Tatra Mountains Math. Publ., 1 (1992), 51–55

TATRA MOUNTAINS
Mathematical Publications

LINGUISTIC EVIDENCE AND DECISION MAKING

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1. Introduction

In this paper we study the decision making problem under uncertainty generated by linguistic evidence. This evidence has the form: Pr(V is A) is Q, where V is fuzzy variable, A is fuzzy event and Q is fuzzy probability. First, we study the fuzzy subset $\mathcal{M}_{A,Q}$ of space of all probability measures induced by above evidence. Next, we investigate the fuzzy subset $E_1(\mathcal{M}_{A,Q})$ called the expected payoff under linguistic evidence. If a payoff function l is one-to-one we can compute the edges of an r-cut of $E_1(\mathcal{M}_{A,Q})$ in a simple way. In other cases, we propose the approximate solutions of this problem. We present two such solutions.

2. Fuzzy evidence.

Now we will study the following linguistic statement:

$$G\colon \mathrm{Pr}(V \text{ is } A) \text{ is } Q,$$

where V is a fuzzy variable taking value in some compact space X, A is a fuzzy subset of X (fuzzy event) and Q is a fuzzy probability [6]. We recall that fuzzy probability is a convex, compact and normal fuzzy subset of $[0,1]$. It is a fuzzy number and every r-cut of Q is a closed interval

$$Q_r = [q_1(r), q_2(r)]\,.$$

For instance the statement:

"The probability that the demand is big is about 0.7",

has the above form, where the fuzzy variable V is the value of the demand, the fuzzy event A is "big" and the fuzzy probability Q is "about 0.7".

This linguistic statement induces a fuzzy subset $\mathcal{M}_{A,Q}$ of space of all probability measures $\mathcal{M}(X)$ on space X with distinguish σ-field B [1], [3], [6]. The membership function of this fuzzy subset is given by

$$\mathcal{M}_{A,Q}(P) = Q(P(A)),$$

where $P \in \mathcal{M}(X)$ and $P(A) = \int A(x)\,\mathrm{d}P(x)$ is a probability of the fuzzy event A.

The r-cut of the fuzzy subset M takes the form

$$\begin{aligned}(\mathcal{M}_{A,Q}) &= \{P \in \mathcal{M}(X) \colon P(A) \in Q_r\} \\ &= \{P \in \mathcal{M}(X) \colon q_1(r) \leq P(A) \leq q_2(r)\}.\end{aligned}$$

THEOREM 1. [4]

(i) *$\mathcal{M}_{A,Q}$ is a convex fuzzy subset of space $\mathcal{M}(X)$.*

(ii) *If A is a compact, continuous fuzzy subset of X then $\mathcal{M}_{A,Q}$ is compact and convex fuzzy subset of $\mathcal{M}(X)$.*

Let f be a continuous, real function determined on space X.

THEOREM 2. [4] *If f is one-to-one then*

$$f(\mathcal{M}_{A,Q}) = \mathcal{M}_{f(A),Q},$$

where $M_{f(A),Q}$ is a fuzzy subset of the space of all probability measures on the real line $\mathbb{R}$.

3. Fuzzy payoff.

Let S be a space of state of nature and l be a payoff function. We assume that our knowledge of the space S is represented by the fuzzy linguistic evidence

$$G \colon \Pr(V \text{ is } A) \text{ is } Q.$$

The payoff function l generates a functional E_l – expected value – on the space of all probability measures on S,

$$E_l(P) = \int l(s)\,\mathrm{d}P(s).$$

It is easy to see that

$$(\mathcal{M}_{A,Q}) = E_A^{-1}(Q_r). \tag{1}$$

DEFINITION 1. [1] The image of the fuzzy subset $\mathcal{M}_{A,Q}$ by the function E_l, i.e., $E_l(\mathcal{M}_{A,Q})$, will be called the *expected payoff* for fuzzy evidence G.

The expected payoff is a fuzzy subset of the real line. It has the following properties:

THEOREM 3. [4]

(i) $E_l(\mathcal{M}_{A,Q}) = E_I(l(\mathcal{M}_{A,Q}))$, *where I is an identity function on $\mathbb{R}$.*

(ii) *If A is continuous then $E_l(\mathcal{M}_{A,Q})$ is a convex and compact fuzzy subset on the real line $\mathbb{R}$.*

The second property follows from (1) and Theorem 1. In this case, every r-cut of expected payoff is a closed interval:

$$\left(E_l(\mathcal{M}_{A,Q})\right)_r = [e_1(r), e_2(r)].$$

The edges of this interval are determined by the solution of the following optimization problem [1], [4]:

$$E_l(P) \to \min(\max) \tag{2}$$

for $P \in \mathcal{M}(S)$ satisfying $q_1(r) \leq P(A) \leq q_2(r)$.

If payoff function is one-to-one we obtain from Theorems 2 and 3(i) the following equation

$$E_l(\mathcal{M}_{A,Q}) = E_I(\mathcal{M}_{l(A),Q}).$$

In this case, we obtain simpler optimizing problem

$$E_I(P) \to \min(\max)$$

for $P \in \mathcal{M}(S)$ satisfying $q_1(r) \leq P(l(A)) \leq q_2(r)$.

We can solve this problem in an elementary way in the case when the space of state of nature S is finite or image $l(A)$ is a fuzzy number [1], [3]. A solution of such problem is the degenerated probability measure, supported on one or two points only.

4. Approximate solutions.

In the general case, if the payoff function l is not one-to-one then the problem become more complicated. We cannot solve the optimization problem (1) or compute the image $l(\mathcal{M}_{A,Q})$ in many cases. In this paper, we propose two approximate solutions of this problem.

(i) Approximation by random sets.

First we compute the expected set of fuzzy event A [5]. It is a crisp set A' which approximates the fuzzy set A. In the case when A is a fuzzy number $N(f,g;a,c,d,b)$, where f, g are sides of A, $\operatorname{supp} A = [a,b]$ and $A = [c,d]$, the expected set A' is a closed interval $[a_1, a_2]$. The ends of this interval are equal

$$a_1 = c - \int_a^c f(x)\,\mathrm{d}x\,, \qquad a_2 = d + \int_d^b g(x)\,\mathrm{d}x\,.$$

Next we derive an expected payoff

$$E_1\big((\mathcal{M}_{A',Q})_r\big) = \{E_1(P) \colon P \in (\mathcal{M}_{A',Q})_r\}$$

for every r-cut. We can do it using the random set representation of subfamily $(\mathcal{M}_{A',Q})$ of space of all probability measures on S. This crisp family generates the simple random set W_r with the following basic probability assignment:

$$m_r(A) = q(r_1)\,, \qquad m_r(S) = q_2(r) - q_1(r),$$
$$m_r(S - A) = 1 - q_2(r)\,.$$

THEOREM 4. *The set of the expected payoff $E_l\big((\mathcal{M}_{A',Q})_r\big)$ is equal the expected value of the payoff function under random set W_r.*

We can derive this expected value in easy way. In the end we can compute the image of random set W_r by payoff function and compute the expected value of random set $l(W)$ or do it using lower and upper distribution [2] (see point (ii)).

(ii) Approximation by lower and upper distributions.

The crisp subfamily of probability measures $(\mathcal{M}_{A,Q})_r$ induces the lower $P_{*,r}$ and upper P_r^* probabilities

$$P_{*,r}(B) = \inf\{P(B) : P \in (\mathcal{M}_{A,Q})_r\}\,,$$
$$P_r^*(B) = \sup\{P(B) : P \in (\mathcal{M}_{A,Q})_r\}\,,$$

where $B \in \mathcal{B}$. These probabilities and payoff function l generate the lower and upper distributions

$$F_{*,r}(x) = P_{*,r}\big(\{s \colon l(s) \le x\}\big)\,,$$
$$F_r^*(x) = P_r^*\big(\{s : l(s) \le x\}\big)\,.$$

We can derive for $r \in (0,1]$ the lower $E_{*,r}$ and the upper E_r^* expected value using the above distributions determined on real line $\mathbb{R}$

$$E_{*,r}(r) = \int x \, \mathrm{d}F(x) , \qquad E_r^*(r) = \int x \, \mathrm{d}F_{*,r}(x) .$$

This procedure led us to construct a fuzzy subset E' on real line. Every r-cut of this fuzzy set is a closed interval

$$E'_r = [E_{*,r}, E_r^*] .$$

If A is a crisp set then E' is an expected value of payoff function [2], but in a fuzzy case we obtain

$$E_l(\mathcal{M}_{A,Q}) \subset E'.$$

In many cases it may be a proper inclusion.

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