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Tatra Mountains
Mathematical Publications
DOI: 10.2478/tmmp-2018-0001
Tatra Mt. Math. Publ. **71** (2018), 1–7

RECENT TRENDS IN DIFFERENTIAL AND DIFFERENCE EQUATIONS
(A SURVEY)

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ABSTRACT. This article briefly describes results of recent publications in the field of differential and difference equations and their applications.

1. Introduction

This special issue contains 15 selected research papers. Most of them is originated from the talks that were presented during the Conference on Differential and Difference Equations and Applications (CDDEA 2017) which was held in the marvelous mountain resort in National Park Low Tatras, Jasná, Slovak Republic, in the term June 26–30, 2017. This international Conference was the 24th continuation of the previous fourteen Summer Schools on Differential Equations, the first of which was organized in 1969, and nine international Conferences on Differential and Difference Equations and Application. The founders of the tradition were university professors Pavol Marušiak and Ladislav Berger.

This conference was a worthy continuation of the tradition and was organized by the Faculty of Mathematics and Informatics, University of Białystok, Poland in cooperation with the Union of Slovak Mathematicians and Physicists—branch Žilina, Slovakia, Faculty of Electrical Engineering and Communication, Brno University of Technology, Czech Republic, Kyiv National Economic University, Ukraine, Poznań University of Technology, Poland and Institute of Mathematics, Łódź University of Technology, Poland. In total, 65 participants from 15 countries and 3 continents co-operated at the conference.

2010 Mathematics Subject Classification: 34Cxx, 34Dxx, 34Fxx, 34Hxx, 34Kxx.
Keywords: differential equations, difference equations.

The conference was prepared by Organizing Committee:
J. Diblík, B. Dorociaková, I. Dzhalladova, M. Kúdelčíková (secretary), *M. Migda, M. Nockowska-Rosiak, M. Růžičková* (chairperson), *E. Schmeidel, Z. Šmarda, M. Záborský;*

Scientific Committee:
I. V. Astashova, L. Berezansky, A. Boichuk, A. Cabada, J. Čermák, J. Diblík, A. Domoshnitsky, I. Dzhalladova, M. Galewski, I. Győri, F. Hartung, S. Hilger, R. Hilscher, J. Jaroš, T. Krisztin, D. Khusainov, M. Pituk, I. Rachůnková, V. Răsvan, A. Reinfelds, Yu. Rogovchenko, M. Rontó, M. Růžičková, F. Sadyrbaev, E. Schmeidel, S. Siegmund, S. Staněk, M. Tvrdý, A. Zafer.

The conference was a successful and fruitful meeting stimulating scientific contacts and collaborations during nice time in Jasná. The programme contained 18 invited plenary lectures delivered by leading researchers, 34 contributed talks and 6 posters, and covered a broad range of mathematics connected with differential and difference equations and their applications. It was divided into six sessions: Ordinary differential equations, Functional differential equations, Difference equations and dynamic equations on time scales, Partial differential equations, Numerical methods in differential and difference equations and Stochastic differential equations.

For publication in this issue 30 authors from 12 countries contributed to the success of this thematic collection of papers. The issue covers various problems for different classes of ordinary, functional, impulsive, stochastic, fractional, partial differential equations, as well as difference and integro-differential equations and dynamic equations on time scales.

2. Main results

Going briefly through the topics of presented articles we find stability problems. The algorithm for estimating the stability domain of zero equilibrium to the system of nonlinear differential equations with a quadratic part and a fractional part is proposed in the paper by I. A. Dzhalladova and M. Růžičková. The second Lyapunov method with quadratic Lyapunov functions is used as a method for studying such systems. Systems with the quadratic right-hand side are considered in the vector-matrix form $x'(t) = a\,x(t) + X^T(t)\,B\,c(t)$, where A is an $n \times n$ constant matrix, $B = (B_1, B_2, \dots, B_n)^T$, B_i, $i = 1, 2, \dots, n$ are $n \times n$ constant matrices, and $X^T = \big(X_1(t), X_2(t), \dots, X_n(t)\big)$, $X_i(t)$, $i = 1, 2, \dots, n$ are $n \times n$ matrices in which only the ith row is nonzero. Nonlinear system of two differential equations with fractional parts depending on parameters $\alpha, \varepsilon, \gamma, \mu \in \mathbb{R}$

is considered in the form

$$
\begin{aligned}
x'(t) \;&=\; x(t) - \frac{x(t)\,y(t)}{1+\alpha\,x(t)} - \varepsilon\,x^2(t),\\
y'(t) \;&=\; -\,\gamma\,y(t) + \frac{x(t)\,y(t)}{1+\alpha\,x(t)} + \mu\,y^2(t).
\end{aligned}
$$

Stability of integro-differential equations in connection with model of testosterone regulation is presented in the paper by O. Pinhasov. The model describes an interaction of the concentration of hormone GnRH–x_1 with the concentration of the hormone LH–x_2 and the concentration of testosterone Te–x_3 and can be written in the form

$$
\begin{cases}
x_1'(t) + b_1\,x_1(t) = 0,\\
x_2'(t) + b_2\,x_2(t) - g_1\,x_1(t) = 0, & t \ge 0.\\
x_3'(t) + b_3\,x_3(t) - c_1 \int\limits_0^t \mathrm{e}^{-\alpha_1\,(t-s)}\,x_2(s)\,ds = 0,
\end{cases}
$$

The values $b_i,\ i = 1,2,3$ correspond to the respective half-life times of GnRH, LH and Te and can be calculated according to the biomedical data for typical hormones levels. The aim is to propose a concept to hold the concentration of testosterone above a corresponding level using distributed input control in the form of integral term.

In the paper by J. R. Graef, L. D. Oudjedi and M. Remili, there are established sufficient conditions to guarantee the square integrability of all solutions and the asymptotic stability of the zero solution of a non-autonomous third-order neutral delay differential equation. In particular, it is examined the uniform asymptotic stability of solutions to

$$
\begin{aligned}
\big[x(t) + \beta\,x(t-\tau)\big]''' &+ a(t)\Big(Q\big(x(t)\big)\,x'(t)\Big)'\\
&+ b(t)\Big(R\big(x(t)\big)\,x'(t)\Big) + c(t)\,f\big(x(t-\tau)\big) = 0
\end{aligned}
$$

as well as the boundedness and square integrability of solutions of the corresponding forced equation

$$
\begin{aligned}
\big[x(t) + \beta\,x(t-\tau)\big]''' &+ a(t)\Big(Q\big(x(t)\big)\,x'(t)\Big)'\\
&+ b(t)\Big(R\big(x(t)\big)\,x'(t)\Big) + c(t)\,f\big(x(t-\tau)\big) = h(t),
\end{aligned}
$$

where β and τ are constants with $0 \le \beta \le 1$ and $\tau \ge 0$, the functions $a, b, c\colon [0,\infty) \to [0,\infty)$, $Q, R\colon \mathbb{R} \to [0,\infty)$, $h\colon [0,\infty) \to \mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$ are continuous, and $x\,f(x) > 0$ for $x \ne 0$.

Systems of ordinary differential equations that arise in the theory of gene regulatory networks are considered in the paper by E. Brokan and F. Sadyrbaev. The goal is to clarify the structure of an attractive set for several choices of regulatory matrices W. Mathematically this means that the nonlinear system of differential equations is studied, conclusions on the number and location of critical points are made and the character of critical points is revealed. The structure of attractors for two-dimensional systems with uniform (non-negative or non-positive elements) regulatory matrices is simple and attractors cannot contain critical points of the type focus. In low-dimension inhibition-activation systems only one critical point was detected. For n-dimensional systems with even n the characteristic equation for a single critical point has pairs of conjugate complex eigenvalues and the real parts of all eigenvalues are equal to -1. Therefore a critical point is the stable focus in all 2D-subspaces. For n odd all eigenvalues but one are pairs of complex values with real parts equal to -1. The remaining eigenvalue is -1. A critical point is the stable focus in all 2D-subspaces and attracted in the remaining dimension.

For partial differential equations in the paper by V. Răsvan there are investigated controlled boundary value problems for conservation laws arising from energy co-generation, hydraulic flows and water-hammer for hydroelectric power plants and control of the open channel flows. The novelty of these models is that they are described by nonlinear hyperbolic partial differential equations of the conservation laws with (possibly) nonlinear boundary conditions. At their turn these boundary conditions are controlled by some systems of ordinary differential equations. The engineering requirements for such systems are asymptotic stability and disturbance rejection: these properties have to be achieved by feedback control. The main tool for tackling these problems is a suitable Lyapunov functional arising from the energy identity.

In the paper by I. V. Astashova and A. V. Filinovskiy a control problem for one-dimensional heat equation with quadratic cost functional is considered. The existence and uniqueness of a control function from a prescribed set is proved and the structure of the set of accessible temperature functions is studied. It is also proved the dense controllability of the problem for some set of control functions.

Also some discrete pseudo-differential equations in discrete Sobolev-Slobodetskii spaces are considered in the paper by A. V. Vasilyev and V. B. Vasilyev. For a discrete half-space and certain values of an index of periodic factorization for an elliptic symbol there are introduced additional potential-like unknowns and existence and uniqueness theorem in appropriate discrete Sobolev-Slobodetskii spaces is proved.

Authors I. Bock and M. Kečkemétyová deal with an optimal control problem governed by a nonlinear hyperbolic initial-boundary value problem describing the perpendicular vibrations of a simply supported anisotropic viscoelastic plate against a rigid obstacle. A variable thickness of a plate plays the role of a control variable. It is verified the existence of an optimal thickness function.

Some papers are devoted to investigation of nonoscillatory solutions. One of them by J. Pasáčková deals with a system of four nonlinear difference equations, where the first equation is of a neutral type, of the form

$$\begin{aligned}\Delta(x_n + p_n\, x_{n-\sigma}) &= A_n\, f_1(y_n),\\ \Delta y_n &= B_n\, f_2(z_n),\\ \Delta z_n &= C_n\, f_3(w_n),\\ \Delta w_n &= D_n\, f_4(x_{\gamma_n}),\end{aligned}$$

where $n \in \mathbb{N}_0 = \{n_0, n_0+1, \ldots\}$, n_0 is a positive integer, σ is a nonnegative integer, $\{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$ are positive real sequences defined for $n \in \mathbb{N}_0$. Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$. The sequence $\gamma\colon \mathbb{N} \to \mathbb{N}$ satisfies $\lim_{n\to\infty} = \infty$, the sequence $\{p_n\}$ is a sequence of the real numbers and it satisfies $\lim_{n\to\infty} = P$, $|P| < 1$, and functions $f_i\colon \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, 4$ satisfy $f_i(u)/u \geq M$, $u \in \mathbb{R} \setminus \{0\}$, $M \in \mathbb{R}$ and $M > 0$. Sufficient conditions for the system to have weak property B are presented.

In the paper by U. Ostaszewska, E. Schmeidel, M. Zdanowicz the system of three dynamic equations with neutral term of the form

$$\begin{cases}\Big(x(t) + p(t)\, x\big(u_1(t)\big)\Big)^{\Delta} = a(t)\, f\Big(y\big(y_2(t)\big)\Big),\\ y^{\Delta}(t) = b(t)\, g\Big(z\big(u_3(t)\big)\Big),\\ z^{\Delta}(t) = c(t)\, h\Big(x\big(u_4(t)\big)\Big)\end{cases}$$

on a time scale $\mathbb{T}$ is considered. $x, y, z\colon \mathbb{T} \to \mathbb{R}$ are unknown functions, $p, a, b, c\colon \mathbb{T} \to \mathbb{R}$, $f, g, h\colon \mathbb{R} \to \mathbb{R}$, $u_i\colon \mathbb{T} \to \mathbb{T}$ is such that $\lim_{t\to\infty} u_i(t) = \infty$ for $i = 1, 2, 3, 4$. The aim is to present sufficient conditions for the existence of positive bounded solutions to considered system for $0 < p(t) \leq const < 1$. The main tool of the proof of presented result is Krasnoselskii's fixed point theorem. Also, the useful generalization of the Arzèla-Ascoli theorem on times scales to the three dimensional case is proved.

In the paper by A. Domoshnitsky and V. Raichik the Sturm separation theorem in the case of impulsive delay differential equations is proposed and assertions about its validity are obtained. Wronskian is one of the classical objects in the theory of ordinary differential equations. Properties of Wronskian lead to important conclusions on behavior of solutions of delay equations.

For instance, non-vanishing Wronskian ensures validity of Sturm's separation theorem (between two adjacent zeros of any solution there is one and only one zero of every other nontrivial linearly independent solution) for delay equations.

Sufficient conditions for oscillation and nonoscillation of a class of the forced first order neutral impulsive difference equations with deviating arguments of the form

$$(E_1)\begin{cases} \Delta|y(n)+p(n)\,y(n-\tau)|+q(n)G\big(y(n-\sigma)\big)=f(n), & \\ \underline{\Delta}|y(m_j-1)+p(m_j-1)\,y(m_j-\tau-1)| & n\neq m_j,\ j\in\mathbb{N}. \\ \qquad\qquad +r(m_j-1)G\big(y(m_j-\sigma-1)\big)=h(m_j-1), & \end{cases}$$

and also fixed moments of impulsive effect, are established in the work by A. K. Tripathy and G. N. Chhatria.

In the paper by G. E. Chatzarakis, P. Gokulraj and T. Kalaimani there is studied the oscillatory behavior of solutions to the fractional difference equation of the form

$$\Delta\Big(r(t)\,g\big(\Delta^{\alpha}x(t)\big)\Big)+p(t)\,f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0,\quad t\in\mathbb{N}_{t_0+1-\alpha},$$

where Δ^{α} denotes the Riemann-Liouville difference operator of order α, $0<\alpha<1$ and $\gamma>0$ is a quotient of odd positive integers. There are established some oscillatory criteria for the above equation, using the Riccati transformation and Hardy type inequalities. Also examples are provided to illustrate the theoretical results.

The asymptotic properties of solutions to nonautonomous difference equations of the form

$$\Delta^m x_n = a_n\,f(n,x_{\sigma(n)})+b_n,\quad f\colon\mathbb{N}\times\mathbb{R}\to\mathbb{R},\ \sigma\colon\mathbb{N}\to\mathbb{N}$$

are studied in the paper by J. Migda and M. Migda. Using the iterated remainder operator and asymptotic difference pairs there are established some results concerning approximative solutions and approximations of solutions. This approach allows to control the degree of approximation.

Stochastic differential equations describing systems effected by coloured noise are investigated by E. Kolářová and L. Brančík. In electrical systems it can be the case, when, e.g., transmission line is modeled by means of proper higher-order ladder network. The mathematical representation of the coloured noise is defined as a solution of the Langevin equation and the corresponding Itô type stochastic differential equation is formulated. Applying this theory the stochastic model of the network is derived and sets of individual stochastic trajectories are found numerically via a stochastic version of the backward Euler scheme. Afterwards respective confidence intervals are computed statistically while utilizing Student's t-distribution. The theoretical results are illustrated by an example of a higher-order ladder network. Numerical simulations in the example are carried out using Matlab.

Overall, this special issue represents a combination of theory and applications in the fields of differential and difference equations. We hope that readers will find all presented topics interesting and also stimulating for their own scientific research.

Received October 28, 2018

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