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NONTRIVIAL EXAMPLE OF AN ASSOCIATIVE CONVOLUTION.

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In this paper we study a convolution of functions of the form:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, K(x,y)\, \mathrm{d}y\,,$$

where K is a given kernel. B. Harman studies in [1] an equivalent convolution of the form

$$(f * g)(x) = \int_{0}^{\infty} f(y)\, g\left(\frac{x}{y}\right)\, \phi(x,y)\, \mathrm{d}y\,.$$

It is easy to see that all results of this paper may be reformulated for Harman's convolution, because the additive group of all real numbers is isomorphic to the multiplicative group of all positive numbers. We shall assume that K is a measurable locally bounded function. When the functions f and g are measurable bounded functions with a compact support, then $f * g$ is defined correctly and has the same property.

We are interested in conditions under which convolution $*$ is associative, i.e.

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)\,.$$

Denote $f_{12} = f_1 * f_2$ and $f_{23} = f_2 * f_3$.

Then

$$((f_1 * f_2) * f_3)(x) = (f_{12} * f_3)(x) =$$
$$= \int_{-\infty}^{\infty} f_{12}(y)\, f_3(x-y)\, K(x,y)\, \mathrm{d}y =$$
$$= \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} f_1(u)\, f_2(y-u)\, K(y,u)\, f_3(x-y)\, K(x,y)\, \mathrm{d}u =$$
$$= \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} f_1(u)\, f_2(y-u)\, K(y,u)\, f_3(x-y)\, K(x,y)\, \mathrm{d}y =$$
$$= \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} f_1(u)\, f_2(z)\, K(z+u,u)\, f_3(x-z-u)\, K(x,z+u)\, \mathrm{d}z =$$
$$= \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} f_1(u)\, f_2(z)\, f_3(x-z-u)\, K(z+u,u)\, K(x,z+u)\, \mathrm{d}z\,,$$

and

$$((f_1 * (f_2 * f_3))(x) = (f_1 * f_{23})(x) =$$
$$= \int_{-\infty}^{\infty} f_1(u)\, f_{23}(x-u)\, K(x,u)\, \mathrm{d}y =$$
$$= \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} f_1(u)\, f_2(z)\, f_3(x-u-z)\, K(x-u,z)\, K(x,u)\, \mathrm{d}z\,.$$

Therefore condition

$$K(z+u,u)\, K(x,z+u) = K(x-u,z) K(x,u)\,, \tag{1}$$

for almost all x, u and z, is sufficient for the asociativity of convolution $*$.

Now, we shall show that this condition is also necessary. Integrating $(f_1 *$

$f_2) * f_3$ and $f_1 * (f_2 * f_3)$ we obtain

$$\int_{-\infty}^{\infty} ((f_1 * f_2) * f_3)(x)\,\mathrm{d}x =$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} f_1(u) f_2(z) f_3(x-z-u) K(z+u,u) K(x,z+u)\,\mathrm{d}z =$$

$$= \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{\infty} f_1(u) f_2(z) f_3(x-z-u) K(z+u,u) K(x,z+u)\,\mathrm{d}x =$$

$$= \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{\infty} f_1(u) f_2(z) f_3(t) K(z+u,u) K(t+u+z,z+u)\,\mathrm{d}t\,,$$

and

$$\int_{-\infty}^{\infty} (f_1 * (f_2 * f_3))(x)\,\mathrm{d}x =$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} f_1(u) f_2(z) f_3(x-u-z) K(x-u,z) K(x,u)\,\mathrm{d}z =$$

$$= \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{\infty} f_1(u) f_2(z) f_3(x-u-z) K(z-u,z) K(x,u)\,\mathrm{d}x =$$

$$= \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{\infty} f_1(u) f_2(z) f_3(t) K(t+z,z) K(t+u+z,u)\,\mathrm{d}t\,.$$

Taking f_1, f_2 and f_3 characteristic functions of arbitrary intervals we see that condition

$$K(z+u,u)\,K(t+u+z,z+u) = K(t+z,z)\,K(t+u+z,u) \tag{2}$$

for almost all t, u, z (with respect to three-dimensional Lebesgue measure) is necessary for the associativity of the convolution $*$. Equivalence of (1) and (2) is obvious after substitution $x = t+u+z$.

The following theorem yields examples of solutions of the functional equation (2).

THEOREM 1.
(i) Let k be a real function which is nonzero everywhere. Put

$$K(x,y) = \frac{k(x-y)\,k(y)}{k(x)}.$$

Then K is a solution of (2).

(ii) If K_1 and K_2 are solutions of (2) then $K = K_1 \cdot K_2$ is a solution as well.

(iii) Put $K_0(x,y) = \begin{cases} 1 & \text{for } y(x-y) \geq 0\,, \\ 0 & \text{for } y(x-y) < 0\,. \end{cases}$

Let $r > 0$. Put $K_1(x,y) = \begin{cases} 1 & \text{for } y \geq r \text{ and } x-y \geq r\,, \\ 0 & \text{otherwise.} \end{cases}$

Let $r < 0$. Put $K_2(x,y) = \begin{cases} 1 & \text{for } y \leq r \text{ and } x-y \leq r\,, \\ 0 & \text{otherwise.} \end{cases}$

Then K_0, K_1 and K_2 are solutions of (2).

P r o o f. Parts (i) and (ii) may be verified in a direct way. We shall show that K_0 is a solution of (2).
Denote $a = uz$, $b = t(u+z)$, $c = tz$ and $d = u(t+z)$. Note that

$$K_0(z+u,u) = \begin{cases} 1 & \text{for } a \geq 0\,, \\ 0 & \text{for } a < 0\,, \end{cases}$$

$$K_0(t+u+z,z+u) = \begin{cases} 1 & \text{for } b \geq 0\,, \\ 0 & \text{for } b < 0\,, \end{cases}$$

$$K_0(t+z,z) = \begin{cases} 1 & \text{for } c \geq 0\,, \\ 0 & \text{for } c < 0\,, \end{cases}$$

$$K_0(t+u+z,u) = \begin{cases} 1 & \text{for } d \geq 0\,, \\ 0 & \text{for } d < 0\,, \end{cases}$$

If $z = 0$ then $a = c = 0$ and $b = d$ and (2) is satisfied. Let $z \neq 0$. Then $u = \dfrac{a}{z}$, $t = \dfrac{c}{z}$, $b = c\left(\dfrac{a}{z^2}+1\right)$ and $d = a\left(\dfrac{c}{z^2}+1\right)$.

If $a \geq 0$ and $c \geq 0$ then $b \geq 0$ and $d \geq 0$.
If $a \geq 0$ and $c < 0$ then $b < 0$.

If $a < 0$ and $c \geq 0$ then $d < 0$. In all cases (2) is satisfied.

If $a < 0$ and $c < 0$ then (2) is satisfied as well.

Now, we shall show that K_1 is a solution of (2) (for $r > 0$).

Let $K_1(z+u,u)K_1(t+u+z,z+u) = 1$. Then $z \geq r$, $u \geq r$, $t \geq r$ and $z+u \geq r$. We have $t+z \geq 2r \geq r$. It means $K_1(t+z,z)K_1(t+u+z,u) = 1$. Let $K_1(t+z,z)K_1(t+u+z,u) = 1$. Then $t \geq r$, $z \geq r$, $t+z \geq r$ and $u \geq r$. We have $z+u \geq 2r \geq r$. It means $K_1(z+u,u)K_1(t+u+z,z+u) = 1$.

The proof for K_2 is similar.

CORROLARY. *Put* $\phi(x,y) = \dfrac{k(x)\,k(x/y)}{y\,k(x)}$, *where* k *is a real function which is nonzero everywhere. Then* ϕ *satisfies the functional equation:*

$$x\phi(xy,x)\,\phi(xyz,xy) = \phi(yz,y)\,\phi(xyz,x)\,,$$

which appears in [1].

It would be interesting to obtain all solutions of (2), particularly it is interesting whether every associative convolution is also commutative.A necessary and sufficient condition for the commutativity is :

$$K(x,y) = K(x,x-y) \qquad \text{almost for all } x \text{ and } y\,.$$

All kernels described in Theorem 1. yield a commutative convolution.

REFERENCES

[1] HARMAN, B.: *Some remarks on the associativity of the product of the the modified fuzzy numbers*, In: Abstracts of Internacional Conference on Fuzzy Theory and Applications, Czecho-Slovakia, L. Mikuláš, February 17–21, 1992.

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