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T–FUZZY OBSERVABLES

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ABSTRACT. In the paper the notion of T-fuzzy observable is given and the properties of T-fuzzy observables are studied. The relation between T-fuzzy observables and random variables with values in the fuzzy real line, see, e.g. [4], is shown, especially, the one-to-one correspondence between T_∞-fuzzy observables and finite fuzzy valued random variables is proved. The last section of the paper concerns with the calculus of T-fuzzy observables.

1. Introduction

The notion of T-fuzzy observable arose as a common generalization of two types of fuzzy observables. The first type was introduced by B. R i e č a n in [10]. In his concept the Zadeh fuzzy connectives are used (e.g., $x \wedge y = \min(x,y)$, $x \vee y = \max(x,y)$). Recently J. P y k a c z [9] has suggested (from a physical point of view) to use the Giles connectives ($x \odot y = \max(x+y-1,0)$, $x \oplus y = \min(x+y,1)$). Again, the corresponding notion of observable was introduced by R i e č a n [11]. The $\min(x,y)$ and $\max(x+y-1,0)$ are only the special types of triangular norms, namely $\min(x,y) = T_0(x,y)$ and $\max(x+y-1,0) = T_\infty(x,y)$, which was the motivation for introducing T-fuzzy observables for any triangular norm T.

Let $T : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$ be a *triangular norm* (t-norm), i.e., a binary operation which is commutative, associative, nondecreasing in each component and $T(x,1) = x$ for each $x \in \langle 0,1\rangle$ and let S be its *dual conorm*, i.e. a function $S : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$ defined by $S(x,y) = 1 - T(1-x, 1-y)$.

A continuous t-norm T is said to be strict if for each $x \in (0,1\rangle$ $T(x,y) < T(x,z)$ whenever $y < z$ and *Archimedean* if $T(x,x) < x$ for each $x \in (0,1)$.

There is an important system $\{T_s\}_{s \in \langle 0,\infty\rangle}$ of Frank's t-norms T_s which are called fundamental t-norms.These t-norms T and their t-conorms S are given by:

$$T(x,y) = \begin{cases} \min(x,y) & s = 0\,, \\ \max(x+y-1,0) & s = \infty\,, \\ x.y & s = 1\,, \\ \log_s\left(1 + \frac{(s^x-1)\cdot(s^y-1)}{s-1}\right) & s \in (0,1) \cup (1,\infty)\,, \end{cases}$$

$$S(x,y) = \begin{cases} \max(x,y) & s = 0\,, \\ \min(1,x+y) & s = \infty\,, \\ x+y-x\cdot y & s = 1\,, \\ 1-\log_s\left(1 + \frac{(s^{1-x}-1)\cdot(s^{1-y}-1)}{s-1}\right) & s \in (0,1) \cup (1,\infty)\,, \end{cases}$$

Remark 1. Let us notice that the t-norms T_s for $s \in (0,\infty)$ are strict (and so Archimedean), T is not Archimedean and the t-norm T_∞ is Archimedean, but not strict.

In the following we give a definition of a fuzzy observable in a more general form than it was done in [10] and [11].

Let $(\Omega, \mathcal{S})$ be a measurable space and let $\tau \subset \langle 0,1\rangle^\Omega$ be the generated fuzzy σ-algebra, i.e. the set of all $\mathcal{S} - \mathcal{B}(\langle 0,1\rangle)$-measurable functions and let T be a measurable t-norm.

DEFINITION 1. A T-fuzzy observable of (Ω,τ) is a mapping $\mathsf{x} : \mathcal{B}(\mathbb{R}) \to \tau$ satisfying the conditions

(i) $\mathsf{x}(E^c) = \mathsf{x}(E)' = 1 - \mathsf{x}(E)$ for each $E \in \mathcal{B}(\mathbb{R})$

(ii) $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$, $E_i \bigcap E_j = \emptyset$, $i \neq j \implies \mathsf{x}(\bigcup\limits_{n\in\mathbb{N}} E_n) = \mathop{S}\limits_{n\in\mathbb{N}} \mathsf{x}(E_n)$

where $\mathop{S}\limits_{n\in\mathbb{N}} u_n = \lim\limits_{n\to\infty} \mathop{S}\limits_{i=1}^{n} u_i$ and $\mathop{S}\limits_{i=1}^{n} u_i = S(\mathop{S}\limits_{i=1}^{n-1} u_i, u_n)$ ($\mathcal{B}(\langle 0,1\rangle)$ is the system of all Borel subsets of the interval $\langle 0,1\rangle$ and $\mathcal{B}(\mathbb{R})$ denotes the system of all Borel subsets of the real line).

Note that the cases mentioned above (when Zadeh's or Giles' fuzzy connectives are used) are, of course, included in this definition.

Directly from this definition and the properties of t-norms we obtain the following results.

PROPOSITION 1. *Let T be an Archimedean t-norm and let x be a T-fuzzy observable of (Ω,τ). Then $\mathsf{x}(\mathbb{R}) = 1_\Omega$.*

Proof. Let $\omega \in \Omega$. Then $\mathsf{x}(\emptyset)(\omega) = \mathsf{x}(\emptyset \cup \emptyset)(\omega) = S(\mathsf{x}(\emptyset)(\omega), \mathsf{x}(\emptyset)(\omega))$. Since T is an Archimedean t-norm, $\mathsf{x}(\emptyset)(\omega) \neq (0,1)$. Let $E, F \in \mathcal{B}(\mathbb{R})$,

$E \subset F$. As $x(F-E)(\omega) \geq 0$ and a t-conorm S is monotone, there holds: $S(x(E)(\omega), x(F-E)(\omega)) \geq S(x(E)(\omega), 0)$. If we use (ii) of Definition 1 and the dual property to $T(a,1) = 1$ (i.e. $S(a,0) = a$), we get $x(F)(\omega) \geq x(E)(\omega)$. It means that the T-fuzzy observable x is a non-decreasing function. Hence $x(\emptyset) = 0_\Omega$ and $x(\mathbb{R}) = 1_\Omega$. □

Note that when a t-norm T is not Archimedean, then a T-fuzzy observable need not preserve the maximal and minimal elements.

PROPOSITION 2. *Let T be a strict t-norm and let x be a T-fuzzy observable of (Ω, τ). Then $x(E)(\omega) \in \{0,1\}$ for each $E \in \mathcal{B}(\mathbb{R})$ and $\omega \in \Omega$.*

Proof. Let $E \in \mathcal{B}(\mathbb{R})$, $\omega \in \Omega$. By Definition 1, $x(E^c)(\omega) = 1 - x(E)(\omega)$ and $S(x(E)(\omega), x(E^c)(\omega)) = x(\mathbb{R})(\omega)$. Let us denote $x(E)(\omega) = a$. As T is a strict t-norm, it is also Archimedean and so, from Proposition 1, we obtain $S(a, 1-a) = 1$. This property can be valid only for $a \in \{0,1\}$. Actually, if e.g. $a \in (0,1)$ and $a < 1-a$, then

$$T(a,a) < T(a, 1-a) = 1 - S(a, 1-a) = 0,$$

which is in contradiction with the properties of T. Analogously for the other cases. □

By Proposition 2, if T is a strict t-norm and x is a T-fuzzy observable, then $x(E)$ is a crisp subset of Ω for each $E \in \mathcal{B}(\mathbb{R})$ and hence $x = f^{-1}$, where f is a random variable on $(\Omega, \mathcal{S})$. So, the most interesting T-fuzzy observables are those which are induced by Archimedean not strict t-norms. In the case of Frank's system $\{T_s\}_{s \in \langle 0, \infty \rangle}$ of fundamental t-norms it means T_∞-fuzzy observables.

2. Fuzzy real line and fuzzy-valued random variables

Following the ideas of Höhle [2, 3], Rodabaugh [12] and others, Klement introduced a concept of fuzzy-valued functions [4, 5]. We recall some basic notions. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $I = \langle 0, 1 \rangle$. *The extended fuzzy real line* $\overline{\mathbb{R}}(I)$ is the set of all functions $p : \overline{\mathbb{R}} \to I$ such that

(i) $p(-\infty) = 0$ and $p(+\infty) = 1$

(ii) $p(r) = \sup\{p(s); s < r, s \in \mathbb{R}\}$ for each $r \in \mathbb{R}$.

Note that the fuzzy real number $p \in \overline{\mathbb{R}}(I)$ is the cumulative distribution function on $\overline{\mathbb{R}}$. A fuzzy number p can be interpreted as follows: $p(r)$ is a degree at which p is less than (non-fuzzy) number r. A non-fuzzy number r is identified with the characteristic function of the set (r,∞). *A fuzzy number p is said to be finite* if $\inf\{p(r); r \in \mathbb{R}\} = 0$ and $\sup\{p(r); r \in \mathbb{R}\} = 1$. A finite fuzzy number is a cumulative distribution on $\mathbb{R}$ and vice versa. The set of all finite fuzzy numbers will be denoted by $\mathbb{R}(I)$.

The partial ordering $\angle$ on $\overline{\mathbb{R}}(I)$ is given by

$$p \angle u \iff \forall r \in \overline{\mathbb{R}} : p(r) \geq u(r).$$

Now, let $f : \langle a,b\rangle \to \langle c,d\rangle$ be a non-decreasing function, left-continuous in (a,b) with $f(a) = c$. Then the *quasi-inverse* of f is a function $[f]^q : \langle c,d\rangle \to \langle a,b\rangle$ defined by

$$[f]^q(s) = \sup\{r \in \langle a,b\rangle; f(r)\langle s\},$$
$$[f]^q(c) = a.$$

The quasi-inverse of f is again a non-decreasing function, left-continuous in (c,d) and $[[f]^q]^q = f$. The set of all quasi-inverses of fuzzy numbers $p \in \overline{\mathbb{R}}(I)$ will be denoted by $\overline{\mathbb{R}}^q(I)$.

Due to the fact that the mapping $q : p \mapsto [p]^q$ is an involution from $\overline{\mathbb{R}}(I)$ onto $\overline{\mathbb{R}}^q(I)$, it is possible to introduce an algebraic structure on $\overline{\mathbb{R}}(I)$ as follows:

Let $p, u \in \overline{\mathbb{R}}(I)$. Then

$$p \angle u \iff [p]^q(\alpha) \leq [u]^q(\alpha) \qquad \text{for all} \quad \alpha \in I, \tag{1}$$

$$[p \oplus u]^q(\alpha) = [p]^q(\alpha) + [u]^q(\alpha) \tag{2}$$
$$\begin{aligned}[p \otimes u]^q(\alpha) = \sup\{&[p^+]^q(\beta)\cdot[u^+]^q(\beta) + [p^+]^q(1-\beta)\cdot[u^-]^q(\beta) + \\ &+ [p^-]^q(\beta)\cdot[u^+]^q(1-\beta) + [p^-]^q(1-\beta)\cdot[u^-]^q(1-\beta); \\ &\beta < \alpha\},\end{aligned}$$

where $p^+(r) = \begin{cases} 0, & r \leq 0, \\ p(r), & r > 0, \end{cases}$ and $p^-(r) = \begin{cases} p(r), & r \leq 0, \\ 1, & r > 0. \end{cases}$

The previous formulae for $p \oplus u$ and $p \otimes u$ can be used if their right-hand sides make sense.

$\overline{\mathbb{R}}(I)$ can be considered as a subspace of $\langle 0,1\rangle^{\overline{\mathbb{R}}}$. Thus we can endow it with the product σ-algebra and it make sense to consider measurable functions $X : \Omega \to \mathbb{R}(I)$, which we will call *fuzzy-valued random variables* (measurability of these functions is defined as usually).

By Proposition 2.1. in [5], the measurability of a function $X : \Omega \to \mathbb{R}(I)$ is equivalent to the existence of a Markov kernel $\mathcal{K}$ from $(\Omega, \mathcal{S})$ to $(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ such that for all $(\omega, t) \in \Omega \times \overline{\mathbb{R}}$, $X(\omega)(t) = \mathcal{K}(\omega, \langle -\infty, t))$.

Recall that a *Markov kernel* $\mathcal{K}$ from $(\Omega, \mathcal{S})$ to $(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{B}}))$ is a function $\mathcal{K} : \Omega \times \mathcal{B}(\overline{\mathbb{R}}) \to \langle 0, 1\rangle$ fulfilling the properties:

$\mathcal{K}(\cdot, E) : \Omega \to \langle 0, 1\rangle$ is a measurable function for each $E \in \mathcal{B}(\overline{\mathbb{R}})$

$\mathcal{K}(\omega, \cdot) : \mathcal{B}(\overline{\mathbb{R}}) \to \langle 0, 1\rangle$ is a probability distribution for each fixed $\omega \in \Omega$.

Note that [5] deals only with non-negative fuzzy numbers. The extension to $\overline{\mathbb{R}}(I)$ is evident.

3. T_s-fuzzy observables

Now let us deal in more detail with the properties of the T_s-fuzzy observables.

If $s = 0$, then the T_0-fuzzy observable is a fuzzy observable introduced by Riečan in [10]. The T_0-fuzzy observable x can be considered as a special kind of random variables X with values in the extended fuzzy real line. Namely,

$$X : \Omega \to \overline{\mathbb{R}}(I), \quad X(\omega)(t) = x(\langle -\infty, t))(\omega) = \begin{cases} x(\mathbb{R})(\omega), & t > f(\omega), \\ x(0)(\omega), & t \le f(\omega), \end{cases}$$

for any $t \in \mathbb{R}$. f is a real random variable on $(\Omega, \mathcal{S})$. For more details see, e.g., [8].

As it was shown above, T_s-fuzzy observables for $s \in (0, \infty)$ are inverses of random variables (Remark 1 and Proposition 2).

Recently Mesiar [7] pointed out the relation between T_∞-fuzzy observables and fuzzy-valued random variables. We have obtained the following results.

PROPOSITION 3. *Let x be a T_∞-fuzzy observable of $(\Omega, \mathcal{S})$. Then for any fixed $\omega \in \Omega$ the set function $x(\cdot)(\omega) : \mathcal{B}(\mathbb{R}) \to \langle 0, 1\rangle$ is a probability measure on $\mathcal{B}(\mathbb{R})$.*

Proof. If we fix $\omega \in \Omega$, then $x(\cdot)(\omega)$ is a set function defined on $\mathcal{B}(\mathbb{R})$ with the following properties:

(i) $x(\mathbb{R})(\omega) = 1$,

(ii) $x(E^c)(\omega) = 1 - x(E)(\omega)$ for each $E \in \mathcal{B}(\mathbb{R})$,

(iii) $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$, $E_i \cap E_j = \emptyset$ for $i \ne j \implies x\big(\bigcup_{n\in\mathbb{N}} E_n\big) = \sum_{n\in\mathbb{N}} x(E_n)$.

The property (i) is valid since T_∞ is an Archimedean t-norm and (ii) is a part of the definition of any T-fuzzy observable. We have still to prove (iii). There holds

$$x\Big(\bigcup_n E_n\Big) = \underset{n}{S_\infty}\, x(E_n) = \min(1, \Sigma_n x(E_n)).$$

If $\sum_n x(E_n)(\omega) \le 1$ for any ω, then $\min(1, \sum_n x(E_n)(\omega)) = \sum_n x(E_n)(\omega)$ and so (iii) holds.

Let there exist $\omega \in \Omega$ such that $\sum_n x(E_n)(\omega) > 1$. Let us choose k, m such that: $0 < \sum_{i=1}^{n} x(E_i)(\omega) \le 1$; $0 \le \sum_{i=k+1}^{m} x(E_i)(\omega) \le 1$, but $\sum_{i=1}^{k} x(E_i)(\omega) + \sum_{i=k+1}^{m} x(E_i)(\omega) = \sum_{i=1}^{m} x(E_i)(\omega) > 1$. Let us put $F = \bigcup_{i=1}^{k} E_i$. Then there holds:

$$\begin{aligned} x(F)(\omega) &= x\Big(\bigcup_{i=1}^{k} E\Big)(\omega) = \underset{i=1}{\overset{k}{S_\infty}}\, x(E_i)(\omega) = \\ &= \min\Big(1, \sum_{i=1}^{k} x(E_i)(\omega)\Big) = \sum_{i=1}^{k} x(E_i)(\omega). \end{aligned}$$

Further, since $\bigcup_{i=k+1}^{m} E_i \subseteq F^c$ there holds:

$$\begin{aligned} x(F^c)(\omega) &= 1 - x(F)(\omega) = 1 - \sum_{i=1}^{k} x(E_i)(\omega) \ge \\ &\ge x\Big(\bigcup_{i=k+1}^{m} E_i\Big)(\omega) = \sum_{i=k+1}^{m} x(E_i)(\omega), \end{aligned}$$

which implies

$$\sum_{i=1}^{k} x(E_i)(\omega) + \sum_{i=k+1}^{m} x(E_i)(\omega) = \sum_{i=1}^{m} x(E_i)(\omega) \le 1.$$

The obtained result is in contradiction with the assumption and so (iii) is proved. Hence $x(\cdot)(\omega)$ is a probability measure on $\mathcal{B}(\mathbb{R})$. □

COROLLARY 1. *Let x be a T_∞-fuzzy observable. Then for any fixed $\omega \in \Omega$ the function $x\big((-\infty,\cdot)\big)(\omega) : \mathbb{R} \to \langle 0,1\rangle$ is a finite fuzzy number.*

P r o o f. The function $x\big((-\infty,\cdot)\big)(\omega)$ is by Proposition 3 a probability distribution on $\mathbb{R}$ for each fixed $\omega \in \Omega$. Thus by Section 2 it is a finite fuzzy number. □

By the previous results the function $X : \Omega \to \mathbb{R}(I)$ defined by $X(\omega) = p_\omega$, where $p_\omega(t) = x\big((-\infty,t)\big)(\omega)$, is a finite fuzzy-valued function. In the following theorem the relation between T_∞-fuzzy observables and finite fuzzy-valued random variables is given.

THEOREM 1. *A mapping $x : \mathcal{B}(\mathbb{R}) \to \tau$ is a T_∞-fuzzy observable of (Ω,τ) if and only if $x\big((-\infty,t)\big)(\omega) = X(\omega)(t)$, $t \in \mathbb{R}$ defines a finite fuzzy-valued random variable X on $(\Omega,\mathcal{S})$.*

P r o o f. (i) Let x be a T_∞-fuzzy observable of (Ω,τ). Let us define a mapping $\mathcal{K} : \Omega \times \mathcal{B}(\overline{\mathbb{R}}) \to \langle 0,1\rangle$ by $\mathcal{K}(\omega,E) = x(E - \{-\infty,+\infty\})(\omega)$. For any $E \in \mathcal{B}(\overline{\mathbb{R}})$ $\mathcal{K}(\cdot,E)$ is an element of τ, i.e. it is an $\mathcal{S}$-measurable function. Further for any fixed $\omega \in \Omega$, $\mathcal{K}(\omega,\cdot)$ is a probability distribution on $\mathcal{B}(\overline{\mathbb{R}})$, i.e., K is a Markov kernel. The finitness of X follows from Corollary 1.

(ii) Let X be a finite fuzzy-valued random variable on $(\Omega,\mathcal{S})$. The Markov kernel $\mathcal{K}$ corresponding to X induces, for any fixed $\omega \in \Omega$, a probability distribution on $\mathbb{R}$. Hence $\mathcal{K}(\omega,E^c) = 1 - \mathcal{K}(\omega,E)$ for any $E \in \mathcal{B}(\mathbb{R})$ and $\mathcal{K}(\omega,\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \mathcal{K}(\omega,E_n) = S_\infty \mathcal{K}(\omega,E_n)$ for any sequence $\{E_n\} \subset \mathcal{B}(\mathbb{R})$ of pairwise disjoint sets. It is obvious that if we put

$$x(E) = \mathcal{K}(\cdot,E), E \in \mathcal{B}(\mathbb{R})$$

then x is a T_∞-fuzzy observable of (Ω,τ). □

4. Calculus for T_s-fuzzy observables

The calculus for T_0-fuzzy observables was introduced by D v u r e č e n s k i j and T i r p á k o v á [1]. They defined the sum z of two T_0-fuzzy observables x,y by

$$z\big((-\infty,t)\big) = \bigvee_{r\in Q} \Big[x\big((-\infty,r)\big) \wedge y\big((-\infty,t-r)\big)\Big]. \tag{4}$$

If we define the sum for T_∞-fuzzy observables x,y in the same way as it was done for T_0-fuzzy observables, we get the following statement.

PROPOSITION 4. *If x, y are T_∞-fuzzy observables, then $z = x + y$ defined by (4) is also a T_∞-fuzzy observable.*

Proof. First we show that $z((-\infty,\cdot))(\omega)$ is for any fixed $\omega \in \Omega$ a probability distribution on $\mathbb{R}$.

It is evident that for any fixed $\omega \in \Omega$, $z((-\infty,\cdot))(\omega)$ is a non-decreasing and left-continuous function on $\mathbb{R}$.

Since $x((-\infty,\cdot))(\omega)$ and $y((-\infty,\cdot)(\omega)$ are probability distributions on $\mathbb{R}$, for each $\varepsilon > 0$ there exists $t_0 \in \mathbb{R}$ such that

$$x((-\infty,t))(\omega) < \frac{\varepsilon}{2} \quad \text{and} \quad y((-\infty,t))(\omega) < \frac{\varepsilon}{2} \quad \text{for all} \quad t \le t_0 .$$

Let $r \in Q$. Then either $r \le t_0$ or $t_0 \le r$. If $r \le t_0$, then $x((-\infty,r))(\omega) < \frac{\varepsilon}{2}$. If $t_0 \le r$, which is the same as $2t_0 - r \le t_0$, then $y((-\infty,2t_0-r))(\omega) < \frac{\varepsilon}{2}$. It means that

$$x((-\infty,r))(\omega) \wedge y((-\infty,2t_0-r))(\omega) < \frac{\varepsilon}{2}$$

for each $r \in Q$ and therefore

$$z((-\infty,2t_0))(\omega) = V_{r\in Q}[x((-\infty,r))(\omega) \wedge y((-\infty,2t_0-r))(\omega)] \le \frac{\varepsilon}{2} < \varepsilon .$$

So we have shown that for each $\varepsilon > 0$ there exists $t^* = 2t_0$ such that $z((-\infty,t))(\omega) < \varepsilon$ for all $t \le t^*$. It means that

$$\lim_{t\to-\infty} z((-\infty,t))(\omega) = 0 .$$

The second boundary condition $\lim_{t\to\infty} z((-\infty,t))(\omega) = 1$ can be proved analogously.

Since $z((-\infty,\cdot))(\omega)$ is a probability distribution on $\mathbb{R}$, it is a finite fuzzy number and the function

$Z : \Omega \to \mathbb{R}(I), Z(\omega)(t) = z((-\infty,t))(\omega)$, $t \in \mathbb{R}$, is a fuzzy-valued random variable (the measurability can be proved as in Theorem 1) and by Theorem 1, $z = x + y$ is a T_∞-fuzzy observable.

Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel measurable mapping and let x be a T_s-fuzzy observable. Then the mapping defined by

$$h(x)(E) = x(h^{-1}(E)) \quad \text{for any} \quad E \in \mathcal{B}(\mathbb{R})$$

is a T_s-fuzzy observable.

Using this fact, the product of $\mathcal{T}_s$-fuzzy observables $\mathbf{x}, \mathbf{y}$ can be defined by

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}\left[(\mathbf{x}+\mathbf{y})^2 - \mathbf{x}^2 - \mathbf{y}^2\right]. \tag{5}$$

For more details see e.g. [7].

As we have shown above, there is a one-to-one correspondence between $\mathcal{T}_\infty$-fuzzy observables and finite fuzzy valued random variables. The sum and product of fuzzy valued random variables X and Y is given pointwise

$$(X+Y)(\omega) = X(\omega) \oplus Y(\omega), \qquad \omega \in \Omega, \tag{6}$$

$$(X \cdot Y)(\omega) = X(\omega) \otimes Y(\omega). \tag{7}$$

On the right-side the addition and multiplication of fuzzy numbers defined by (2) and (3) is used.

The natural question arises, whether both types of calculus for $\mathcal{T}_\infty$-fuzzy observables are equivalent. The following statement holds.

Proposition 5. *Let $\mathbf{x}, \mathbf{y}$ be $\mathcal{T}_\infty$-fuzzy observables and let X, Y be finite fuzzy valued random variables corresponding to $\mathbf{x}$ and $\mathbf{y}$ by Theorem 1. Then the sum $X+Y$ defined by (6) corresponds to sum $\mathbf{x}+\mathbf{y}$ defined by (4).*

We omit the details of the proof.

The analogous statement for the product of $\mathcal{T}_\infty$-fuzzy observables given by (5) and the product of finite fuzzy valued random variables given by (7) is not true in general. More details can be found in another paper.

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