x,t) \in C^\infty\big([0,2\pi] \times [\varepsilon,T]\big) \quad \text{for any} \quad \varepsilon \in (0,T)$$

and it is a classical solution of equation (82) in $Q_T^{(2\pi)}$. Then

$$0 \leq \widetilde{P}(x,t) \leq 1, \quad 0 \leq x \leq 2\pi, \quad \varepsilon < t \leq T. \tag{86}$$

Let us suppose that

$$\widetilde{P}(c,t) = 0, \quad 0 < t < \beta \leq T, \tag{87}$$

and consider the function $P(x,t)$ in the domain $Q_{T,\beta/2}^{(2\pi)} = (0,2\pi) \times (\beta/2,T)$. Note that by (86), (87)

$$P(c,\beta) \ = \ 0 \ = \inf_{(x,t)\in Q_{T,\beta/2}^{(2\pi)}} P(x,t) \tag{88}$$

and

$$(c,\beta) \in Q_{T,\beta/2}^{(2\pi)}.$$

By the strong maximum principle [20, Ch. 7, § 7.1, Theorem 11] we obtain that

$$P = 0 \quad \text{in } \ Q_{\beta,\beta/2}^{(2\pi)} = (0,2\pi) \times (\beta/2,\beta)$$

which is impossible due to the boundary conditions (83), (84). These contradiction means that $\beta = 0$. So, by the inequality

$$\alpha + \beta \geq T$$

we have

$$\alpha \geq T \quad \text{and} \quad z_2(t) = 0 \quad \text{a.e. in } \ (0,T).$$

Now, $z_1(t) = 0$ almost everywhere in $(0,T)$.

Therefore, by the Lemma 3 we obtain the equality (46). Theorem 3 is proved. □

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Received December 11, 2017

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