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ON A REPRESENTATION OF OBSERVABLES IN D–POSETS OF FUZZY SETS

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1. Introduction

The functional calculus for observables in fuzzy quantum spaces is based on a representation of these observables by a suitable observable and by suitable Borel measurable functions [1].

F. Kôpka suggested how to build up the probability theory on the so-called D–poset of fuzzy sets [3].

Let F be a family of fuzzy sets (i.e. a family of real functions defined on a nonempty set X with their values in the interval $[0,1]$). We say that there is a difference defined on the family F, if there is just one element from F for any $f,g \in F$, $f \le g$, we denote it $g \setminus f$, such that the following conditions hold

(1) $g \setminus f \le g$;

(2) $g \setminus (g \setminus f) = f$;

(3) if $f \le g \le h$, then $h \setminus g \le h \setminus f$ and $(h \setminus f) \setminus (h \setminus g) = g \setminus f$.

A D-poset of fuzzy sets is a family F of fuzzy sets such that

(4) there exists a difference on F;

(5) if $1_x(t) = 1$ for every $t \in X$, then $1_x \in F$;

(6) if $\{f_n\}_{n\in\mathbb{N}} \subseteq F$, $f_n \nearrow f$, then $f \in F$.

An observable on a D–poset F is a mapping $x : \mathcal{B}(\mathbb{R}) \to F$ with the following properties:

(7) $x(\mathbb{R}) = 1_x$;

(8) if $\{A_n\}_{n\in N} \subset \mathcal{B}(\mathbb{R})$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$;

(9) if $A, B \in \mathcal{B}(\mathbb{R})$, $A \subseteq B$, then $x(B \setminus A) = x(B) \setminus x(A)$,

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of the real line $\mathbb{R}$. In particular, for $f \in F$, the mapping $x_f : \mathcal{B}(\mathbb{R}) \to F$ defined by

$$x_f(E) = \begin{cases} 1_x, & \text{if } \{0,1\} \cap E = \{0,1\}, \\ f, & \text{if } \{0,1\} \cap E = \{1\}, \\ 1_x \setminus f, & \text{if } \{0,1\} \cap E = \{0\}, \\ 0_x, & \text{if } \{0,1\} \cap E = \emptyset, \end{cases}$$

is an observable on F called the indicator of the element f.

A set $\mathcal{R}(x) = \{x(E) \colon E \in \mathcal{B}(\mathbb{R})\}$ is said to be a range of an observable x. The range of an observable on a D-poset is not closed with respect to the difference of its elements, in general. Indeed, if f be the constant function identically equal to 0.8 (i.e. $f = 0.8_x$), then the range $\mathcal{R}(x_f) = \{1_x, 0.8_x, 0.2_x, 0_x\}$ does not contain the difference of 0.8_x and 0.2_x.

2. Representation of observables

LEMMA 1. *Let x be an observable on F. Then the following assertions are true.*

(i) $x(A \cup B) \setminus x(B) = x(A) \setminus x(A \cap B)$ *for all* $A, B \in \mathcal{B}(\mathbb{R})$.

(ii) *If* $x(A) = 1_x$, *then* $(x(A) \setminus x(B)) \in \mathcal{R}(x)$ *and moreover* $x(A \cap B) = x(B)$ *for any* $B \in \mathcal{B}(\mathbb{R})$.

(iii) *If* $x(B) = 0_x$, *then* $(x(A) \setminus x(B)) \in R(x)$ *and moreover* $x(A \cup B) = x(A)$ *for any* $A \in \mathcal{B}(\mathbb{R})$.

(iv) *If* $x(A) \le x(B)$, *then* $x(B) \setminus x(A) \le x(B \setminus A)$.

LEMMA 2. *Let $x \colon \mathcal{B}(\mathbb{R}) \to F$ be an observable on F and let $T \colon \mathbb{R} \to \mathbb{R}$ be a Borel measurable mapping. Then the mapping $y \colon \mathcal{B}(\mathbb{R}) \to F$ defined by the formula $y(E) = x(T^{-1}(E))$, $E \in \mathcal{B}(\mathbb{R})$, is also an observable (and we write $y = x \circ T^{-1}$).*

DEFINITION 1. Let x be an observable on F and let G be a subset of the range $\mathcal{R}(x)$. We say that the observable x has a *V-property* on G, if for every two Borel sets A, B, $A \subseteq B$, and for every element $c \in G$ such that $x(A) \le c \le x(B)$, there exists a Borel set C such that $x(C) = c$ and $A \subseteq C \subseteq B$.

EXAMPLE 1. Let x, y be two observables on F defined via

$$x(E) = \begin{cases} 1_x, & \text{if } 1,2,3 \in E, \\ (3/4)_x, & \text{if } 1,3 \in E \text{ and } 2 \notin E \text{ or } 2,3 \in E \text{ and } 1 \notin E, \\ (1/2)_x, & \text{if } 1,2 \in E \text{ and } 3 \notin E \text{ or } 3 \in E \text{ and } 1,2 \notin E, \\ (1/4)_x, & \text{if } 1 \in E \text{ and } 2,3 \notin E \text{ or } 2 \in E \text{ and } 1,3 \notin E, \\ 0_x, & \text{if } 1,2,3 \notin E, \end{cases}$$

and

$$y(F) = \begin{cases} 1_x, & \text{if } a,b,c,d \in F, \\ (3/4)_x, & \text{if } a,b,c \in F \text{ and } d \notin F, \\ (1/2)_x, & \text{if } a,b \in F \text{ and } c,d \notin F, \\ (1/4)_x, & \text{if } a \in F \text{ and } b,c,d \notin F, \\ 0_x, & \text{if } a,b,c,d \notin F, \end{cases}$$

where a, b, c, d are mutually different elements from the set $\{4,5,6,7\}$. The observable x has not the V-property on the range $\mathcal{R}(x)$ and the observable y has the V-property on the range $\mathcal{R}(y)$.

LEMMA 3. *Let x be an observable such that the following implication holds: If $x(A) \le x(B)$, then $x(A \cup B) = x(B)$ and $x(A \cap B) = x(A)$. Then the observable x has the V-property on the range $\mathcal{R}(x)$.*

Proof. Let $A, B \in \mathcal{B}(\mathbb{R})$, $A \subseteq B$ and $c \in \mathcal{R}(x)$ such that $x(A) \le c \le x(B)$. Then there exists a set $C_1 \in \mathcal{B}(\mathbb{R})$ such that $c = x(C_1)$. Put $C = A \cup (B \cap C_1)$. Since $x(B \cap C_1) = x(C_1) = c$, we get $x(C) = x(A \cup (B \cap C_1)) = x(B \cap C_1) = c$. It is clear that $A \subseteq C \subseteq B$. □

LEMMA 4. *Let x be an observable with the V-property on $\mathcal{R}(x)$. If $x(A) \le x(B)$, then $(x(B) \setminus x(A)) \in \mathcal{R}(x)$.*

Proof. We have $x(A) \le x(B) \le x(A \cup B)$. The V-property of the observable x implies the existence of a set $B_1 \in \mathcal{B}(\mathbb{R})$ such that $A \subseteq B_1 \subseteq A \cup B$ and $x(B) = x(B_1)$, moreover, $x(B) \setminus x(A) = x(B_1) \setminus x(A) = x(B_1 \setminus A) \in \mathcal{R}(x)$. □

THEOREM 1. *Let x and y be two observables such that the following conditions hold:*

(i) $\mathcal{R}(x) \subseteq R(y)$;

(ii) *the observable y has the V-property on $\mathcal{R}(x)$.*

Then there exists a Borel measurable function $T\colon \mathbb{R} \to \mathbb{R}$ *such that* $x(E) = y(T^{-1}(E))$ *for every* $E \in \mathcal{B}(\mathbb{R})$.

Proof. The proof of this theorem is analogous to the proof of the Theorem 4 in [4]. It is based on two steps:
1. Let $\mathbb{Q}$ be the set of all (mutually different) rational numbers. We construct (by mathematical induction) a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$ such that

a) $x((-\infty, r_i)) = y(A_i)$ for any $r_i \in \mathbb{Q}$;

b) if $r_i, r_j \in \mathbb{Q}$, $r_i < r_j$, then $A_i \subseteq A_j$;

c) $\bigcap\limits_{n\in\mathbb{N}} A_n = \emptyset$.

2. We define the real function $T\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$T(t) = \begin{cases} 0, & \text{if } t \in \mathbb{R} \setminus \bigcup\limits_{n\in\mathbb{N}} A_n, \\ \inf\{r_i \in \mathbb{Q}\colon t \in A_i\}, & \text{if } t \in \bigcup\limits_{n\in\mathbb{N}} A_n,\ t \in \mathbb{R}. \end{cases}$$

□

Let $y, x_1, x_2, \dots, x_n$, $n \geq 2$, be observables on P such that

(i) $\bigcup\limits_{i=1}^{n} \mathcal{R}(x_i) \subseteq \mathcal{R}(y)$;

(ii) the observable y has the V-property on $\bigcup\limits_{i=1}^{n} \mathcal{R}(x_i)$.

Then the observable z defined by the formula

$$z = y \circ (f_1 + f_2 + \dots + f_n)^{-1}, \quad \text{where} \quad x_i = y \circ f_i^{-1},$$

is called the sum of observables $x_1, \dots, x_n$ and we write $z = x_1 + x_2 + \dots + x_n$.

We can define various other "functions" of observables in a similar manner, for example, the difference or the product. Besides, the representation theorem enables to introduce a so called joint observable and prove, for example, the weak law of large numbers [2].

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