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ON FUZZY FUNCTIONS BETWEEN FUZZY SETS

László Filep

ABSTRACT. First a critical survey of existing fuzzy function notions is given. Then a new concept of fuzzy function is introduced, the basic properties of it are established and its equivalence to Zadeh's extension principle is shown.

I. Introduction

It is well known that if X is a set and $I = [0,1]$ the real unit interval, then the map $\mu\colon X \to I$ is called a *fuzzy set* on X or a *fuzzy subset* of X. Denote their family by I^X, which is a complete lattice under the "pointwise" extension of the natural ordering of I. The elements of X for which $\mu(x) > 0$ give the support of μ: $\operatorname{supp}(\mu) = S \subseteq X$.

A fuzzy relation on X is simply defined as a fuzzy set on $X \times X$. In this paper we will use the concept of fuzzy relation between fuzzy sets introduced by Rosenfeld in [5] and rarely studied in literature.

Let $\mu, \nu \in I^X$. Then any $r \in I^{X\times X}$ with

$$r(x,y) \le (\mu \times \nu)(x,y) = \min\bigl(\mu(x), \nu(y)\bigr), \qquad \forall x,y \in X\,,$$

is called a *fuzzy relation* between μ and ν. Let their family be denoted by $R(\mu,\nu)$, specially, if $\mu = \nu$, by $R(\mu)$. By the inverse of an $r \in R(\mu,\nu)$ we mean the following fuzzy relation r^{-1} between ν and μ:

$$r^{-1}(x,y) = r(y,x), \qquad \forall x,y \in X\,.$$

The (sup-min) product of $r \in R(\mu,\nu)$ and $q \in R(\nu,\eta)$, where $\mu,\nu,\eta \in I^X$, is denoted by $r \circ q$ and defined by

$$(r \circ q)(x,y) = \sup_{z\in X} \min\bigl(r(x,z), q(z,y)\bigr), \qquad \forall x,y \in X\,.$$

In [2] it was proved that this product is well-defined (i.e., $r \circ q \in R(\mu,\eta)$) as well as associative and isotone. It is also easy to show that for any $r,q \in R(\mu,\nu)$: $(r \circ q)^{-1} = q^{-1} \circ r^{-1}$.

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We define and denote the domain and range of a fuzzy relation $r \in R(\mu, \nu)$ as follows:

$$\operatorname{dom} r = \delta \in I^X : \ \delta(x) = \sup_{y \in X} r(x,y)\,, \qquad \forall x \in X\,;$$
$$\operatorname{ran} r = \varrho \in I^X : \ \varrho(y) = \sup_{x \in X} r(x,y)\,, \qquad \forall y \in X\,.$$

Clearly: $\delta \le \mu$, $\varrho \le \nu$.

To study the connections between fuzzy and crisp concepts, the notion of α-cut of a fuzzy set or relation is very useful:

$$\mu_\alpha = \{x \in X,\ \mu(x) \ge \alpha\}\,, \quad 0 \le \alpha \le 1\,,$$
$$r_\alpha = \{(x,y) \in X \times X,\ r(x,y) \ge \alpha\}\,, \quad 0 \le \alpha \le 1\,.$$

By the diagonal relation d_μ in $R(\mu)$ we mean the following:

$$d_\mu(x,y) = \begin{cases} \mu(x)\,, & \text{if } x = y\,, \\ 0\,, & \text{if } x \ne y\,, \end{cases} \qquad \forall x, y \in X\,.$$

II. Study of existing fuzzy function concepts

There are several definitions of *fuzzy function* in the literature (see, e.g., [1], [3]), but none of them meet all the following natural requirements: they should be special fuzzy relations; their product should be again a fuzzy function, their α-cuts should be crisp functions.

Here we list and study these definitions proving some properties of them and showing some connections between them. Then we give a new fuzzy function notion satisfying the above requirements.

Definition 1. An $r \in I^{X \times Y}$ is called a fuzzy function from X to Y [3].

Definition 2. An $r \in I^{X \times Y}$ is a fuzzy function, if $X = Y$ and r is reflexive and symmetric [7].

Definition 3. An $r \in I^{X \times Y}$ is a fuzzy function from X to Y, if for all $x \in X$ there exists $y \in Y$ such that $r(x,y) > 0$ [1].

Definition 4. An $r \in I^{X \times Y}$ is called a fuzzy function from X to Y, if for all $x \in X$ there exists a unique $y \in Y$ such that $r(x,y) = 1$ [4].

Definition 5. An $r \in I^{X \times Y}$ is said to be a fuzzy function from X to Y, if for all $x \in X$ there exists exactly one $y \in Y$ such that $r(x,y) = 1$, and if each $\alpha \in [0,1]$ appears at most once as a membership value of r [6].

Definition 6. An $r \in R(\mu, \nu)$, where $\mu \in I^X$ and $\nu \in I^Y$ is called a fuzzy function from μ to ν [3].

Definition 7. A fuzzy function from $\mu \in I^X$ to $\nu \in I^Y$ denoted by $f\colon \mu \to \nu$ is a usual function $f\colon X \to Y$ such that $\mu = \nu \circ f$, i.e., $\mu(x) = \nu\big(f(x)\big)$ for all $x \in X$ [3].

Definition 8. The same as the previous one, but with $\mu \le \nu \circ f$ [3].

Definition 9. (Zadeh's extension principle, [8]) Any crisp function $f\colon X \to Y$ $(x \mapsto y)$ can be extended to a fuzzy function $I^X \mapsto I^Y$ $(\mu \mapsto \nu)$, where μ is given and ν is defined in the following way

$$\nu(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x), & \text{if } y \in \operatorname{ran} f, \\ 0, & \text{if } y \in Y \setminus \operatorname{ran} f, \end{cases}$$

ν is called the direct image of ν under f, while μ the inverse image of ν under f.

Proposition 1. *The map f in Definition 7 exists iff for all $x \in X$ there exists $y \in Y$ such that $\mu(x) = \nu(y)$.*

Proof. Suppose that for some $x \in X$ there does not exist $y \in Y$ with $\mu(x) = \nu(y)$. Let $f(x) = z \in Y$, and let $\nu(z) = \alpha \ne \beta = \mu(x)$. Then

$$\beta = \mu(x) = (\nu \circ f)(x) = \nu\big(f(x)\big) = \nu(z) = \alpha\,,$$

which is a contradiction.

If $\mu(x) = \nu(y)$ for some $y \in Y$, then taking $f(x) = y$ we can construct f. □

Proposition 2. *The map f in Definition 8 exists iff*

$$\sup_{x \in X} \mu(x) \le \sup_{y \in Y} \nu(y)\,.$$

Proof. Suppose that $\sup\limits_{x \in X} \mu(x) > \sup\limits_{y \in Y} \nu(y)$. Then there exists $x_0 \in X$ with $\mu(x_0) > \nu(y)$ for all $y \in Y$. Since $f(x_0) \in Y$, so specially $\mu(x_0) > \nu\big(f(x_0)\big)$ also holds, which contradicts the condition made for f.

The converse statement is trivial. □

Proposition 3. *Definition 6 is an extension of Definition 1.*

Proof. Immediate. □

Proposition 4. *If $r \in I^{X \times Y}$ is a fuzzy function by Definition 4, then r generates a map f from X to Y.*

Proof. The following construction for f readily proofs the statement:

$$f\colon X \to Y : x \mapsto y; \qquad y = f(x) \iff r(x, y) = 1\,.$$

□

III. A new notion for fuzzy function

DEFINITION 10. An $r \in R(\mu, \nu)$ is said to be a *fuzzy function* (from μ to ν) if $r \circ r^{-1} \geq d_\mu$ and $r \circ r^{-1} \leq d_\nu$.

LEMMA 1. *$r \circ d_\mu = d_\mu \circ r = r, \quad r \circ d_\nu = d_\nu \circ r = r$, for any $r \in R(\mu, \nu)$.*

Proof.

$$(d_\mu \circ r)(x,y) = \sup_{z \in X} \min\big(d_\mu(x,z), r(z,y)\big) = \min\big(d_\mu(x,x), r(x,y)\big) =$$
$$= \min\big(\mu(x), r(x,y)\big) = r(x,y)\,, \text{ so really } d_\mu \circ r = r\,.$$

The other statement can be proved similarly. □

THEOREM 1. *If $r, q \in R(\mu, \nu)$ are fuzzy functions from μ to ν, so is $r \circ q$.*

Proof. Using Lemma 1, Definition 10 and the property of product inverse we can write:

$$(r \circ q) \circ (r \circ q)^{-1} = r \circ q \circ q^{-1} \circ r^{-1} \geq r \circ d_\mu \circ r^{-1} = r \circ r^{-1} \geq d_\mu\,;$$
$$(r \circ q)^{-1} \circ (r \circ q) = q^{-1} \circ r^{-1} \circ r \circ q \leq q^{-1} \circ d_\nu \circ q = q^{-1} \circ q \leq d_\nu\,,$$

which means that $r \circ q$ is also a fuzzy function. □

This theorem together with the previous results proves that the fuzzy functions from μ to μ form a lattice ordered monoid under the product operation.

The proof of the following lemma is obvious:

LEMMA 2. *Let $r \in R(\mu, \nu)$ be a fuzzy function. Then r^{-1} is a fuzzy function iff $r \circ r^{-1} = d_\mu$ and $r^{-1} \circ r = d_\nu$.* □

LEMMA 3. *Let $r \in R(\mu, \nu)$ be a fuzzy function, then $\delta = \mu$.*

Proof.

$$\delta(x) = \sup_{y \in X} r(x,y) = \sup_{y \in X} \min\big(r(x,y), r(x,y)\big) = \sup_{y \in X} \min\big(r(x,y), r^{-1}(y,x)\big) =$$
$$= (r \circ r^{-1})(x,x) \geq d_\mu(x,x) = \mu(x)\,,$$

which together with $\delta \leq \mu$ proves the lemma. □

THEOREM 2. *An $r \in R(\mu, \nu)$ is a fuzzy function iff r is of the following form*

$$r(x,y) = \begin{cases} \min\big(\mu(x), \nu(y)\big)\,, & \textit{for } \forall x \in S \textit{ and } \exists!\; y \in S\,, \\ 0\,, & \textit{otherwise}\,, \end{cases} \tag{1}$$

for all $x, y \in X$.

Proof. First suppose r having the form (1). If $x \in X \setminus S$, then $r(x,z) = r(z,x) = 0$ for all $z \in X$, therefore

$$(r \circ r^{-1})(x,y) = (r^{-1} \circ r)(x,y) = 0, \quad \forall y \in X.$$

Since in this case $d_\mu(x,y) = 0$ is also true, so $r \circ r^{-1} = r^{-1} \circ r = d_\mu$ trivially holds.

If $x \in S$, then for any $z \neq x$ $d_\mu(x,y) = 0$, thus clearly $(r \circ r^{-1})(x,z) \geq d_\mu(x,z)$. If $z = x$, then

$$(r \circ r^{-1})(x,x) = \sup_{y \in X} \min\bigl(r(x,y), r^{-1}(y,x)\bigr) =$$
$$= \sup_{y \in X} r(x,y) = \delta(x) = \mu(x) = d_\mu(x,x).$$

With this we have proved that $r \circ r^{-1} \geq d_\mu$. Now we show the validity of $r^{-1} \circ r \leq d_\nu$. Clearly it is enough to consider only elements from S. If $x \in S$ and $z \neq x$, then

$$(r \circ r^{-1})(x,z) = \sup_{y \in X} \min\bigl(r^{-1}(x,y), r(y,z)\bigr) =$$
$$= \sup_{y \in X} \min\bigl(r(y,x), r(y,z)\bigr) = 0 = d_\nu(x,z),$$

since at least one of $r(y,x)$ and $r(y,z)$ is zero because of (1).

If $z = x$, then

$$(r^{-1} \circ r)(x,x) = \sup_{x \in X} \min\bigl(r^{-1}(x,y), r(y,x)\bigr) = \sup_{y \in X} r(y,x) \leq$$
$$\leq \sup_{x \in X} \min\bigl(\mu(y), \nu(x)\bigr) \leq \min\bigl(\sup_{x \in X} \mu(y), \nu(x)\bigr) \leq \nu(x) = d_\nu(x,x),$$

that is, our statement is valid in all cases.

Now, conversely, let r be a fuzzy function. If $x \in X \setminus S$, then $r(x,y) = 0$ for all $y \in X$. If $x \in S$, then we show that there exists at least one $y_0 \in S$ such that

$$r(x, y_0) = \min\bigl(\mu(x), \nu(y_0)\bigr) > 0.$$

Suppose that for all $y \in S$

$$r(x,y) < \min\bigl(\mu(x), \nu(y)\bigr).$$

Then using the first part of this proof we get

$$\mu(x) = (r \circ r^{-1})(x,x) = \sup_{y \in X} r(x,y) < \sup_{y \in X} \min\bigl(\mu(x), \nu(y)\bigr) \leq$$
$$\leq \min\bigl(\mu(x), \sup_{y \in X} \nu(y)\bigr) \leq \mu(y),$$

which is a contradiction.

Finally we show, also by indirect way, that there is no other $y_1 \in S$ $(y_1 \neq y_0)$ with $r(x, y_1) > 0$. Namely:

$$0 = d_\nu(y_0, y_1) \geq (r^{-1} \circ r)(y_0, y_1) = \sup_{z \in X} \min\bigl(r^{-1}(y_0, z), r(z, y_1)\bigr) =$$
$$= \sup_{z \in X} \min\bigl(r(z, y_0), r(z, y_1)\bigr) \geq \min\bigl(r(x, y_0), r(x, y_1)\bigr) > 0\,,$$

which is a contradiction and completes the proof of the theorem. □

Lemma 4. *Any fuzzy function $r \in R(\mu, \nu)$ induces a mapping (crisp function) f from X to X.*

Proof. From Theorem 2 one can easily deduce that the following correspondence

$$y = \phi(x) \iff r(x, y) = \min\bigl(\mu(x), \nu(y)\bigr), \qquad \forall x, y \in S\,,$$

defines a map from S to S. It can be extended to X as follows

$$f\colon X \to X \quad \begin{pmatrix} x \mapsto \phi(x), & \text{if} & x \in S\,, \\ x \mapsto x, & \text{if} & x \in X \setminus S\,. \end{pmatrix} \tag{2}$$

□

Lemma 5. *If $r \in R(\mu, \nu)$ is a fuzzy function, then for any $\alpha \in [0, 1]$*

$$(x, y) \in r_\alpha \Longrightarrow x \in \mu_\alpha\,,\ y \in \nu_\alpha\,, \quad \forall x, y \in X\,.$$

Proof. If $\alpha = 0$, the statement is trivial. If $\alpha > 0$, then $x \in S$, further by Theorem 2 for a unique $y_0 \in S$ we have

$$r(x, y_0) = \min\bigl(\mu(x), \nu(y_0)\bigr) > 0\,.$$

Moreover from

$$\alpha \leq r(x, y_0) = \min\bigl(\mu(x), \nu(y_0)\bigr)$$

follows that $\alpha \leq \mu(x)$ and $\alpha \leq \nu(y_0)$, that is $x \in \mu_\alpha$ and $y \in \nu_\alpha$. □

Lemma 6. *If $r \in R(\mu, \nu)$ is a fuzzy function, then*

$$\forall x \in S\,,\ \forall \alpha \in (0, 1]\colon x \in \mu_\alpha \Longrightarrow \exists!\ y_0 \in S\colon (x, y_0) \in r_\alpha\,,\ y_0 \in \nu_\alpha\,.$$

Proof. This lemma is a consequence of Lemma 3 and Theorem 2. Namely:

$$\alpha \leq \mu(x) = \delta(x) = \sup_{x \in X} r(x, y) = r(x, y_0) \Longrightarrow (x, y_0) \in r_\alpha\,;$$
$$\alpha \leq r(x, y_0) \leq \sup_{x \in X} r(x, y_0) = \varrho(y_0) \leq \nu(y_0) \Longrightarrow y_0 \in \nu_\alpha\,.$$

□

THEOREM 3. *An $r \in R(\mu, \nu)$ is a fuzzy function iff each α-cut r_α $(0 < \alpha \leq 1)$ is a crisp function from μ_α to ν_α.*

Proof. Let r be a fuzzy function and consider an arbitrary element (x, y) from r_α. Then by Lemma 5 and 6 r_α really establishes a map from μ_α to ν_α.

Conversely, let each α-cut r_α $(0 < \alpha \leq 1)$ be a map from μ_α to ν_α. If $x \in X \setminus S$, then $(x, y) \in r_\alpha$ cannot be true whatever α and y are. So we may suppose that $x \in S$. Let $\alpha = \mu(x) > 0$. For this α $x \in \mu_\alpha$ and since r_α is a map from μ_α to ν_α, then there exists a unique $y_0 \in \nu_\alpha$ for which $(x, y_0) \in r_\alpha$. Thus we have

$$r(x, y_0) \geq \alpha = \mu(x) \qquad \text{and} \qquad \nu(y_0) \geq \alpha = \mu(x).$$

Moreover

$$\alpha = \mu(x) \leq r(x, y_0) \leq \min(\mu(x), \nu(y_0)) = \mu(x) = \alpha,$$

which implies

$$r(x, y_0) = \min(\mu(x), \nu(y_0)).$$

If $x \in X \setminus S$, then $r(x, y) = 0$ for all $y \in X$. Consequently r is of the form (1). So by Theorem 2 it is a fuzzy function. □

THEOREM 4. *If $r \in R(\mu, \nu)$ is a fuzzy function, then $\operatorname{ran} r = \varrho$ is of the form:*

$$\varrho(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall y \in X,$$

where f denotes the map induced by r and having the form (2).

Proof. For $y \in X$ with $f^{-1}(y) \neq \emptyset$ we can apply Theorem 2 and 3:

$$\varrho(y) = \sup_{x \in X} r(x, y) = \sup_{x \in f^{-1}(y)} r(x, y) = \sup_{x \in f^{-1}(y)} \min(\mu(x), \nu(y)) = \sup_{x \in f^{-1}(y)} \mu(x).$$

If $f^{-1}(y) = \emptyset$, then by Theorem 2 $r(x, y) = 0$ for all $x \in X$, therefore

$$\varrho(y) = \sup_{x \in X} r(x, y) = 0.$$

This completes the proof of our theorem. □

Theorem 4 shows that our notion of fuzzy function leads to Zadeh's extension principle (Definition 9). A fuzzy function (from μ to ν) includes a crisp one, under which the direct image of μ in sense of Zadeh's principle is ν.

The proof of the following lemma (being a converse of Theorem 4) is left to the reader:

LEMMA 7. *If we are given a map $f\colon X \to X$, a fuzzy set μ on X and the direct image ν of μ under f, then the following fuzzy relation*

$$r(x,y) = \begin{cases} \min\bigl(\mu(x),\nu(y)\bigr), & \text{if } y = f(x), \\ 0, & \text{otherwise}, \end{cases} \qquad \forall x,y \in X,$$

defines a fuzzy function from μ to ν. □

The discovery of this connection between our fuzzy function notion and Zadeh's extension principle was only possible, because we used the concept of fuzzy relation between fuzzy sets. It also turned out that our fuzzy function notion is a natural generalization of the crisp one.

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Bessenyei College
Department of Mathematics
Nyíregyháza
P. O. Box 166
H–4401
HUNGARY
E-mail: filepl@ny1.bgytf.hu