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LAW OF LARGE NUMBERS ON D–POSET OF FUZZY SETS

FERDINAND CHOVANEC — MÁRIA JUREČKOVÁ

1. Introduction

F. Kôpka defined the new structure a *D-poset* of fuzzy sets [1] and suggested how to construct the probability theory on it .

The D–poset of fuzzy sets is a family $F \subset [0,1]^x$ on which is defined the difference (we denote $g \setminus f$) for any $f, g \in F, f \leq g$ such that:

(1) $g \setminus f \leq g$

(2) $g \setminus (g \setminus f) = f$

(3) if $f \leq g \leq h$, then $h \setminus g \leq h \setminus f$ and $(h \setminus f) \setminus (h \setminus g) = g \setminus f$ and F satisfies the following properties

(4) if $1_x(t) = 1$ for every $t \in X$, then $1_x \in F$

(5) if $\{f_n\}_{n \in \mathbb{N}} \subseteq F$, $f_n \nearrow f$, then $f \subseteq F$.

An observable on a D–poset F is a mapping $x\colon \mathcal{B}(\mathbb{R}) \to F$ with properties:

(6) $x(\mathbb{R}) = 1_x$

(7) if $\{A\}_{n \in \mathbb{N}} \subset \mathcal{B}(\mathbb{R})$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$

(8) if $A, B \in \mathcal{B}(\mathbb{R})$, $A \subseteq B$, then $x(B \setminus A) = x(B) \setminus x(A)$,

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of the real line $\mathbb{R}$.

A state is a mapping $m\colon F \to [0,1]$, such that:

(9) $m(1_x) = 1$

(10) if $(f_n)_{n \in \mathbb{N}} \subset F$, $f_n \nearrow f$, then $m(f) = m(f_1) + \sum_{n=1}^{\infty} m(f_n \setminus f_{n-1})$.

The mapping $m_x\colon \mathcal{B}(\mathbb{R}) \to [0,1]$, defined by the formula $m_x(A) := m(x(A))$ is a probability measure.

The mean value of the observable x in the state m is defined by the integral

$$E(x) := \int_{\mathbb{R}} t \, \mathrm{d}m_x \qquad \text{(if there exists)},$$

and the dispersion of x is integral

$$D(x) := \int_{\mathbb{R}} (t - E(x))^2 \, dm_x \qquad \text{(if there exists).}$$

We denote $L^2 = \{x,\ x\text{-observable: } \int t^2 \, dm_x < \infty\}$.

2. Definitions and notions

DEFINITION 1. We shall say that the *observables* $x_1, x_2, \dots x_n$ *are compatible*, if there exists the observable y and the real-valued Borel measurable functions $f_1, f_2, \dots, f_n$, such that $x_i = y \circ f_i^{-1}$ for $i = 1, 2, \dots, n$.

By this definition we can introduce the calculus for compatible observables.

$$x_1 + x_2 + \dots + x_n = y \circ (f_1^{-1} + f_2^{-1} + \dots + f_n^{-1}),$$
$$x_1 \cdot x_2 \dots x_n = y \circ (f_1 \cdot f_2 \dots f_n)^{-1} x_i = y \circ f_i^{-1}.$$

In [2] it is shown, when the observables are compatible.

DEFINITION 2. Let $x_1, \dots, x_n$, $n \geq 2$, be the observables from F. Let the mapping $p_i : \mathbb{R}^n \to \mathbb{R}$ be the projection $p_i(t_1, \dots t_n) = t_i t_i \in \mathbb{R}$, $i = 1, 2, \dots n$. We shall say that the observables $x_1, x_2, \dots x_n$ have a *joint observable* $w : B(\mathbb{R}^n) \to F$, if

(i) $w(\mathbb{R}^n) = 1_x$.

(ii) if $(A_n)_{n \in \mathrm{N}} \subset B(\mathbb{R}^n)$, $A_n \subset A_{n+1}$, $n \in \mathrm{N}$, then $\bigvee_{n \in \mathrm{N}} w(A_n) \in F$ and
$$w(\bigcup_{n \in \mathrm{N}} A_n) = \bigvee_{n \in \mathrm{N}} w(A_n)$$

(iii) If $A, B \in B(\mathbb{R}^n)$, $A \subseteq B$, then $w(B \setminus A) = w(B) \setminus w(A)$

(iv) $w\big(p_i^{-1}(E)\big) = x_i(E)$ for every $E \in B(\mathbb{R})$, $i = 1, 2, \dots, n$.

It is easy to prove the next theorem.

THEOREM 1. *If the observables $x_1, \dots, x_n$ are compatible, then there exists a joint observable for $x_1, \dots, x_n$.*

If the joint observable for $x_1, \dots, x_n$ exists, then we are able to construct some operations with $x_1, \dots, x_n$. E. g.,

$$\frac{1}{n}\sum_{i=1}^{n} x_i = w \circ g^{-1}, \quad \text{where } g\colon \mathbb{R}^n \to \mathbb{R},\ g(t_1, \dots, t_n) = \frac{1}{n}\sum_{i=1}^{n} t_i\,,$$

$$x_i \cdot x_j = w \circ h^{-1}, \quad \text{where } h(u_1, \dots u_n) = u_i \cdot u_j\,, \quad i, j = 1, \dots n.$$

DEFINITION 3. We shall say that $(x_n)_{n\in\mathbb{N}}$ of observables is

(i) ***finitely compatible*** if $x_1, \dots x_n$ are compatible for every n,
(ii) ***pairwise uncorrelated*** if $E(x_i \cdot x_j) = E(x_i) \cdot E(x_j)$.

3. Law of large numbers.

THEOREM 2. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of finitely compatible pairwise uncorrelated observables from L^2 such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} D(x_i) = 0$. Then for every $\varepsilon > 0$*

$$\lim_{n\to\infty} m\Big(\frac{1}{n}\sum_{i=1}^{n}(x_i - E(x_i))\,((-\varepsilon, \varepsilon))\Big) = 1\,.$$

Proof. Let $J = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, $J \subseteq N$ and let w be a joint observable for $x_1, \dots, x_n$.

We define the mapping $P_J : B(\mathbb{R}^n) \to [0,1]$, $P_J(A) = m(w(A))$, for every $A \in B(\mathbb{R}^n)$. By the Kolmogorov theorem there exists exactly one probability measure P on measurable space $(\mathbb{R}^N, \sigma(S))$, ($S$ being the algebra of all measurable cylinders in $\mathbb{R}^N$) such that $P(\pi_J^{-1}(A)) = P_J(A)$, for every $A \in B(\mathbb{R}^n)$ ($\pi_J : \mathbb{R}^N \to \mathbb{R}^n$ being the corresponding projection).
We define the mapping $\xi_i : \mathbb{R}^N \to \mathbb{R}$ by the formula $\xi_i((t_n)_{n\in\mathbb{N}}) = t_i$.

We denote $\bar{t} = (t_n)_{n\in\mathbb{N}} \in \mathbb{R}^n$. Then
$(\xi_1, \dots, \xi_N)^{-1}(A) = \{\bar{t} \in \mathbb{R}^N : (\xi_1(\bar{t}), \dots, \xi_n(\bar{t})) \in A\} = \{\bar{t} \in \mathbb{R}^N : (t_1, \dots, t_n) \in A\} = \{\bar{t} \in \mathbb{R}^N : \pi_J(\bar{t}) \in A\} = \pi_J^{-1}(A)\}$.

We obtain that $P((\xi_1, \dots, \xi_n)^{-1}(A)) = P(\pi_J^{-1}(A)) = m(w(A))$ for every $A \in B(\mathbb{R}^N)$.

It follows that $P\Big((\xi_1, \dots, \xi_n)^{-1}(g^{-1}(F))\Big) = m\Big(w(g^{-1}(F))\Big)$ for every $F \in B(\mathbb{R})$ and every Borel function $g : \mathbb{R}^n \to \mathbb{R}$.

If $g(t_1,\dots,t_n) = \frac{1}{n}\sum_{i=1}^{n}(t_i - k_i)$, where $k_i = E(\xi_i)$ and $F = (-\varepsilon,\varepsilon)$, we obtain $P\Big(\frac{1}{n}\sum_{i=1}^{n}(\xi_i - E(\xi_i))^{-1}((-\varepsilon,\varepsilon))\Big) = m\Big(\frac{1}{n}\sum_{i=1}^{n}(x_i - E(x_i))((-\varepsilon,\varepsilon))\Big)$. By the classical theory

$$\lim_{n\to\infty} P\Big(\frac{1}{n}\sum_{i=1}^{n}(\xi_i - E(\xi_i))^{-1}((-\varepsilon,\varepsilon))\Big) = 1 .$$

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Technical University of Liptovský Mikuláš
031 19 Liptovský Mikuláš
CZECHO-SLOVAKIA