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A REPREENTATION OF FUZZY QUANTUM POSETS OF TYPE I, II

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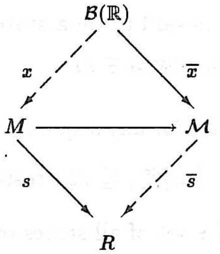
ABSTRACT. Let (Ω, M) be a fuzzy quantum poset of type I, II, or FQP of type I,II for short. For Boolean representations of fuzzy quantum spaces, see [5]. By a representation of (Ω, M) we mean a quantum logic $\mathcal{M}$ (i.e. an orthocomplemented $\sigma - orthocomplete$ orthomodular poset, see [7]) with a homomorphism h: $M \xrightarrow{\text{onto}} \mathcal{M}$ such that for any state s on M and any observable $\overline{X}$ on $\mathcal{M}$ there is a state $\overline{s}$ on $\mathcal{M}$ and observable X on M such that the following diagram commutes (where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of the real line $\mathbb{R}$).



(1)

We prove that a representation of FQP of type I always exists and representation of FQP of type II exists in some cases.

1. Preliminaries.

We recall that two fuzzy sets a,b are said to be *fuzzy orthogonal*, we write $a \perp_F b$, iff $a \cap b := \inf(a, b) \leqslant 1/2$ and *orthogonal*, we write $a \perp b$, iff $a \leqslant b^{\perp}$.

Let Ω be a non-empty set, M be a system of fuzzy sets, $M \subseteq [0,1]^{\Omega}$, such that

(i) $\mathbf{1}(\omega) = 1$ for any $\omega \in \Omega$ then $\mathbf{1} \in M$;

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(ii) $a \in M$ then $a^{\perp} := 1 - a \in M$;
(iii) $\mathbf{1/2}(\omega) = 1/2$ for any $\omega \in \Omega$ then $\mathbf{1/2} \notin M$;

a set $M \subseteq [0,1]^{\Omega}$ satisfying conditions (i)–(iii) is said to be *an FQP of type I (of type II)* if it is closed with respect to a union of any sequence of mutually fuzzy orthogonal (mutually orthogonal) fuzzy sets, respectively, where by union we mean the union of Zadeh's connective. If M is closed with respect to a union of any sequence of fuzzy sets from M then M is said to be a *fuzzy quantum space*, or FQS for short.

It is clear that $a \perp b$ then $a \perp_F b$ for any $a, b \in M$. So, an FQP of type I is an FQP of type II and an FQS is an FQP of type I. See [1,3,4,6].

An observable X on (Ω, M) is a mapping $X : \mathcal{B}(\mathbb{R}) \longrightarrow M$ such that
(i) $X(E^c) = X(E)^{\perp}$ for any Borel set $E \in \mathcal{B}(\mathbb{R})$;
(ii) $X\left(\bigcup\limits_{i=1}^{\infty} E_i\right) = \bigcup\limits_{i=1}^{\infty} X(E_i)$ for any sequence $\{E_i\}_{i=1}^{\infty} \in \mathcal{B}(\mathbb{R})$.

Denote $\mathfrak{O}(M)$ be the set of all observables on (Ω, M).

A mapping $m : M \longrightarrow [0,1]$ is said to be a state of type I,II on (Ω, M) if
(i) $m(a) + m(a^{\perp}) = 1$ for any $a \in M$;
(ii) $m\left(\bigcup\limits_{i=1}^{\infty} a_i\right) = \sum\limits_{i=1}^{\infty} m(a_i)$ for any sequence of mutually fuzzy orthogonal, orthogonal fuzzy sets $\{a_i\}_{i=1}^{\infty} \subseteq M$, respectively.

Denote $\mathfrak{S}_I(M)$, $\mathfrak{S}_{II}(M)$ be the set of all states of type I,II on (Ω, M), respectively.

Proposition 1. *Let (Ω, M) be an FQP of type I then,*
(i) $\mathfrak{S}_I(M) \subseteq \mathfrak{S}_{II}(M)$;
(ii) *if (Ω, M) is an FQS then $\mathfrak{S}_I(M) = \mathfrak{S}_{II}(M)$.*

Now let (Ω, M) be an FQP of type I or FQP of type II such that

$$a \cap c \in M \quad \text{for any } a, c \in M,\ c \geqslant 1/2. \tag{2}$$

Consider a relation $\sim \subseteq M \times M$ defined by

$$a \sim b \quad \text{iff} \quad a \cap b^{\perp},\quad a^{\perp} \cap b \leqslant 1/2.$$

It is clear that (i) $a \sim a$ for any a from M; (ii) if $a \sim b$, then $a^{\perp} \sim b^{\perp}$; (iii) if $a \sim b$, then $b \sim a$, but $\sim$ is not transitive, in general. Let $\approx$ be the

transitive closure of $\sim$, i.e., the smallest quivalence relation on M containing $\sim$. It is obvious that $a \approx b$ iff there are $a_1, a_2, \dots, a_n \in M$ such that $a \sim a_1$, $a_1 \sim a_2, \dots, a_n \sim b$.
It can be proved that $a \approx b$ iff there is an $c \in M$, $c \geqslant 1/2$ such that

$$a \cap b^\perp \cap c,\ a^\perp b \cap c \leqslant 1/2;$$

or equivalently $\{a \cap b^\perp > 1/2\} \bigcup \{a^\perp \cap b > 1/2\} \subseteq \{c = 1/2\}$,
where $\{a \cap b^\perp > 1/2\} := \{\omega \in \Omega; (a \cap b^\perp)(\omega) > 1/2\}$, etc..

Note that if we consider $\Omega = [0,1]$;

$$a(\omega) = \begin{cases} 0,7 & \text{if } 0 \le \omega < 0,6 \\ 0,3 & \text{if } 0,6 \le \omega \le 1; \end{cases} \qquad b(\omega) = \begin{cases} 0,4 & \text{if } 0 \le \omega < 0,6 \\ 0,6 & \text{if } 0,8 \le \omega \le 1; \end{cases}$$

$c = a \cup a^\perp$; $d = b \cup b^\perp$; $e = d \cap a$; $f = d \cap a^\perp$; $g = a \cup f$; $h = b \cup e$; $i = e \cup d^\perp$; $k = e \cup e^\perp$ then

$$M = \{0, 1, a, b, c, d, e, f, g, h, i, k, a^\perp, b^\perp, c^\perp, d^\perp, e^\perp, f^\perp, g^\perp, h^\perp, i^\perp, k^\perp\}$$

is an FQP of type II with (2) but not a type I.

The following results can be proved by the same ways as proofs in [2].

PROPOSITION 2. *The transitive closure $\approx$ is a proper congruence relation in M.*

Now, for any $a \in M$, we put $\overline{a} := \{b \in M;\ b \approx a\}$, and $\mathcal{M} := \{\overline{a};\ a \in M\}$. In $\mathcal{M}$ we define a relation $\le$ via

$$\overline{a} \le \overline{b} \ \text{ iff there is an } \ c \geqslant 1/2 \ \text{ and } \ a \cap b^\perp \cap c \leqslant 1/2$$

and the mapping $\perp : \mathcal{M} \longrightarrow \mathcal{M}$ defined via $\overline{a} \mapsto \overline{a}^\perp, a \in M$ then $\le$ and $\perp$ are well-defined. It is easy to check that $\le$ is an order relation and $\perp$ is an orthocomplementation on $\mathcal{M}$.

LEMMA 3. *Let (Ω, M) be an FQP of type I or FQP of type II with (2)*

(i) *for any $a, c \in M, c \geqslant 1/2; a \approx a \cap c \cup c^\perp$;*
(ii) *for any $a, b \in M, \overline{a} \le \overline{b}$ then there are $a_1, b_1 \in M$ such that $a_1 \approx a, b_1 \approx b$ and $a_1 \le b_1$.*

P r o o f. (i) is clear. (ii) Since $\overline{a} \le \overline{b}$, there is $c \in M; c \geqslant 1/2$ such that $a \cap b^\perp \cap c \leqslant 1/2$, then $a_1 := a \cap c \cup c^\perp$ and $b_1 := b \cap c \cup c^\perp$ satisfy conditions of theorem. □

THEOREM 4. *Let (Ω, M) be an FQP of type I or FQP of type II with (2) then $\mathcal{M}$ equipped with an order relation $\leq$ and an orthocomplementation $\perp$ is a quantum logic with the least element $\overline{0}$ and the greatest element $\overline{1}$ and $h : M \longrightarrow \mathcal{M}$ defined via $a \mapsto \overline{a}$ is an $\sigma-$homomorphism from M onto $\mathcal{M}$ (i.e. $h(a^{\perp}) = h(a)^{\perp}$ and $h\left(\bigcup\limits_{i=1}^{\infty} a_i\right) = \bigcup\limits_{i=1}^{\infty} h(a_i)$ for any sequence of mutually fuzzy orthogonal, orthogonal fuzzy sets, respectively).*

Let (Ω, M) be an FQP of type II, we put

$$\mathcal{K}(M) = \{A \subseteq \Omega; \exists a \in M; \{a > 1/2\} \subseteq A \subseteq \{a \geqslant 1/2\}\} \qquad (3)$$

$$\mathcal{I}(M) = \{A \subseteq \Omega;\ \exists a \in M;\ A \subseteq \{a = 1/2\}\}.$$

There are two constructions of representations of FQP [1,2]. The following Proposition shows that they are equivalent.

PROPOSITION 5. *Let (Ω, M) be an FQP of type I or FQP of type II with (2) then*

(i) *$\mathcal{K}(M)$ is a q-σ-algebra and $\mathcal{I}(M)$ is an σ-ideal of $\mathcal{K}(M)$ (i.e. $\mathcal{K}(M)$ is a system of subsets of Ω which is closed with respect to complementation and countable union of mutually disjoint subsets, $\mathcal{I}(M)$ is a nonempty subset of $\mathcal{K}(M)$ closed with respect to countable union of mutually disjoint subsets and if $A \in \mathcal{K}(M)$, $B \in \mathcal{I}(M)$, $A \subseteq B$ then $A \in \mathcal{I}(M)$).*

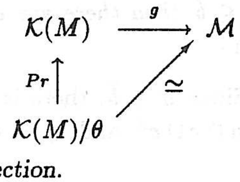
(ii) *Consider a mapping $g : \mathcal{K}(M) \longrightarrow \mathcal{M}$, defined via $A \mapsto \overline{a}$, where A,a satisfy (3) then g defines well a $\sigma-$homomorphism from $\mathcal{K}(M)$ onto $\mathcal{M}$ and $g^{-1}(\overline{0}) = \mathcal{I}(M)$. Moreover, we consider on $\mathcal{K}(M)$ a relation θ: for any $A, B \in \mathcal{K}(M)$, $A\theta B$ iff $A \setminus B$, $B \setminus A \in \mathcal{I}(M)$ then θ is a congruence relation on $\mathcal{K}(M)$. Put, for any $A \in \mathcal{K}(M)$,*

$$\overline{A} := \{B \in \mathcal{K}(M);\ B\theta A\} \quad \text{and} \quad \mathcal{K}(M)/\theta := \{\overline{A};\ A \in \mathcal{K}(M)\}$$

Define

$$\overline{A}^{\perp} := \overline{A^c} \quad \text{and} \quad \overline{A} \leq \overline{B} \quad \text{iff} \quad A \setminus B \in \mathcal{I}(M),$$

then $\perp, \leq$ is well defined an orthocomplementation and an order relation on $\mathcal{K}(M)/\theta$ such that $\mathcal{K}(M)/\theta$ with $\perp, \leq$ is a quantum logic and the following diagram commutes



where Pr is a projection.

2. A representation of type I, II FQP

THEOREM 6. *Let (Ω, M) be an FQP of type I or FQP of type II with (2) then for any observable $\overline{X}$ on $\mathcal{M}$ there is an observable X on M such that $\overline{X} = h \circ X$, where h $\mathcal{M}$ from Theorem 4.*

Proof. Let $\overline{X}$ be an observable on $\mathcal{M}$ and $\mathbb{Q}$ be the set of rational numbers. Consider $\overline{a_r} := \overline{X}((-\infty, r))$, $r \in \mathbb{Q}$, then $\overline{a_r} \leq \overline{a_s}$ if $r \leq s$. Owing to Lemma 3, we can set up a sequence $b_r, r \in \mathbb{Q}$ such that h(b_r) $= \overline{a_r}$ and $b_r \leq b_s$ if $r \leq s$. Due to Theorem 1.4 [7], Theorem 4.5 [4], there is an observable X on M such that $X((-\infty, r)) = b_r$. Therefore, $\overline{X} = h \circ X$. □

THEOREM 7. *Let (Ω, M) be an FQP of type I then for any $m \in \mathfrak{S}_I(M), \overline{m} : \mathcal{M} \longrightarrow [0,1]$ defined by $\overline{m}(\overline{a}) = m(a), a \in M$ is a state on M. Conversely, for any $s \in \mathfrak{S}_1(M)$ there is a state $m \in \mathfrak{S}_1(M)$ such that $\overline{m} = s$.*

Proof. The theorem can be proved by the same ways as proofs in [1]. □

COROLLARY 8. *Let (Ω, M) be an FQP of type II with (2) such that for any $m \in \mathfrak{S}_{II}(M)$; for any $a, b \in M; a \cap b^\perp \leqslant 1/2, a^\perp \cap b \leqslant 1/2$ imply $m(a) = m(b)$ then for any $m \in \mathfrak{S}_{II}(M), \overline{m} : \mathcal{M} \longrightarrow [0,1]$ defined by $\overline{m}(\overline{a}) = m(a), a \in M$ is a state on M.*
Conversely, for any $s \in \mathfrak{S}(\mathcal{M})$ there is a state $m \in \mathfrak{S}_2(M)$ such that $\overline{m} = s$.

THEOREM 9. *Let (Ω, M) be an FQP of type I or FQP of type II satisfying conditions of Corollary 8 then $\mathcal{M}$ with σ-homomorphism h from Theorem 4 is a representation of M.*

3. Conclusion

We have solved the problem of representation of FQP of type I and some kinds of FQP of type II. We can also point out that there is an FQP of type II which has no representation. Finally, natural questions arise: Is any quantum logic a representation of some FQP? We note that in [7], Th. 2.2.5 it is proved that every logic is a surjective homomorphic image of a concrete logic. But the conditions of a representation in our sense are not satisfied in general.

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