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ON ASSOCIATIVITY OF THE PRODUCT OF MODIFIED REAL FUZZY NUMBERS

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The notion of the modified real fuzzy numbers was defined in [1], [2], [3]. This paper takes up again paper [3] where the commutativity property was analyzed. In order to get a better survey let us recall the notion of the modified real fuzzy numbers.

Let $\mathbb{R}$ be the set of the real numbers endowed with usual topology. Let $\mathcal{F} = \{f \in \langle 0,1\rangle^{\mathbb{R}};\ f$–piecewise continuous, $\overline{\operatorname{supp}(f)}$ be compact, $0 < \operatorname{essup}(f) < \infty\}$. $\mathcal{F}^+ = \{f \in \mathcal{F};\ \operatorname{supp}(f) \subset \langle 0,+\infty)\}$. The binary relation $E \subset \mathcal{F} \times \mathcal{F}$, $(f,g) \in E \overset{\text{def}}{\Longrightarrow} \exists\, \alpha \in (0,+\infty)\colon f = \alpha g$ is an equivalence relation. The elements of the factor set $\Phi = \mathcal{F}/E$ resp. $\Phi^+ = \mathcal{F}^+/E$ will be called the modified proper real fuzzy numbers or nonnegative modified proper real fuzzy numbers respectively. In order to assume the improper real fuzzy numbers, which are representatives of the crisp ones, we shall consider the class $\mathcal{D}$ of the Dirac δ-functions $\delta_a(x)$ for $a \in \mathbb{R}$. Elements of the set $\tilde{\Phi} = (\mathcal{F} \cup \mathcal{D})/E$ will be called modified real fuzzy numbers or nonnegative ones. Because every decomposition class $F \in \Phi^+$ (or $\tilde{\Phi}^+$) is uniquely determined by an arbitrary representant $f \in F$, we can without fear of being confused write f instead of F. We shall write $F = E(f)$ to emphasize the relation between f and F.

One of the ways how to define the binary operation $\tilde{\cdot}$ defined on the set Φ^+ which corresponds to the ordinary crisp multiplication is as follows:

$$(f \,\tilde{\cdot}\, g)(x) = \int_0^{+\infty} f(u)\, g\Big(\frac{x}{u}\Big)\, \varphi(x,u)\, du\,. \tag{1}$$

This relation depends essentially on the choice of function φ. The natural question may arise—how to choose the function φ so that $\tilde{\cdot}$ should have the special required properties? The commutativity property was studied in [3]. The result is the satisfaction of the functional equation

$$\varphi(x, x/y)\, x/y^2 = \varphi(x,y)\,; \quad x \geq 0,\ y > 0\,. \tag{2}$$

The formal application of relation (1) on the set $(\mathcal{F}^+ \cup \mathcal{D}^+)$ could cause some confusions. In order to avoid them we must implement some rearrangement in the following definition.

DEFINITION 1. Let $F, G \in \Phi^+$. Then we define $\ddot{\cdot} : \tilde{\Phi}^+ \times \tilde{\Phi}^+ \to \tilde{\Phi}^+$, $(F, G) \mapsto F \ddot{\cdot} G = E(h)$, where $F = E(f)$, $G = E(g)$ and

i) $h(x) = \int\limits_0^{+\infty} f(u)\, g(\frac{x}{u}) \varphi(x, u)\, du$ if $(f, g) \in (\mathcal{F}^+ \cup \mathcal{D}^+) \times (\mathcal{F}^+ \cup \mathcal{D}^+) \setminus (\mathcal{D}^+ \times \mathcal{D}^+)$

ii) $h(x) = a\, \varphi(ab, a)\, \delta_{ab}(x)$ if $(f, g) \in \mathcal{D}^+ \times \mathcal{D}^+$, $f = \delta_a$, $g = \delta_b$. (3)

Remark 1. We shall assume that the function $\varphi(x, u)$ is locally integrable with respect to u and piecewise continuous with respect to x. In this case $h \in \mathcal{F}^+$ for $f, g \in \mathcal{F}^+$.

Remark 2. Relation (3) is based on the following motivation:

$$\text{If} \quad f_n \to \delta_a,\ g_n \to \delta_b,\ f_n, g_n \in \mathcal{F}^+ \quad \text{then}$$
$$(f_n \ddot{\cdot} g_n) \to a\, \varphi(ab, a)\, \delta_{ab} \quad \text{with respect to (1).}$$

Remark 3. Due to the above it is easy to see, that the binary operation from Definition 1 is defined correctly. Moreover, we can further write $f \ddot{\cdot} g$ instead of $F \ddot{\cdot} G$.

THEOREM 1. *The binary operation* $\ddot{\cdot}$ *is associative if and only if the following functional equation is fulfilled:*

$$x\, \varphi(xyz, xy)\, \varphi(xy, x) = \varphi(xy, z)\, \varphi(yz, y) \quad \text{for} \quad x, y, z > 0\,. \tag{4}$$

Proof. We will work with the expressions of the types $f \ddot{\cdot} (g \ddot{\cdot} h)$ or $(f \ddot{\cdot} g) \ddot{\cdot} h$ which are appropriated to the triad (f, g, h) where $f, g, h \in \mathcal{F}^+ \cup \mathcal{D}^+$. All the eight possible situations written symbolically are (f, g, h), (δ, g, h), (f, δ, h), (f, g, δ), (f, δ, δ), (f, g, δ), (δ, δ, h), (δ, δ, δ). We shall prove the first, second and fifth cases only. The other proofs are quite analogical ones. The eighth case is a straightforward consequence of relation (3).

a) Let $f, g, f \in \mathcal{F}^+$. Due to the properties of the elements of $\mathcal{F}^+$ we can use the Fubini theorem on double integrals which represents $f \ddot{\cdot} (g \ddot{\cdot} h)$ and $(f \ddot{\cdot} g) \ddot{\cdot} h$. After regular transformation

$$(0, +\infty) \times (0, +\infty) \longleftrightarrow (0, +\infty) \times (0, +\infty), \quad (s, t) \longleftrightarrow (u, w) = \left(s, \frac{s}{t}\right),$$

we obtain consequently

$$[f \dot{\cdot} (g \dot{\cdot} h)](x) = \int_0^{+\infty}\int_0^{+\infty} \Big\{ f(u)\, g(w)\, h\Big(\frac{x}{uv}\Big) \varphi\Big(\frac{x}{u}, w\Big) \varphi(x,u) \Big\} \mathrm{d}u\, \mathrm{d}v =$$

$$= \int_0^{+\infty}\int_0^{+\infty} \Big\{ f(s)\, g\Big(\frac{t}{s}\Big) h\Big(\frac{x}{t}\Big) \varphi\Big(\frac{x}{s}, \frac{t}{s}\Big) \varphi(x,s)\, \frac{1}{s} \Big\} \mathrm{d}s\, \mathrm{d}t =$$

$$= \int_0^{+\infty}\int_0^{+\infty} \Big\{ f(u)\, g\Big(\frac{w}{u}\Big) h\Big(\frac{x}{w}\Big) \varphi(w,u) \varphi(x,w) \Big\} \mathrm{d}u\, \mathrm{d}v =$$

$$= \big[(f \dot{\cdot} g) \dot{\cdot} h\big](x).$$

Due to the independence of the integral variable denotation the above equality is equivalent to the condition

$$\varphi(w,u)\, \varphi(x,w) = \varphi\Big(\frac{x}{u}, \frac{w}{u}\Big) \varphi(x,u)\, \frac{1}{u},$$

which is equivalent to (4) due to the regular transformation $w = xyz$, $u = x$.

b) Let $g, h \in \mathcal{F}^+$, $a > 0$. After a short computation we have

$$[\delta_a \dot{\cdot} (g \dot{\cdot} h)](x) = \int_0^{+\infty} g(w)\, h\Big(\frac{x}{aw}\Big) \varphi\Big(\frac{x}{a}, w\Big) \varphi(x,a)\, \mathrm{d}w,$$

$$[(\delta_a \dot{\cdot} g) \dot{\cdot} h](x) = \int_0^{+\infty} g\Big(\frac{u}{a}\Big) h\Big(\frac{x}{u}\Big) \varphi(u,a)\, \varphi(x,u)\, \mathrm{d}u =$$

$$= a \int_0^{+\infty} g(w)\, h\Big(\frac{x}{aw}\Big) \varphi(aw,a)\, \varphi(x,aw)\, \mathrm{d}w.$$

The equality of the above expressions is equivalent to the equation $a\,\varphi(x,aw)\, \varphi(aw,a) = \varphi(\frac{x}{a}, w)\, \varphi(x,a)$. This equation is equivalent to (4) due to the substitution $a = x, w = yz$.

c) Let $h \in \mathcal{F}^+$; $a, b > 0$. It is easy to compute

$$[\delta_a \dot{\cdot} (\delta_b \dot{\cdot} h)](x) = h\Big(\frac{x}{ab}\Big) \varphi\Big(\frac{x}{a}, b\Big) \varphi(x,a),$$

$$[(\delta_a \dot{\cdot} \delta_b) \dot{\cdot} h](x) = h\Big(\frac{x}{ab}\Big) a\, \varphi(ab,a)\, \varphi(x,ab).$$

After substitution $x = abz$ we obtain the equation which is equivalent to (4). □

THEOREM 2. *Let $\varphi(x,y) = u(x)\,v(y)$ where $u(x)$ is a continuous and $v(y)$ a differentiable function. The function $\varphi(x,y)$ satisfies condition (4) iff $u(x)$ is a constant function and $v(y) = \frac{c}{y}$, $c = const$.*

Proof. It is easy to see that the function $\varphi(x,y) = \frac{\alpha}{y}$; $\alpha = \text{const}$ satisfies the equation (4). In order to prove the necessity let us put the function $\varphi(x,y) = u(x)\,v(y)$ into (4). We obtain

$$u(y,z) = \frac{xu(xy)v(xy)}{v(y)}, \tag{5}$$

and then $u(t) \equiv \text{const}$ for all $t \in (0,+\infty)$. Further, reduction of (5) yields

$$x\,v(xy) = v(y). \tag{6}$$

Using (6) we obtain

$$v'(y) = \lim_{x\to 1} \frac{v(xy) - v(y)}{xy - y} = -\frac{v(y)}{y}.$$

Solution of this differential equation is of the form $v(y) = -\frac{\alpha}{y}$. □

The concluding remarks will deal with some open questions. What is the set of all solutions of equation (4)? Because the solution from Theorem 2 satisfies equation (2) we see, that the classical Mellin convolution

$$(f \,\tilde{\cdot}\, g)(x) = \int_0^{+\infty} f(u)\,g\Big(\frac{x}{u}\Big)\,\frac{du}{u}$$

can serve as a base for defining the associative and commutative product of the modified real fuzzy numbers. The second question – is this result a chance or every associative convolution of the type (1) is commutative? The last question concerns the possibilities of an effective technique of computing these convolutions. How to define the special integral transformation $I_\varphi : \mathcal{F}^+ \to \mathcal{C}^{\mathcal{C}}$ ($\mathcal{C}$ is the set of complex numbers) so that

$$I_\varphi(f \,\tilde{\cdot}\, g) = I_\varphi(f)\, I_\varphi(g)\,?$$

What are its possible properties ?

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