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FUZZY SETS AND PROBABILITY THEORY

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1. Introduction

The background of classical probability theory is based on the axiomatic model of Kolmogorov, where events are understood as Cantorian subsets of a given universe X. These events form a σ-algebra $\mathcal{A}$. The probability P is presented as a real-valued set function defined on $\mathcal{A}$ fulfilling the boundary conditions, $P(\emptyset) = 0$ and $P(X) = 1$, and the σ-additivity property, i.e. $P(\bigcup A_n) = \Sigma P(A_n)$ for any sequence of mutually exclusive events $\{A_n\} \subset A$. Generalizing the boundary condition $P(X) = 1$ we get the notion of a measure. One important branch of the "fuzzy" theory deals with further generalizations of the probability P (and possibly of the σ-algebra A), while the concept of Cantorian subsets remains unchanged. This direction is not the main topic of this paper. However, we discuss some of such generalizations in Section 2.

Fuzzy sets were introduce by Zadeh in 1965 [40] as a generalization of Cantorian sets (represented by their characteristic functions) to be functions from the universe X into the unit interval [0,1]. We will omit the other possible generalizations (for a deeper review on fuzzy set theory and its applications see, e.g., [27]). The extension of operations of intersection, union and complementation in ordinary set theory to fuzzy sets is usually done pointwise - one considers two twoplace functions $T : [0,1] \times [0,1] \to [0,1]$, $S : [0,1] \times [0,1] \to [0,1]$ and a oneplace function $c : [0,1] \to [0,1]$ and extend them in the usual way:

if A, B are two fuzzy subsets of X, then for any $x \in X$ we put

$$(A \cap B)(x) = T\big(A(x), B(x)\big)\,,$$
$$(A \cup B)(x) = S\big(A(x), B(x)\big)\,,$$
$$A^c(x) = c(A(x))\,.$$

According to some natural requirements T becomes a triangular norm of Schweizer and Sklar [30]. Similarly S is a triangular conorm. T and S are discussed in Section 3. The complementation function c and its connections with T and S are discussed in Section 4. Note that element dependent intersections and unions were studied by K l e m e n t [12] in a category framework; similarly L o w e n [16] studied the element-dependent complementations. Throughout this paper, we deal with pointwise defined fuzzy connectives only.

A couple $(\boldsymbol{X},\mathcal{A})$, where $\mathcal{A}$ is a σ-algebra of Cantorian subsets of the universe $\boldsymbol{X}$, forms a classical measurable space. In Section 5, we discuss several fuzzy generalizations of a measurable space such as generated fuzzy algebras (tribes), fuzzy σ-algebras, $\boldsymbol{T}$-tribes or g-$\boldsymbol{T}$-tribes. After a short review of this topic we present some recent results and open problems.The last Section 6 deals with measures of fuzzy events (fuzzy probability measures, $\boldsymbol{T}$-measures, $\perp$-decomposable measures etc.). Again it includes a historical review, some latest results and some open problems.

2. Fuzzy measures

Fuzzy measures were first introduced by S u g e n o [35] in 1974 in his Ph.D. thesis. A fuzzy measure is a set function defined on a system $\mathcal{D}$ of Cantorian subsets of a universe $\boldsymbol{X}$ (for $\boldsymbol{X}$ finite, $\mathcal{D}$ is usually taken to be the power set of X, $\mathcal{D} = 2^{\boldsymbol{X}}$). The only necessary condition for $\mathcal{D}$ is that it includes the empty set, $\emptyset \in \mathcal{D}$.Often $\mathcal{D}$ is supposed to be a σ-algebra. A fuzzy measure $\boldsymbol{m} : \mathcal{D} \to \mathbb{R}$ ($\mathbb{R}$ is the real line) fulfills the following conditions:

1) $\boldsymbol{m}(\emptyset) = 0$,
2) $A, B \in \mathcal{D}, A \subset B \Rightarrow \boldsymbol{m}(A) \leq \boldsymbol{m}(B)$,
3) for monotone sequences $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{D}, \lim A_n = A \in \mathcal{D}$ implies $\lim \boldsymbol{m}(A_n) = \boldsymbol{m}(A)$.

Condition 3) is rather strong—e.g., a lot of possibility measures do not fit the continuity from above. This is the main reason why in later papers the continuity condition is omitted, see, e.g., [22, 23, 24]. Hence now a fuzzy measure is a monotone set function on $\mathcal{D}$ vanishing in the empty set, i.e. a fuzzy measure fulfills 1) and 2). If additionally the condition 3) is fulfilled, m is called a continuous fuzzy measure. The integration with respect to a fuzzy measure is provided using the Choquet integral,

$$(C) - \int f \,\mathrm{d}\boldsymbol{m} = \int_0^\infty \boldsymbol{m}(f \geq r)\,\mathrm{d}r\,,$$

where f is a nonnegative $\mathcal{D}$-measurable function and the right-hand side integral is an ordinary Lebesque integral. Note that in 1978, Š i p o š [32] introduced an integration theory with respect to a pre-measure, which is independent of both Lebesque and Choquet integrals. A premeasure coincides with a fuzzy measure and the Šipoš integral is a generalization of the Choquet integral (it is defined for any measurable function under some natural restrictions). For more details see, e.g., [32, 33, 34].

A large class of fuzzy measures posses some pseudo-additivity property, i.e., there is a pseudo-addition $\oplus$ such that for disjoint events A and B, A, B, $A \cup B \in \mathcal{D}$, we have $\boldsymbol{m}(A \cup B) = \boldsymbol{m}(A) \oplus \boldsymbol{m}(B)$. Often the continuity from below is supposed to be satisfied by $\boldsymbol{m}$. Take, e.g., $\oplus = \sup$ (supremum). Then we get a possibility measure. Pseudo-additive measures in a general framework were studied, e.g., by Murofushi and Sugeno [23] in 1987. Their integral is built similarly as the Lebesque integral, starting from simple functions and using the usual limit procedure. Some interesting result concerning this topic can be found, e.g., in [14].

If a pseudo-addition $\oplus$ is generated by an additive generator g, then (following Weber [38]) we will denote it by $\perp$ (see also Sections 4 and 6). The corresponding pseudo-additive measures are called $\perp$-decomposable measures. They form a proper subfamily of pseudo-additive measures and they were introduced by Weber [38] in 1984. Weber's integral with respect to a $\perp$-decomposable measure is based on the Lebesque integral with respect to $g \circ \boldsymbol{m}$. If $g \circ \boldsymbol{m}$ is an ordinary finite σ-additive measure, then Weber's approach coincides with that of Murofushi and Sugeno. Some more details can be found, e.g., in [22]. A similar but slightly modified approach was used also by Pap [28].

The last, most general integral with respect to a fuzzy measure was introduced by Murofushi and Sugeno in 1991 [26]. Under some restrictions on the range of functions and measures this integral includes both the Choquet integral and the Sugeno integral [35].

3. Triangular norms and conorms

The problem of finding appropriate connectives for the union and intersection of fuzzy sets has turned out to be an important issue from several points of view. At the fundamental level it must be solved in order to provide a sound basis to fuzzy set theory. The choice of a functional representation of a set-theoretic operation must be justified not only empirically but also axiomatically. Actually, most results on fuzzy set-theoretic operations are nothing but reinterpretation of results on functional equations (especially the associativity equations).

Let the intersection $\cap$ and the union $\cup$ of fuzzy sets be defined pointwise by means of binary operations $\boldsymbol{T}$ and $\boldsymbol{S}$ on the unit interval [0,1]. It is natural to require the commutativity, the associativity and the monotonicity (non-decreasingness) of both intersection and union, and hence of both $\boldsymbol{T}$ and $\boldsymbol{S}$. Further, we put $\boldsymbol{T}(a,1) = a$ (this corresponds to $A \cap \boldsymbol{X} = A$ in the ordinary set theory) and $S(0,a) = a$ (from $A \cup \emptyset = A$) for any $a \in [0,1]$. But then T is a triangular norm, shortly a t-norm. Similarly, S is a triangular conorm, shortly a t-conorm. Note that the concept of a triangular norm goes back to 1942 to Menger [17], and it was introduced in the present form by Schweizer and Sklar in 1960 [30].

Let T be a given t-norm. Then

$$S_T(a,b) = 1 - T(1-a, 1-b)$$

defines a t-conorm S_T. In the same way, any t-conorm S induces a t-norm $\underset{S}{T}$. There holds $\underset{S_T}{T} = T$ and $S_{\underset{S}{T}} = S$, i.e., there is a one-to-one correspondence between t-norms and t-conorms. A couple (T,S), where $S = S_T$ (or equivalently $T = \underset{S}{T}$) is called a pair of dual t-norm and t-conorm. Given a t-norm T and its dual t-conorm S their associativity allows to extend them to n-ary operations on the unit interval. There holds that $S_{i=1}^n a_i = 1 - \overset{n}{\underset{i=1}{T}} (1-a_i)$. For any sequence $\{a_i\}_{n \in \mathbb{N}}$ in [0,1] the sequence $\{\overset{n}{\underset{i=1}{T}} a_i\}$ is nonincreasing; therefore its limit $\underset{n \in \mathbb{N}}{T} a_n$ always exists. Again the duality of T and S is preserved. If no confusion is possible, we will use a shorter notation form $T a_n$ and $S a_n$.

In what follows we will deal only with (Borel-)measurable t-norms and t-conorms. A t-norm T is called strict if it is continuous and strictly increasing, i.e. $T(a,b) < T(a,c)$ for any $a \in]0,1]$ whenever $0 \leq b < c \leq 1$. T is called Archimedean if for any $a, b \in]0,1[$ there is an integer n such that $\overset{n}{\underset{i=1}{T}} a_i < b$, where $a_i = a$, $i = 1, 2, \ldots, n$. If T is continuous, then it is Archimedean if and only if $T(a,a) < a$ for any $a \in]0,1[$. A dual of a strict (Archimedean) t-norm is called a strict (Archimedean) t-conorm. It is evident that any strict t-norm is Archimedean t-norm, too (the opposite implication is false). Continuous Archimedean t-norms which are not strict are called nilpotent. Note that t-norms and t-conorms are widely used not only in the probabilistic metric theory and the fuzzy set theory, but also in the evaluation procedures for weights in the artificial intelligence.

There are many examples of t-norms and t-conorms. A most complete list of them is contained in Mizumoto's paper [21] from 1989. As dual t-conorms are easy to be derived from their corresponding t-norms, we give some examples of t-norms only.

A most important family of t-norms is the Frank family of fundamental t-norms $\{\underset{s}{T}, s \in [0,\infty]\}$, see [6]:

$$\begin{aligned}\underset{s}{T}(a,b) &= \min(a,b)\,, && \text{if } s=0\,,\\ &= a.b\,, && \text{if } s=1\,,\\ &= \max(0, a+b-1)\,, && \text{if } s=\infty\,,\\ &= \log\big(1+(s^a-1)\cdot(s^b-1)/(s-1)\big)\,, && \text{otherwise}.\end{aligned}$$

Note that $\underset{0}{T}$ and its dual S_0 lead to the originally Zadeh fuzzy intersection and union. $\underset{1}{T}$ is product and S_1 is the probabilistic sum $(S_1(a,b) = a+b-a\cdot b)$, $\underset{\infty}{T}$ is the bounded (bold) product and S_∞ is the bounded (bold) sum. All $\underset{s}{T}$ are continuous, $\underset{0}{T}$ is the only non-Archimedean fundamental t-norm, $\underset{\infty}{T}$ is the only nilpotent (i.e. Archimedean nonstrict) fundamental t-norm, all $\underset{s}{T}$ for $s \in]0,\infty[$ are strict t-norms. Frank's family is continuous in the sense that $\lim_{s\to t} \underset{s}{T} = \underset{t}{T}$. Further, this family is decreasing, i.e., $\underset{s}{T} \geq \underset{t}{T}$ whenever $s \leq t$. Each dual pair $(\underset{s}{T}, S_s)$ satisfies the functional equation

$$T(a,b) + S(a,b) = a+b \quad \text{for any} \quad a,b \in [0,1]\,. \tag{1}$$

Another important t-norm is $\underset{w}{T}$,

$$\begin{aligned}\underset{w}{T}(a,b) &= \min(a,b)\,, && \text{if} \quad \max(a,b)=1\,,\\ &= 0\,, && \text{otherwise}.\end{aligned}$$

$\underset{w}{T}$ is Archimedean, but it is not continuous. It is the "smallest" t-norm, and the fundamental t-norm $\underset{0}{T}$ is the "largest" t-norm, i.e., for any t-norm T we have

$$\underset{w}{T} \leq T \leq \underset{0}{T}\,.$$

The only strict t-norms which can be expressed as rational functions were found by Hamacher [7]:

$$H_\gamma(a,b) = \frac{a.b}{\gamma + (1-\gamma)(a+b-a\cdot b)}\,, \qquad \gamma \geq 0\,.$$

Note that $H_1 = \underset{1}{T}$ and that $\lim_{\gamma\to\infty} H_\gamma = \underset{w}{T}$.

Other interesting families of t-norms are, e.g., Yager's nilpotent t-norms [39],

$$Y_q(a,b) = \max\Big(0, 1 - \big((1-a)^q + (1-b)^q\big)^{\frac{1}{q}}\Big), \qquad 0 < q < \infty$$

(here $Y_1 = \underset{\infty}{T}$, $Y_0 = \lim_{q\to 0} Y_q = \underset{w}{T}$, $Y_\infty = \lim_{q\to\infty} Y_q = \underset{0}{T}$), and the family of Dubois and Prade [5] of non-Archimedean continuous t-norms

$$D - \alpha(a,b) = a \cdot b/\alpha \qquad \text{if } a, b \leq \alpha$$

$$= \min(a,b) \qquad \text{otherwise} \quad , \alpha \in [0,1[$$

(here $D_0 = \underset{0}{T}$ and $D_1 = \lim_{\alpha\to 1} D_\alpha = \underset{1}{T}$).

The only distributive dual pair (T, S) (i.e. for any triple A, B, C of fuzzy subsets of a universe X there holds $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, where $\cap$ and $\cup$ are induced by T and S) is $(\underset{0}{T}, S_0)$. It means that only the original Zadeh's fuzzy intersection (= min) and union (= max) satisfy the distributivity properties.

The following results on the structure of t-norms (and by duality of t-conorms) are based mostly on the results of A c z e l [1] and L i n g [15].

THEOREM 3.1. *Any strict t-norm T is isomorphic to the product $\underset{1}{T}$, i.e., there is a continuous strictly increasing function (a multiplicative generator) $\phi : [0,1] \to [0,1]$, $\phi(0) = 0$, $\phi(1) = 1$, such that*

$$T(a,b) = \phi^{-1}\big(\phi(a) \cdot \phi(b)\big).$$

Similarly any strict t-conorm S is isomorphic to the probabilistic sum S_1, $S_1(a,b) = a + b - a \cdot b$. □

Note that ϕ^r, $r > 0$, induces the same T as ϕ does. If we put $f(x) = -\log(\phi(x))$., then any strict t-norm T can be represented in the following form:

$$T(a,b) = f^{-1}\big(f(a) + f(b)\big),$$

where $f : [0,1] \to [0,\infty]$ (continuous, strictly decreasing, $f(0) = +\infty$, $f(1) = 0$) is called an additive generator of T. Similarly any strict t-conorm S is generated by an additive generator $h : [0,1] \to [0,\infty]$ (continuous, strictly increasing, $h(0) = 0$, $h(1) = +\infty$), $S(a,b) = h^{-1}\big(h(a) + h(b)\big)$.

If T and S form a dual pair and if f is an additive generator of T, then $h(x) = f(1-x)$ is an additive generator of S. Note that the additive generators are determined uniquely by $T(S)$ up to a positive multiplicative constant.

Some examples:

1) for Frank's family $\{\underset{s}{T}\}$ we have $\phi_s(x) = (s^x - 1)/(s-1)$; $f_s(x) = \log\big((s-1)/(s^x-1)\big)$ for $s \in]0,1[\cup]1,\infty[$; $\phi_1(x) = x$ and $f_1(x) = -\log x$.

2) for Hamacher's family $\{H_\gamma\}$ we have $\phi_\gamma(x) = \dfrac{x}{\gamma+(1-\gamma)x}$; $f_\gamma(x) = \log(\gamma/x + 1 - \gamma)$ for $\gamma \in]0,\infty[$; $\phi_0(x) = \exp(\frac{x-1}{x})$ and $f_0(x) = \dfrac{1-x}{x}$.

THEOREM 3.2. *Nilpotent t-norms (t-conorms) are isomorphic to the bold (bounded) product* $\underset{\infty}{T}$ *(bold (bounded) sum* S_∞ *).* □

Similarly as strict operations are, also nilpotent operations are generated by additive generators. While for strict operations these generators are unbounded, for the nilpotent operations the additive generators are bounded. But then we have to replace the inverses of generators by their quasi-inverses. Any nilpotent t-norm T is generated by a continuous strictly decreasing generator $f : [0,1] \to [0,M]$, $M \in]0,\infty[$, via

$$T(a,b) = \overline{f}^{-1}\big(f(a) + f(b)\big),$$

where $\overline{f}^{-1}(x) = f^{-1}\big(\min(M,x)\big)$ for $x \in]0,\infty[$. Note that the bounded product $\underset{\infty}{T}$ is generated by $f(x) = 1-x$, $x \in [0,1]$. Similarly any nilpotent t-conorm S is generated by an increasing continuous generator $h : [0,1] \to [0,M]$ via

$$S(a,b) = \overline{h}^{-1}\big(h(a) + h(b)\big),$$

where $\overline{h}^{-1}(x) = h^{-1}(\min(M,x))$, $x \in]0,\infty[$. The bounded sum S_∞ is generated by $h(x) = x$, $x \in [0,1]$. Again the generators are unique up to a positive multiplicative constant and the duality of t-norms and t-conorms corresponds to the duality of generators $(h(x) = f(1-x))$.

EXAMPLE. For the Yager family of nilpotent t-norms $\{Y_q\}$ we have

$$f_q(x) = (1-x)^q, x \in [0,1], q \in]0,\infty[\,.$$

Of course, non-Archimedean t-norms (t-conorms) are not generated—this is, e.g., the case of $\{D_\alpha\}$, $\alpha \in [0,1[$. The same is true for non-continuous t-norms (t-conorms), such as $\underset{w}{T}$. From the practical point of view, the most important t-norms and t-conorms are those which are continuous. Their structure is fully described by the following result of Alsina, Trillas and Valverde [2] from 1983.

THEOREM 3.3. *T is a continuous t-norm if and only if T is an ordinal sum of continuous Archimedean t-norms $\{\underset{(i)}{T}\}_{i\in I}$, i.e., there is a mutually disjoint system $\{]\alpha_i,\beta_i[\}_{i\in I}$ of open subintervals of the unit interval $[0,1]$ such that*

$$T(a,b) = \alpha_i + (\beta_i - \alpha_i)\cdot \underset{(i)}{T}\big((a-\alpha_i)/(\beta_i-\alpha_i),(b-\alpha_i)/(\beta_i-\alpha_i)\big), \qquad (2)$$

if both a and b are contained in some $]\alpha_i,\beta_i[$,

$$= \min(a,b), \qquad \textit{otherwise}.$$

□

Let f_i be an additive generator of $\underset{i}{T}$ for $i \in I$. Denote by p_i a continuous strictly decreasing function on $[\alpha_i,\beta_i]$ defined through

$$p_i(x) = f_i\big((x-\alpha_i)/(\beta_i-\alpha_i)\big),$$

and by $\overline{p}_i^{\,-1}$ its inverse (quasi-inverse) defined through

$$\overline{p}_i^{\,-1}(x) = \alpha_i + (\beta_i-\alpha_i)\cdot f_i^{-1}\big(\min(x,f_i(0)\big), \quad x\in[0,\infty[.$$

Then the ordinal sum (2) can be rewritten in the next form:

$$T(a,b) = \overline{p}_i^{\,-1}\big(p(a)+p(b)\big), \qquad \text{if } a,b\in\,]\alpha_i,\beta_i[\text{ for some } i\in I,$$
$$= \min(a,b), \qquad \text{otherwise}.$$

The representation of continuous t-norms (or by duality of continuous t-conorms) allows to extend $T(S)$ even for uncountably many arguments $\{a_u\}_{u\in U}$,

$$\underset{u\in U}{T}\, a_u = \inf_{u\in U} a_u, \quad \text{if } \inf_{u\in U} a_u \notin \bigcup_{i\in I}\,]\alpha_i,\beta_i[$$

$$= \alpha_i, \quad \text{if } \inf_{u\in U} a_u \in\,]\alpha_i,\beta_i[\text{ for some } i\in I \text{ and}$$

$$U_i = \{u\in U, a_u\in\,]\alpha_i,\beta_i[\} \text{ is uncountable},$$

$$= \underset{u\in U}{T}\, a_u, \quad \text{if } \inf_{u\in U} a_u \in\,]\alpha_i,\beta_i[\text{ and } U_i \text{ is countable}.$$

R e m a r k 3.1. F r a n k [6] proved that the only dual pairs (T,S) of continuous t-norms and t-conorms solving Eq.(1) are the fundamental t-norms $\underset{s}{T}$ and their ordinal sums together with corresponding dual t-conorms. The only continuous Archimedean solutions of Eq.(1) are just $(\underset{s}{T},S_s)$ for $s\in]0,\infty]$.

4. Complementations

Let a complementation operation on fuzzy sets be pointwisely defined by a mapping $c : [0,1] \to [0,1]$. B e l l m a n and G i e r t z [3] suggested the following axioms as being natural for mapping c, which will be also called a complementation:

(C1) $c(1) = 0$ and $c(0) = 1$

(C2) c is a strictly decreasing continuous mapping

(C3) c is involution, i.e. $c(c(a)) = a$ for any $a \in [0,1]$.

If only (C1) and (C2) are fulfilled, then c is called a negation. Trillas [36] in 1979 has solved the functional equations (C1)–(C3).

THEOREM 4.1. *A mapping $c : [0,1] \to [0,1]$ is a complementation if and only if there is a continuous increasing generator $g : [0,1] \to [0,1]$, $g(0) = 0$ and $g(1) = 1$, such that*

$$c(a) = g^{-1}(1 - g(a)), a \in [0,1].$$

□

Hence any complementation c is isomorphic to c^*, $c^*(a) = 1 - a$, which induces the original Zadeh fuzzy complementation $A' = 1 - A$. Note hat any continuous increasing function g with $g(0) = 0$ can be taken as a complementation generator. If $g(1)$ differs from 1, we use the formula $c(a) = g^{-1}\big(g(1) - g(a)\big)$. A complementation generated by a generator g will be denoted by c_g. Hence $c^* = c_i$, where i is the identity on $[0,1]$. The crossover point s_{c_g} $(c_g(s_{c_g}) = s_{c_g})$ is defined by $g^{-1}(1/2)$ (or by $g^{-1}\big(g(1)/2\big)$). For a given complementation c, the corresponding generator g may be chosen arbitrarily on $[0, s_c]$ ($g(0) = 0$, g is continuous and increasing). On $]s_c, 1]$, g is defined by $g(x) = 1 - g\big(c(x)\big)$. For example, Zadeh's complementation c^* is generated not only by the identity i on $[0,1]$, but also by $g(x) = 1 - \cos(\Pi x)$.

S u g e n o [36] introduced a class of complementations $\{c_\lambda\}_{\lambda > -1}$, $c_\lambda(a) = \frac{1-a}{1+\lambda a}$. Note that these are the unique rational complementations. Sugeno's complementations are generated, e.g., by

$$g_\lambda(x) = \log(1 + \lambda x)/\log(1 + \lambda), \quad \lambda > -1, \ \lambda \neq 0, \quad x \in [0,1].$$

If we take $g(x) = x^2$, then the corresponding complementation is

$$c_g(a) = (1 - a^2)^{1/2}.$$

Any complementation c establishes a one-to-one correspondence between t-norms and t-conorms in a similar way as Zadeh's complementation c^*. A couple $(\boldsymbol{T},\boldsymbol{S})$ is called a pair of g-dual t-norm and t-conorm, if $\boldsymbol{S}=\boldsymbol{S}_{\boldsymbol{T}}^{(g)}$, i.e. $\boldsymbol{S}(a,b)=c_g\Big(\boldsymbol{T}\big(c_g(a),c_g(b)\big)\Big)$ for any $a,b\in[0,1]$. It is evident that then $\boldsymbol{T}=\overset{(g)}{\underset{\boldsymbol{S}}{\boldsymbol{T}}}$, i.e. $\boldsymbol{T}(a,b)=c_g\Big(\boldsymbol{S}\big(c_g(a),c_g(b)\big)\Big)$. Further note that g-duality preserves continuity, strictness and Archimedean property. Hence a given couple $(\boldsymbol{T},\boldsymbol{S})$ need not be a g-dual pair for any g. The same is true even if both $\boldsymbol{T}$ and $\boldsymbol{S}$ are strict. On the other hand, there are some $\boldsymbol{T}$ and $\boldsymbol{S}$, which are dual for more complementations. The pairs $(\underset{0}{\boldsymbol{T}},\boldsymbol{S}_0)$ and $(\underset{w}{\boldsymbol{T}},\boldsymbol{S}_w)$ (i.e. the limit cases) are g-dual for any g. Hamacher's t-norm $\boldsymbol{H}_0$ induces the same t-conorm using any of Sugeno's complementations c_λ, $\lambda>-1$.

From now, if we will speak about fuzzy sets, we will define the fuzzy connectives pointwisely by a complementation operator c_g and by a pair of g-dual t-norm $\boldsymbol{T}$ and t-conorm $\boldsymbol{S}$. Hence the De Morgan laws will be fulfilled, $(A\cup B)^c=A^c\cap B^c$ and $(A\cap B)^c=A^c\cup B^c$.

Let g be an additive generator on $[0,\infty]$, i.e. g is a continuous strictly increasing function, $g(0)=0$ and $g(\infty)=\infty$. Then $g/[0,1]$ is a complementation generator, $c_g(a)=g^{-1}(g(1)-g(a))$ for $a\in[0,1]$. Further, g induces a pseudo-addition $\oplus_g$ 0n $[0,\infty]$ via

$$u\oplus_g v=g^{-1}(g(u)+g(v)),\quad u,v\in[0,\infty].$$

If $g=i$ is the identity, then this pseudo-addition turns to be an ordinary addition on $[0,\infty]$. We have proved the following result [19]:

THEOREM 4.2. *The only continuous g-dual solutions $(\boldsymbol{T},\boldsymbol{S})$ of the functional equation,*

$$\boldsymbol{T}(a,b)\oplus_g\boldsymbol{S}(a,b)=a\oplus_g b$$

are the pairs of g-fundamental t-norms and t-conorms $\{(\underset{s,g}{\boldsymbol{T}},\boldsymbol{S}_{s,g})\}$, $s\in[0,\infty]$, and their ordinal sums, where $\underset{s,g}{\boldsymbol{T}}(a,b)=g^{-1}(\underset{s}{\boldsymbol{T}}(g(a),g(b)))$ and $\boldsymbol{S}_{s,g}(a,b)=g^{-1}\Big(\boldsymbol{S}_s\big(g(a),g(b)\big)\Big)$. □

Recall that $\underset{s}{\boldsymbol{T}}=\underset{s,i}{\boldsymbol{T}}$, $s\in[0,\infty]$, are Frank's fundamental t-norms. Further, $\underset{0,g}{\boldsymbol{T}}=\underset{0}{\boldsymbol{T}}$ and $\boldsymbol{S}_{0,g}=\boldsymbol{S}_0$ for any g.

The majority of notions and results for a general complementation $c=c_g$ differs from those for Zadeh's complementation $c^*=c_i$ only by the transformation g (see, e.g., [18]). This is the reason why the mostly used complementation is original Zadeh's c^*.

5. Fuzzy tribes

The generalization of Cantorian subsets to the fuzzy subsets (of a universe X) have brought on the problem of generalization of the measurability concept. In short, how to generalize the σ-algebras in the fuzzy theory? In his pioneer work from 1965 [40], Z a d e h dealt with the system $\mathcal{F}(X)$ of all fuzzy subsets of X, i.e. with the power set, $\mathcal{F}(X) = [0,1]^X$. In 1968 [41], Zadeh introduced a generated fuzzy σ-algebra. Let $(X, \mathcal{A})$ be a classical measurable space, i.e. $\mathcal{A}$ is a σ-algebra of crisp subsets of X. The system $\mathcal{F}(\mathcal{A})$ of all $\mathcal{A}$-measurable fuzzy subsets of X is called a *generated fuzzy σ-algebra*, or a *generated tribe.* Its members are called *fuzzy events.* It is obvious that $\mathcal{F}(\mathcal{A})$ is closed under any complementation of fuzzy subsets. However, it might not be closed under fuzzy unions (intersections) induced by a t-conorm S (t-norm T), see e.g [11]. The measurability of S (and hence of its dual T) excludes this failure. This is the reason why we will deal with measurable t-conorms and t-norms only.

In 1979, K h a l i l i [8] defined first axiomatically a *fuzzy σ-algebra* $\tau \subseteq \mathcal{F}(X)$ as a system of fuzzy subsets of X fulfilling the next axioms:

A1) $0_X \in \tau$,

A2) $A \in \tau$ implies $A' = 1 - A \in \tau$,

A3) $\{A_n\}_{n\in\mathbb{N}} \subset \tau$ implies $\sup_{n\in\mathbb{N}} A_n$ $(= S_0 A_n) \in \tau$.

Note that for a constant $a \in [0,1]$, a_X denotes a constant fuzzy subset of X, $a_X(x) = a$ for any $x \in X$. It is evident that the Khalili approach is a straightforward generalization of the notion of a σ-algebra.

K l e m e n t [9] in 1980 defined a fuzzy σ-algebra in a similar way as Khalili, only the first axiom was stronger, namely

A1*) for any $a \in [0,1]$, $a_X \in \tau$. Note that after some time, Khalili's approach was adopted in the fuzzy probability theory. In 1982, Klement [11] introduced a notion of a *T-fuzzy σ-algebra.* Nowadays, it is called a *T-tribe*, see, e.g., [4]. Let T be a t-norm and let S be its dual t-conorm. A fuzzy σ-algebra τ is closed under Zadeh's fuzzy unions (i.e. under S_0) and hence under the corresponding fuzzy intersection (i.e. under T_0). Let T be a t-norm and let S be its dual t-conorm. A T-tribe $\tau \subset \mathcal{F}(X)$ fulfills the axioms A1) and A2) and is closed under (countable) fuzzy intersection induced by T (we will denote this fuzzy intersection also by T). By the duality and A2), τ is closed under (countable) fuzzy union S, too. Hence a T-tribe τ fulfills A1), A2) and A3*),

A3*) $\{A_n\}_{n\in\mathbb{N}} \subset \tau$ implies $(SA_n) \in \tau$.

In this notation, Khalili's fuzzy σ-algebras correspond to the T_0-tribes.

The second axiom A2) can be generalized replacing Zadeh's complementation c^* by a general complementation $c = c_g$ (then T and S are supposed to be

g-dual),

A2*) $A \in \tau$ implies $A^c = c(A) \in \tau$.

A system $\tau \subset \mathcal{F}(X)$ fulfilling A1), A2*) and A3*) is called a *g-T-tribe*, see, e.g., [19].

In 1985, P i a s e c k i [29] introduced the concept of a *soft fuzzy σ-algebra* $\mathcal{M}$. $\mathcal{M}$ is supposed to be Khalili's fuzzy σ-algebra, i.e., it fulfills A1) – A3). Piasecki defined a W–empty set $A \in \mathcal{M}$ as a fuzzy subset fulfilling $A \leq A'$ (i.e., A is contained in its complement). Similarly a W–universe $B \in \mathcal{M}$ contains its complement, i.e. $B \geq B'$. It wouldn't be sound to admit the existence of such elements in $\mathcal{M}$, which are simultaneously a W-empty set and a W–universe. This idea led Piasecki to the fourth axiom for soft fuzzy σ-algebras, namely

A4) $(1/2)_X \notin \mathcal{M}$.

Now, we recall some results for T-tribes. They can be found, e.g., in [4, 11, 19, 20].

THEOREM 5.1.

a) *Let τ be a T_s-tribe for some $s \in]0, \infty[$. Then τ is a T_∞-tribe, too.*

b) *Any T_∞-tribe τ is also a T_0-tribe, i.e. a fuzzy σ-algebra.*

c) *Let $\tau \subset \mathcal{F}(X)$ be a T_s-tribe for some $s \in]0, \infty]$ and let τ contain all constant fuzzy subsets of X (i.e. τ fulfills (A1*))). Then τ is a generated tribe, $\tau = \mathcal{F}(A)$ fore some σ-algebra $\mathcal{A}$.*

□

It is obvious that a generated tribe is a T-tribe for any (measurable) t-norm T. The opposite assertion is not true. In [20], we have introduced a *semigenerated tribe* $\tau \subset \mathcal{F}(X)$ as a T_0-tribe whose restriction τ/Y (for some crisp subset $Y \subset X$) is a generated tribe on Y and $\tau/(X - Y)$ is a σ-algebra of crisp subsets of $X - Y$. It is evident that any semigenerated tribe τ is a T-tribe, too, and that the generated tribes form a proper subfamily of semigenerated tribes. Our last result on this topic is the following (see [20]):

THEOREM 5.2. *Let X be a denumerable universe. Let $\tau \subset \mathcal{F}(X)$. Then the following statements are equivalent:*

i) *τ is a semigenerated tribe*

ii) *τ is a T-tribe for any (measurable) t-norm T*

iii) *τ is a T_0-tribe and a T-tribe for some strict t-norm T*

iv) *T is a T_s-tribe for some $s \in]0, \infty[$.*

□

For a general universe X, the structure of subfamilies $\tau \subset \mathcal{F}(X)$, which are T-tribes for any T, is an open problem.

EXAMPLE 5.1. Let $X = \{x\}$ be a singleton. Then $\mathcal{F}(X) = [0,1]$ is the only generated tribe. There are two semigenerated tribes, $[0,1]$ and $\{0,1\}$. These semigenerated tribes are the only T_s-tribes, $s \in]0,\infty[$. There are countably many T_∞-tribes: $[0,1], \{0,1\}$ and $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$, $n = 2,3,\ldots$. There are uncountably many T_0-tribes:

$$\mathcal{F}_H = \{a \in [0,1], a \in H \quad \text{or} \quad (1-a) \in H\},$$

where H is any subset of $[0,\frac{1}{2}]$ containing zero.

EXAMPLE 5.2. [11] Let $X = [0,1]$, $\tau = \{a_X, a \in [0,1]\} \cup [1/3, 2/3]^X$. Then τ is both a T_0-tribe and a T_w-tribe, it contains all constant fuzzy subsets of X, but it is not a T_s-tribe for any $s \in]0,\infty]$. If we require further the continuity of elements of τ, then τ is only a T_w-tribe, but it is not more a T_0-tribe. Hence there are T-tribes, containing all constant fuzzy subsets, which are not closed under inf and sup.

6. Measures of fuzzy events

The first step in defining a measure of fuzzy events was made in 1968 by Zadeh [41]. Let $(X, \mathcal{A}, P)$ be a classical Kolmogorovian probability space. Let $\mathcal{F}(\mathcal{A})$ be a generated tribe of fuzzy subsets of X. A mapping $m : \mathcal{F}(A) \to [0,1]$ defined via

$$A \in \mathcal{F}(\mathcal{A}) : m(A) = \int_X A(x)\,\mathrm{d}P(x), \tag{3}$$

where the right-hand side is a Lebesque-Stiltjes integral, is called a (Zadeh) *fuzzy probability measure*. Hence m defined through (3) is an ordinary mean value functional.

The next steps were made mainly by E. P. Klement. In 1980 [9] he defined axiomatically a fuzzy probability measure m on a fuzzy σ-algebra (i.e. on a T_0-tribe) τ as a mapping fulfilling

M1) $m(0_X) = 0$ and $m(1_X) = 1$ (boundary conditions)
M2) $m(A \vee B) + m(A \wedge B) = m(A) + m(B)$ for any $A, B \in \tau$ (valuation)
M3) $\{A_n\} \subset \tau$, $A_n \uparrow A \Rightarrow m(A_n) \uparrow m(A)$ (left-continuity)

Klement proved the following result:

THEOREM 6.1. *Let $\tau = \mathcal{F}(\mathcal{A})$ be a generated tribe on X. A mapping m is a fuzzy probability measure on τ if and only if there is uniquely determined probability P on $\mathcal{A}(P = m/\mathcal{A})$ and a P-a.e. unique Markov-kernel K such that*

$$\forall A \in \tau\colon\ m(A) = \int_X K(x, [0, A(x)[\,)\, dP(x)\,. \tag{4}$$

Recall that here a Markov-kernel K is a mapping, $K : X \times \mathcal{B}([0,1[) \to [0,1]$, such that $K(\circ, E)$ is an A-measurable function for any Borel subset E of the interval $[0,1[, E \in \mathcal{B}([0,1[)$, and $K(x,\cdot)$ is a probability distribution on $\mathcal{B}([0,1[)$ for any $x \in X$. Hence K is a (measurable) family of probability distributions. Note that for Zadeh's fuzzy probability measures, $K(x,\cdot)$ is the uniform distribution on $[0,1[$ for any $x \in X$. Klement, Lowen and Schwyhla [10] showed the necessary and sufficient conditions for a fuzzy probability measure to be a Zadeh fuzzy probability measure.

THEOREM 6.2. *Let m be a fuzzy probability measure on a generated tribe $\tau = \mathcal{F}(\mathcal{A})$. Then the following conditions are equivalent:*

1) *m is a Zadeh fuzzy probability measure;*
2) *for any $\alpha \in [0,1[$ and $A \in \tau$: $\lim_{\beta\downarrow\alpha}(m(A \wedge \beta_X) - m(A \wedge \alpha_X))/(\beta - \alpha) = P(D)$, where $P = m/\mathcal{A}$ and $D = \{x \in X : A(x) > \alpha\} \in \mathcal{A}$;*
3) *m is linear, i.e. if for some reals u and v and some fuzzy events $A, B \in \tau$ it is $(u \cdot A + v \cdot B) \in \tau$, then*

$$m(u \cdot A + v \cdot B) = u \cdot m(A) + v \cdot m(B)\,;$$

4) *m is additive, i.e. if A, B and $(A + B) \in \tau$, then*

$$m(A + B) = m(A) + m(B)\,;$$

5) *m is homogenous, i.e. if $u \in R, A$ and $u \cdot A \in \tau$, then*

$$m(u \cdot A) = u \cdot m(A)\,.$$

□

In the axiom M2), i.e., in the valuation property, the maximum $\vee$ (i.e. S_0) and the minimum $\wedge$ (i.e. T_0) can be replaced by a t-conorm S and its dual t-norm T. Hence we obtain a T-valuation property

M2T) $m(ASB) + m(ATB) = m(A) + m(B)$ for any $A, B \in \tau$.

Klement in 1982 [11] introduced a T-*measure* m defined on a T-tribe τ as a mapping satisfying M1) (here $m(1_X) = 1$ can be generalized to be a positive real), M2T) and M3) (here we have to suppose $\sup A_n \in \tau$, see Example 5.2.). If a T-tribe τ is also a T_0-tribe, then any T-measure m on τ is also a T_0-measure(i.e. if $m(1_X) = 1$),it is a fuzzy probability measure). A short proof of this fact is due to Schwyhla:

$$m(A \vee B) + m(A \wedge B) = m(T(A \vee B, A \wedge B)) + m(S(A \vee B, A \wedge B)) =$$
$$= m(T(A,B)) + m(S(A,B)) = m(A) + m(B).$$

Here the first and the last equalities are true because of the T-valuation property of m, the second equality follows from the simple fact $\{(A\vee B)(x), (A\wedge B)(x)\} = \{A(x), B(x)\}$ for any $x \in X$. It follows that (if τ is a generated tribe) a T-measure m is representable as an integral of a Markov-kernel K with respect to a measure $M = m/\mathcal{A}$, $\tau = \mathcal{F}(\mathcal{A})$, similarly as in (4). If $T = T_s$ is a fundamental t-norm, Klement [11] and B u t n a r i u with Klement [4] have proved the following results:

THEOREM 6.3. *Let $\tau = \mathcal{F}(\mathcal{A})$ be a generated tribe. Let m be a T_s-measure on τ for some $s \in]0, \infty[$. Then there is a unique finite measure M ($M = m/\mathcal{A}$) and an element $B \in \tau$ (M-a.e. unique) such that*

$$\forall A \in \tau : m(A) = \int_{\{A>0\}} (B(x) + (1 - B(x)) \cdot A(x))\, dM(x). \tag{5}$$

□

It is evident that Theorem 6.3. is true for semigenerated tribes τ, too. If $T = T_\infty$, then the axioms M2T) and M3) are equivalent to the σ-additivity, i.e. if $\Sigma A_n \leq 1$, then $m(\Sigma A_n) = \Sigma m(A_n)$. Butnariu (see e.g. [4]) showed that any T_∞-measure is representable in the integral form (2).

THEOREM 6.4. *Let τ be a T_∞-tribe of fuzzy subsets of X and let m be a T_∞-measure on τ. Then there is a unique finite measure M on a σ-algebra $\mathcal{A}$ of all crisp subsets of τ (then $M = m/\mathcal{A}$) such that*

$$\forall A \in \tau : \; m(A) = \int_X A(x)\, dM(x).$$

□

For a general T_s-tribe $\tau, s \in]0, \infty[$, Butnariu and Klement [4] proved in 1991 the following decomposition theorem for T_s-measures:

THEOREM 6.5. *Let* m *be a* $\underset{s}{T}$*-measure on a* $\underset{s}{T}$*-tribe* $\tau, s \in]0,\infty[$*, of fuzzy subsets of* X*. Then* m *can be uniquely decomposed in a monotonically irreducible* $\underset{s}{T}$*-measure* m^* *on* τ *and a generated measure* $\overline{m}$ *on* τ*,*

$$m = m^* + \overline{m}.$$

□

Recall that any $\underset{s}{T}$-tribe $\tau, s \in]0,\infty]$, is contained in a minimal generated tribe $\mathcal{F}(\mathcal{A})$, where $\mathcal{A}$ is a σ-algebra of all crisp subsets of τ. Let M be a finite measure on $\mathcal{A}$ and let $B, C \in \mathcal{F}(\mathcal{A})$. A mapping $\overline{m} : \tau \to [0,\infty[$ defined via

$$\forall A \in \tau:\ \overline{m}(A) = \int\limits_{\{A>0\}} \big(B(x) + C(x)\cdot A(x)\big)\, \mathrm{d}M(x)$$

is called a *generated measure*. A measure m is called monotonically irreducible (on τ) if $m^* - q$ is a monotone measure on τ for a generated measure q only if q is identically equal to zero.

R e m a r k 6.1. Recently we have shown [20] that for denumerable universes X, any $\underset{s}{T}$- measure $(s > 0)$ is generated and it fulfills (5). Our conjecture is that the same is true in general, it means that the monotonically irreducible measure m^* in Butnariu-Klement's decomposition is always identically equal to a zero measure. □

Generalizing the concept of $\perp$-decomposable measures, K l e m e n t and W e b e r [13] introduced in 1991 a *$\perp$-decomposable measure of fuzzy events.* Let $\perp$ be a pseudo-addition on $[0,\infty]$ generated by an additive generator $g : [0,\infty] \to L[0,M]$, $M \in [0,\infty]$, i.e.

$$u \perp v = g^{-1}\Big(\min(M, g(u) + g(v))\Big), \quad u, v \in [0,\infty].$$

Note that $\perp$ is a continuous Archimedean t-conorm on $[0,\infty]$.Let $\tau = \mathcal{F}(\mathcal{A})$ be a generated tribe. A mapping $m : \tau \to L[0,M]$ fulfilling M1) (we replace $m(1_X) = 1$ by $m(1_X) \in]0,M]$,. $m(1_X) < \infty$), M3) and

M2$\perp$) $m(A \wedge B) \perp m(A \vee B) = m(A) \perp m(B)$ for any $A, B \in \tau$ ($\perp$-valuation)

is called a $\perp$-decomposable measure of fuzzy events. Klement and Weber have shown the integral representation of these measures by Markov kernels using an integral with respect to $\perp$-decomposable measures due to W e b e r [38].

Comparing T-measures and $\perp$-decomposable measures of fuzzy events we see that the only difference is in the type of valuation property. We have proposed [19] the following generalization including both T-measures and $\perp$-decomposable measures of fuzzy events.

DEFINITION 6. 1. Let τ be a generated tribe (it is enough to be a g-$\boldsymbol{T}$-tribe), g an additive generator on $[0,\infty]$ and $(\boldsymbol{T},\boldsymbol{S})$ a g-dual pair of a t-norm and a t-conorm.Recall that g generates a complementation c_g, $c_g(a) = g^{-1}(g(1) - g(a))$. Let $\perp$ be a pseudo-addition generated by g. A mapping $\boldsymbol{m}: \tau \to [0,M]$ will be called a $\perp$-*decomposable* $\boldsymbol{T}$-*measure* iff $\boldsymbol{m}$ fulfills

M1*) $\boldsymbol{m}(0_X) = 0$,

M2$\boldsymbol{T}\perp$) $\boldsymbol{m}(\boldsymbol{T}(A,B)) \perp \boldsymbol{m}(\boldsymbol{S}(A,B)) = \boldsymbol{m}(A) \perp \boldsymbol{m}(B)$ for any $A,B \in \tau$ ($\boldsymbol{T}$-$\perp$-valuation)

and M3.

We expect that the study of $\perp$-decomposable $\boldsymbol{T}$-measures will exploit the techniques of both Butnariu-Klement's and Klement-Weber's approaches.

Remark 6.2. Piasečki [29] in 1985 investigated the fuzzy probability measures from the Bayes principle point of view. He showed that the only fuzzy probability measures fulfilling the Bayes principle are so called *fuzzy* $\boldsymbol{P}$-*measures.* Let $\mathcal{M}$ be a soft fuzzy σ-algebra. A mapping $\boldsymbol{p}: \mathcal{M} \to L[0,1]$ is called a fuzzy $\boldsymbol{P}$-measure if it fulfills two following axioms:

P1) $\boldsymbol{p}(A \vee A') = 1$ for any $A \in \mathcal{M}$

P2) $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$, $A_n \le A'_m$ whenever $n \neq m$ implies $\boldsymbol{p}(\vee A_n) = \Sigma \boldsymbol{p}(A_n)$.

Note that the W-disjointness ($A_n \le A'_m$)in the axiom P2) can be replaced by the weaker F-disjointness ($(A_n \wedge A_m)$ is a W-empty set). Some other generalizations on this topic can be found, e.g., in [18].

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