TRIANGULAR NORM–BASED ADDITION
OF LINEAR FUZZY NUMBERS

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ABSTRACT. The addition of linear fuzzy numbers based on triangular norms is studied. In the case of continuous Archimedean t-norms with strictly convex generators a necessary and sufficient condition for linearity of the t-norm-based sum is given.

1. Basic notions

Let us introduce the definitions and basic properties of finite fuzzy numbers and t-norms which will be used in the next part of the paper.

A *finite fuzzy number* is a convex normal fuzzy set \( p \) in the universum of real numbers \( \mathbb{R} \) which has a continuous membership function \( \mu_p \) and for which there exist numbers \( a, b \in \mathbb{R} \) and \( \alpha, \beta \in \mathbb{R}^+ \) such that

(i) \( \mu_p(x) = 1 \) if \( x \in [a,b] \) and \( \mu_p(x) = 0 \) if \( x \leq a - \alpha \) or \( x \geq b + \beta \).

(ii) \( \mu_p \) is increasing in the interval \( [a - \alpha, a] \) and decreasing in \( [b, b + \beta] \).

The interval \( [a, b] \) is the peak of the fuzzy number \( p \) and the interval \( [a - \alpha, b + \beta] \) is its support.

The membership function of a finite fuzzy number \( p \) can be expressed in the following form \(^2\)

\[
\mu_p(x) = \begin{cases} 
1, & \text{for } x \in [a,b], \\
L\left(\frac{a-x}{\alpha}\right), & \text{for } x \in [a - \alpha, a], \\
R\left(\frac{x-b}{\beta}\right), & \text{for } x \in [b, b + \beta], \\
0, & \text{otherwise,}
\end{cases}
\]

where \( L, R : [0,1] \rightarrow [0,1] \) are shape functions which are non-decreasing, continuous and \( L(0) = R(0) = 1, L(1) = R(1) = 0 \).

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For a finite fuzzy number $p$ we will use the notation $p = (a, b, \alpha, \beta)_{LR}$. A finite fuzzy number is said to be a linear fuzzy number if $L(x) = R(x) = 1 - x$, $x \in [0, 1]$. A linear fuzzy number $p$ will be denoted by $p = (a, b, \alpha, \beta)$.

Let us note that for $a \neq b$ we will get trapezoidal fuzzy numbers, and for $a = b$ triangular fuzzy numbers.

The original sum of fuzzy numbers $p, q$ has been defined by Zadeh's extension principle:

$$\mu_{p \oplus q}(z) = \sup_{z=x+y} (\mu_p(x) \land \mu_q(y)), \quad z \in \mathbb{R}. \quad (1)$$

If we use in (1) instead of the operation $\land = \min$, which is only a special kind of a t-norm, some t-norm $T$, we get the sum of fuzzy numbers based on the t-norm $T$:

$$\mu_{p \oplus q}^T(z) = \sup_{z=x+y} T(\mu_p(x), \mu_q(y)), \quad z \in \mathbb{R}, \quad (2)$$

or, in the modified form,

$$\mu_{p \oplus q}^{T'}(z) = \sup_{x \in \mathbb{R}} T(\mu_p(x), \mu_q(z - x)), \quad z \in \mathbb{R}. \quad (3)$$

Recall that a t-norm $T$ is a binary operation, $T: [0, 1] \times [0, 1] \rightarrow [0, 1]$, which is commutative, associative, non-decreasing in each argument and $T(x, 1) = x$ for each $x \in [0, 1]$. For the t-norm $\land = \min$ we will use the notation $T_M$.

Any continuous Archimedean t-norm $T$ (i.e., a continuous t-norm for which $T(x, x) < x$, $x \in (0, 1)$) can be represented by means of its additive generator $f$. Namely,

$$T(x, y) = f^{(-1)}(f(x) + f(y)) \text{ for each } x, y \in [0, 1],$$

where $f^{(-1)}$ is a pseudo-inverse of $f$ given by $f^{(-1)}(u) = f^{-1}(\min\{u, f(0)\})$. Therefore for a continuous Archimedean t-norm $T$ the $T$-sum of $p$ and $q$ can be expressed in the form:

$$\mu_{p \oplus q}^T(z) = \sup_{x \in \mathbb{R}} f^{(-1)}(f(\mu_p(x)) + f(\mu_q(z - x))), \quad z \in \mathbb{R}. \quad (4)$$

Let us note that the additive generator $f$ of a t-norm $T$ is a continuous, strictly decreasing function, $f: [0, 1] \rightarrow [0, \infty]$ with $f(1) = 0$.

2. Results

In [4] a membership function of a $T$-sum $\bigoplus_{i=1}^{n} p_i$ of finite fuzzy numbers $p_i = (a_i, b_i, \alpha, \beta)_{LR}$, $i = 1, 2, ..., n$, in the case of an Archimedean t-norm $T$ having a
strictly convex twice differentiable additive generator $f$ and fuzzy numbers with concave, twice differentiable shape functions $L, R$ is determined.

According to [4]:

$$
\mu_{\frac{p}{T}}(z) = \begin{cases} 
1, & \text{for } A \leq z \leq B, \\
 f^{-1}\left(\frac{n f\left(L\left(\frac{z}{nA}\right)\right)}{1-n}\right), & \text{for } A - nA \leq z \leq A, \\
 f^{-1}\left(\frac{n f\left(R\left(\frac{z-B}{n\beta}\right)\right)}{1-n}\right), & \text{for } B \leq z \leq B + n\beta, \\
0, & \text{otherwise},
\end{cases}
$$

(5)

where $A = \sum_{i=1}^{n} a_i$, $B = \sum_{i=1}^{n} b_i$.

In this paper we will be interested in such $T$-norms $T$ with strictly convex generators for which $T$-sums of linear fuzzy numbers are again linear fuzzy numbers. The following assertion gives a necessary condition for linearity of $T$-sums in such a case.

**Proposition 1.** Let $T$ be a $T$-norm with a strictly convex twice differentiable additive generator $f$. If the $T$-sum $p \oplus_T q$ of any linear fuzzy numbers $p, q$ is a linear fuzzy number, then

$$
f(x) = a(1-x)^s, \quad \text{for some } a \in \mathbb{R}^+, \quad s \in (1, \infty).
$$

**Proof.** Let $T$ be a $T$-norm with a strictly convex additive generator $f$ and let $p = q = (1, 1, 1, 1)$. If we take into account that the shape function $L$ of $p$ and $q$ is given by $L(x) = 1 - x$, $x \in [0, 1]$, then according to (5) for each $z \in [0, 2]$ it holds:

$$
\mu_{\frac{p}{T}\oplus_T q}(z) = f^{-1}\left(\frac{1}{2} - \frac{z}{2}\right) = f^{-1}\left(2f\left(\frac{z}{2}\right)\right).
$$

(6)

Since $p \oplus_T q$ is a linear fuzzy number, there exists a number $c \in [0, 2]$ such that:

$$
\mu_{\frac{p}{T}\oplus_T q}(z) = \frac{1}{2-c}(z-c), \quad z \in [c, 2].
$$

(7)

We exclude $c = 0$ from our considerations, since for $c = 0$ we have $T = T_M$. The $T$-norm $T_M$ has no additive generator and does not satisfy the assumption given in the proposition.

Comparing (6) and (7) we get

$$
f^{-1}\left(2f\left(\frac{z}{2}\right)\right) = \frac{z-c}{2-c}, \quad z \in (c, 2).
$$

(8)
ANNA KOLESÁROVÁ

Let us consider a one-to-one correspondence between intervals \((0, \infty)\) and \((0, 2)\) given by the mapping \(h, h(s) = 2 - 2^{s-1}, s \in (0, \infty)\).

Since for any \(c \in (0, 2)\) there exists a unique element \(s \in (0, \infty)\) such that \(c = 2 - 2^{s-1}\), we can rewrite (8) into the form:

\[
2f\left(\frac{z}{2}\right) = f\left(\frac{z - 2 + 2^{\frac{s-1}{s}}}{2^{\frac{s-1}{s}}}\right),
\]

or

\[
2f\left(\frac{z}{2}\right) = f\left(1 - \left(1 - \frac{z}{2}\right)^2\right)^{\frac{1}{s}}.
\]

Let us denote \(g(x) = f(1 - x)\). Then the previous formula can be written in the form:

\[
2g\left(1 - \frac{z}{2}\right) = g\left(1 - \frac{z}{2}\right)^{2^{\frac{1}{s}}},
\]

Putting \(1 - \frac{z}{2} = u\) and \(2^{\frac{1}{s}} = \lambda\), we get

\[
\lambda^s g(u) = g(u\lambda).
\] (9)

For a given \(s \in (0, \infty)\) the only continuous, strictly increasing, non-negative solutions of the functional equation (9) in the interval \((0, \infty)\) are functions \(g\) given by \(g(u) = au^s\) for some \(a \in \mathbb{R}^+\), see, e.g., [1].

Therefore we have

\[
f(x) = g(1 - x) = a(1 - x)^s \quad \text{for some } a \in \mathbb{R}^+ \text{ and } s \in (0, \infty).
\]

Since \(f\) is by the assumption a strictly convex, twice differentiable function, only the values \(s > 1\) are satisfactory. Note, that the values \(s \in (1, \infty)\) correspond to the values \(c \in (0, 1)\). \hfill \Box

In other words, we have just proved that if a t-norm \(T\) has a strictly convex, twice differentiable additive generator and the \(T\)-sum of any linear fuzzy numbers is a linear fuzzy number, then \(T\) must necessarily be a member of Yager’s family of t-norms for some \(s \in (1, \infty)\).

Recall that the family of t-norms \(\{T_s^Y\}_{s \in (0, \infty)}\), where

\[
T_s^Y(x, y) = \max\left\{0, 1 - \left[(1-x)^s + (1-y)^s\right]^{\frac{1}{s}}\right\} \quad \text{for } x, y \in [0, 1] \text{ and } s \in (0, \infty)
\]

was introduced by Yager in 1980 for modeling fuzzy intersection. The corresponding normed additive generators \(f_s^Y\) are given by \(f_s^Y(x) = (1 - x)^s\).

Now we show that the addition based on each Yager’s t-norm \(T_s^Y\), \(s > 1\) preserves linearity. Before proving this fact, let us make the following remark.

Remark 1. It is easy to see that the support of the sum \(p \oplus q\), where \(p = (a_p, b_p, \alpha_p, \beta_p)_{L_pR_p}\) and \(q = (a_q, b_q, \alpha_q, \beta_q)_{L_qR_q}\) is a subinterval of the interval

78
TRIANGULAR NORM-BASED ADDITION OF LINEAR FUZZY NUMBERS

\[ [a_p + a_q - \alpha_p - \alpha_q, b_p + b_q + \beta_p + \beta_q] \] and the peak of the sum for each t-norm \( T \) is the interval \([a_p + a_q, b_p + b_q]\). Moreover, according to the decomposition rule of finite fuzzy numbers into two separate parts [3], the left part of the membership function of the sum, i.e., its values for \( z \in [a_p + a_q - \alpha_p - \alpha_q, a_p + a_q] \), depend only on the left parts of \( \mu_p \) and \( \mu_q \). It means that in (2) to determine \( \mu_{p \oplus q}(z) \) for such \( z \), it is enough to use values \( x \in [a_p - \alpha_p, a_p] \) and \( y \in [a_q - \alpha_q, a_q] \).

The analogous assertion holds for the right side of \( \mu_{p \oplus q} \).

**PROPOSITION 2.** Let \( T^Y_s \), \( s > 1 \) be a Yager’s t-norm. Then the \( T^Y_s \)-sum of arbitrary linear fuzzy numbers \( p \) and \( q \) is a linear fuzzy number.

**Proof.** By Remark 1 it is sufficient to prove the assertion for triangular fuzzy numbers.

Fix \( s > 1 \). Let \( p = (a_p, a_p, \alpha_p, \beta_p) \), \( q = (a_q, a_q, \alpha_q, \beta_q) \) and \( p \oplus q = r \), where \( T^Y_s \) is the Yager t-norm. For \( z \leq a_p + a_q \) it holds

\[
\mu_r(z) = \sup_{x \in \mathbb{R}} T^Y_s(\mu_p(x), \mu_q(z - x))
\]

\[
= \sup_{x \in [a_p - \alpha_p, a_p]} \left( 1 - \min \left\{ 1, \left[ (1 - \mu_p(x))^s + (1 - \mu_q(z - x))^s \right]^\frac{1}{s} \right\} \right)
\]

\[
= 1 - \min \left\{ 1, \inf \left[ (1 - \mu_p(x))^s + (1 - \mu_q(z - x))^s \right]^\frac{1}{s} \right\}.
\]

Using linearity of fuzzy numbers \( p, q \), i.e., the fact that \( \mu_p(x) = \frac{1}{\alpha_p}(x - a_p + \alpha_p) \) and \( \mu_q(z - x) = \frac{1}{\alpha_q}(z - x - a_q + \alpha_q) \) for \( x \in [a_p - \alpha_p, a_p] \), we get

\[
\mu_r(z) = 1 - \min \left\{ 1, \inf \left[ \frac{1}{\alpha_p}(x - a_p + \alpha_p))^s + \frac{1}{\alpha_q}(z - x - a_q + \alpha_q)^s \right]^\frac{1}{s} \right\}
\]

or

\[
\mu_r(z) = 1 - \min \left\{ 1, \inf \left[ \left( \frac{a_p - x}{\alpha_p} \right)^s + \left( \frac{a_q - z + x}{\alpha_q} \right)^s \right]^\frac{1}{s} \right\}. \quad (10)
\]

Let us denote \( h(x) = \left( \frac{a_p - x}{\alpha_p} \right)^s + \left( \frac{a_q - z + x}{\alpha_q} \right)^s \) and let us look on the minimal value of \( h \) for \( x \in [a_p - \alpha_p, a_p] \).

The derivative of the function \( h \) is:

\[
h'(x) = -\frac{s}{\alpha_p} \left( \frac{a_p - x}{\alpha_p} \right)^{s-1} + \frac{s}{\alpha_q} \left( \frac{x + a_q - z}{\alpha_q} \right)^{s-1}.
\]

The only point \( x \) for which \( h'(x) = 0 \) is

\[
x_0 = \frac{a_p \lambda - a_q + z}{1 + \lambda}, \quad \text{where} \quad \lambda = \left( \frac{\alpha_q}{\alpha_p} \right)^{\frac{s}{s-1}}. \quad (11)
\]

79
ANNA KOLESÁROVÁ

It can be shown that the function \( h \) acquires its minimum in the interval \([a_p - \alpha_p, a_p]\) in the point \( x_0 \) and the minimal value of \( h \) is
\[
h(x_0) = \left[ \frac{a_p + a_q - z}{a_p \alpha_q(1 + \lambda)} \right] \left( \alpha_q^s + \lambda^s \alpha_p^s \right)^{\frac{1}{s}}.
\]
The function \( h^{1/s} \) acquires its minimal value in the point \( x_0 \), too, and it holds:
\[
h^{\frac{1}{s}}(x_0) = \frac{a_p + a_q - z}{a_p \alpha_q(1 + \lambda)} \left( \alpha_q^s + \lambda^s \alpha_p^s \right)^{\frac{1}{s}}. \tag{12}
\]
If we denote
\[
k_s = \frac{a_p \alpha_q(1 + \lambda)}{\left( \alpha_q^s + \lambda^s \alpha_p^s \right)^{\frac{1}{s}}}, \tag{13}
\]
then using (12), (10) can be expressed in the form:
\[
\mu_r(z) = 1 - \min \left\{ 1, \frac{1}{k_s}(a_p + a_q - z) \right\}. \tag{14}
\]
It means that for \( z \in [a_p + a_q - \alpha_p - \alpha_q, a_p + a_q] \) for which \( \frac{1}{k_s}(a_p + a_q - z) \geq 1 \) we get \( \mu_r(z) = 0 \). Further, if \( \frac{1}{k_s}(a_p + a_q - z) < 1 \) it holds
\[
\mu_r(z) = 1 - \frac{1}{k_s}(a_p + a_q - z),
\]
or, in the modified form,
\[
\mu_r(z) = \frac{1}{k_s}(z - a_p - a_q + k_s).
\]
If we express the number \( k \) given by (13) only by means of \( \alpha_p \) and \( \alpha_q \), we get
\[
k_s = \alpha_p \left[ 1 + \left( \frac{\alpha_q}{\alpha_p} \right)^{\frac{s}{s-1}} \right]^{\frac{s-1}{s}}. \tag{15}
\]
It can be easily shown that \( \lim_{s \to \infty} k_s = \alpha_p + \alpha_q \), \( \lim_{s \to 1^+} k_s = \max \{ \alpha_p, \alpha_q \} \) and \( k_s \in (\max \{ \alpha_p, \alpha_q \}, \alpha_p + \alpha_q) \) for all \( s \in (1, \infty) \).

If we sum up the previous results, we can write
\[
\mu_r(z) = \begin{cases} 
0, & \text{for } z \leq a_p + a_q - k_s, \\
\frac{1}{k_s}(z - a_p - a_q + k_s), & \text{for } a_p + a_q - k_s \leq z \leq a_p + a_q,
\end{cases}
\]
where \( k_s \in (\max \{ \alpha_p, \alpha_q \}, \alpha_p + \alpha_q) \) is given by (15).

It means that the left part of the membership function of the sum \( p \oplus q \) is linear. The same procedure can be used for proving linearity of the right part.

Remark 2. The number \( k_s \) defined by (15) expresses the "uncertainty" of the left side of the sum. As we have seen, for given \( \alpha_p, \alpha_q \), the value \( k_s \) depends only on the parameter \( s \) of the used t-norm \( T_s^\gamma \). So, choosing the parameter \( s \) \((s > 1)\), we can change the uncertainty of the sum from \( \max \{ \alpha_p, \alpha_q \} \) to \( \alpha_p + \alpha_q \).

Summarizing, we get our main result.
TRIANGULAR NORM–BASED ADDITION OF LINEAR FUZZY NUMBERS

**Theorem 1.** Let $T$ be a continuous Archimedean $t$-norm with strictly convex, twice differentiable additive generator. Then the $T$–sum of any linear fuzzy numbers is a linear fuzzy number if and only if the $t$-norm $T$ is a Yager’s $t$-norm $T^Y_s$ for some $s > 1$.

The proof of this assertion follows from Propositions 1 and 2. 

**References**


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