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KERNEL LOGICS

MIRKO NAVARA

ABSTRACT. Let $\mathcal{A}$ be a Boolean algebra and m a (group-valued) measure on $\mathcal{A}$. Then the kernel $\operatorname{Ker} m$ is a concrete logic (= set-representable orthomodular poset). We exhibit the efficiency of this technique in constructions of concrete logics with special properties, e.g., the Jauch–Piron property.

1. Motivation

Concrete logics have been studied for many years (see, e.g., [9]) as an alternative structure for the description of events in a system including noncompatibility. Possible areas of applications include quantum mechanics, artificial intelligence, psychology, sociology etc. Concrete logics admit analogues of some results obtained in the measure and integration theory on Boolean algebras (see, e.g., [1, 5]). However, serious difficulties are encountered in these attempts. Kernel logics, which are introduced in this paper, form a class of concrete logics. As they are described in terms of Boolean algebras using measure-theoretic notions, there is a greater chance to generalize classical results for Boolean algebras to kernel logics. As we show here, the class of kernel logics is still rather general. It is closed with respect to products and horizontal sums. We find among them important and quite non-trivial examples of "almost Boolean" logics. Besides this, our approach might be interesting as a special construction technique for concrete logics.

2. Basic definitions and examples

Let us recall the basic definitions. Let X be a nonempty set. A collection $L \subset 2^X$ is called a *concrete logic* if $X \in L$ and $A, B \in L$, $A \subset B$, implies

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$B \setminus A \in L$. Concrete logics may be alternatively defined as orthomodular posets possessing order-determining sets of two-valued measures (see [10]). Notice that a concrete logic is closed with respect to disjoint unions, but not to all unions. Of course, all Boolean algebras are concrete logics. Another typical example is the following.

EXAMPLE 2.1. Let X be a set of an even cardinality and let L be the set of all subsets of X of an even cardinality. Then L is a concrete logic.

Let G be a commutative group. A *G–valued measure* on a concrete logic L is a mapping $m: L \to G$ such that $m(A \cup B) = m(A) + m(B)$ whenever $A \cap B = \emptyset$. Our approach is based on the following observation:

PROPOSITION 2.2. *Let m be a G–valued measure on a Boolean algebra $\mathcal{A}$ such that $m(1) = 0$. Then the kernel of m, $\operatorname{Ker} m = m^{-1}(0)$, is a concrete logic.*

Logics isomorphic to those constructed as in Proposition 2.2 are called *kernel logics.*

EXAMPLE 2.3. Logics from Example 2.1 are kernel logics. It suffices to take a Z_2–valued measure on 2^X (Z_2 is the two-element cyclic group) such that $m(\{x\}) = 1$ for all $x \in X$.

EXAMPLE 2.4. Let $X = \{a,b,c,d,e,f,g,h\}$. We define a Z–valued measure m on 2^X such that $m(\{a\}) = m(\{d\}) = 1$, $m(\{b\}) = m(\{c\}) = -1$, $m(\{e\}) = m(\{f\}) = m(\{g\}) = m(\{h\}) = 0$. The atoms of the kernel logic $\operatorname{Ker} m$ are $\{a,b\}$, $\{a,c\}$, $\{b,d\}$, $\{c,d\}$, $\{e\}$, $\{f\}$, $\{g\}$, $\{h\}$. It is isomorphic to the free orthomodular lattice with 2 free generators (which may be identified with $\{a,b,e,f\}$ and $\{a,c,f,g\}$).

In the following sections we summarize (without proofs) some discoveries concerning kernel logics. Their detailed treatment is provided in [6].

3. Properties of the class of kernel logics

It is natural to ask how large the class of kernel logics is. Further, we may want the respective measure to attain values in some special group. Until now we have only the following partial results.

PROPOSITION 3.1. *Every product of finitely many kernel logics is a kernel logic.*

THEOREM 3.2. *Every horizontal sum of finitely many kernel logics is a kernel logic.*

Janowitz [2] introduced the class of *constructible logics* – it is the smallest class containing all Boolean algebras and closed under products and horizontal sums. According to Proposition 3.1 and Theorem 3.2, each finite constructible logic is a kernel logic. Moreover, we can strengthen the latter result:

THEOREM 3.3. *Every finite constructible logic is the kernel of some integer-valued measure.*

Obviously, neither integer-valued nor real-valued measures suffice to determine all kernel logics.

4. Almost Boolean kernel logics

Important and quite non-trivial applications of kernel logics were found in the study of classes of concrete logics which are "close" to Boolean algebras. The key references to this topic are [7, 4].

A concrete logic P has the *Jauch–Piron property* if, for each **non-negative** finite measure s, each $A, B \in \operatorname{Ker} s$ have an upper bound $C \in \operatorname{Ker} s$. We denote by $\mathcal{C}_{JP}$ the class of concrete logics with the Jauch–Piron property and by $\mathcal{B}$ the class of Boolean algebras. Obviously, $\mathcal{B} \subset \mathcal{C}_{JP}$. The question has arisen whether the latter inclusion is proper. This problem was formulated, e.g., in [7, 8] and remained open for several years. An affirmative answer was given in [3]. Here we provide an example of a kernel logic from $\mathcal{C}_{JP} \setminus \mathcal{B}$.

EXAMPLE 4.1. There is a kernel logic which is not a Boolean algebra and satisfies the Jauch–Piron property.

Let F, G be two disjoint uncountable sets, and let $E = F \cup G$, $X = E^{\mathbb{N}}$. We define $\mathcal{A} \subset 2^X$ as the Boolean algebra generated by all sets of the form

$$G(e_1, \dots, e_k) = \{(x_i)_{i \in \mathbb{N}} \in X : x_1 = e_1, \dots, x_k = e_k\},$$

where $k \in \mathbb{N}$, $e_1, \dots, e_k \in E$. The real-valued set function m defined by

$$m(G(e_1, \dots, e_k)) = (-1)^{\operatorname{card}((e_1, \dots, e_k) \in F)},$$
$$m(X) = 0,$$

extends uniquely to a measure on $\mathcal{A}$, and the corresponding kernel logic belongs to $\mathcal{C}_{JP} \setminus \mathcal{B}$.

Let $n \in \mathbb{N}$. We say that a concrete logic L satisfies the *n-covering property* if for each $A, B \in L$ there exist $C_1, \dots, C_n \in L$ such that $A \cap B = \bigcup_{i \le n} C_i$. We denote by $\mathcal{C}_n$ the class of concrete logics with the n-covering property. Obviously, $\mathcal{C}_1 = \mathcal{B}$. It was proved in [7] that $\mathcal{C}_{JP} \subsetneqq \mathcal{C}_2$. Again the question remained open for some time whether the inclusions $\mathcal{C}_n \subset \mathcal{C}_{n+1}$, $n \in \mathbb{N}$, are proper. An affirmative answer was given in [4]. Here we construct a different example which is a kernel logic; moreover, unlike the example of [4], our technique is universal for all $n \in \mathbb{N}$.

EXAMPLE 4.2. Let $n \in \mathbb{N}$. There is a kernel logic which satisfies the $(n+1)$-covering property, but does not satisfy the n-covering property.

With the notation $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, let $X = \{(x_i)_{i\in\mathbb{N}_0} : x_0 \in \{0,1,2,3\}$ and $x_i \in \{0,\dots,n+1\}$ for $i \geq 1\}$. We define $\mathcal{A} \subset 2^X$ as the Boolean algebra generated by all sets of the form

$$G(e_0,\dots,e_k) = \{(x_i)_{i\in\mathbb{N}_0} \in X : x_0 = e_0,\dots,x_k = e_k\},$$

where $k \in \mathbb{N}$, $e_0 \in \{0,1,2,3\}$, $e_1,\dots,e_k \in \{0,\dots,n+1\}$. The real-valued set function m defined by

$$m(G(e_0,\dots,e_k)) = \frac{(-1)^{e_0+\operatorname{card}\{j:\ e_j=0\}}}{n^k},$$
$$m(X) = 0$$

extends uniquely to a measure on $\mathcal{A}$, and the corresponding kernel logic belongs to $\mathcal{C}_{n+1} \setminus \mathcal{C}_n$.

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Department of Mathematics
Faculty of Electrical Engineering
Czech Technical University
CZ–166 27 Praha 6
CZECH REPUBLIC
E-mail: navara@math.feld.cvut.cz