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THE COMPOSITIONAL RULE OF INFERENCE WITH SEVERAL RELATIONS

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ABSTRACT. The compositional rule of inference with several relations, which is the mainly used inference rule in approximate reasoning, is considered in this paper. Stability results are given and exact computational formulae are provided.

1. Introduction

During the past ten years, expert systems have drawn tremendous attention from researchers and practitioners working in the area of fuzzy information processing. At the same time, approximate reasoning gained importance, especially, after L. A. Zadeh [Zad 83] published "The role of fuzzy logic in the management of uncertainty in expert systems". One of the most widely used inference rule in approximate reasoning is the compositional rule of inference, which has the general form

Observation:	X has property P	
Relation 1:	X and Y are in relation W_1	
	$\dots$	(1)
Relation m:	X and Y are in relation W_m	
Conclusion:	Y has property Q	

where X and Y are linguistic variables taking their values from fuzzy sets in classical sets U and V, respectively, P and Q are unary fuzzy predicates in U and V, respectively, W_i is a binary fuzzy relation in $U \times V$, $i = 1, \dots, m$.

The conclusion Q is determined by [Zad 73]

$$Q = \bigcap_{i=1}^{m} P \circ W_i ,$$

Key words: Compositional rule of inference, fuzzy relation, fuzzy set, fuzzy interval, triangular norm, extension principle.

or in detail,

$$\mu_Q(y) = \min_{i=1,\dots,m} \sup_{x \in U} T(\mu_P(x), \mu_{W_i}(x,y)),$$

where T is a triangular norm [Sch 63], the conclusion is obtained by triggering the antecedents separately, and combining the partial results in the second step (for other approaches, see [Dub 88, Mag 89]). The goal of this paper is (i) to provide exact calculation formulae for the compositional rule with several relations of inference under Archimedean t-norms, when both the observation and the relation parts are given by Hellendoorn's φ-function; (ii) to show some important properties of the compositional rule of inference under t-norms.

2. Preliminaries

In this Section we set up the notations and present a lemma needed in order to calculate the exact calculation formulae and to prove the stability and continuity properties of the compositional rule of inference under t-norms.

DEFINITION 2.1. A *fuzzy interval* A is a fuzzy quantity with a continuous, finite-supported, fuzzy-convex and normalized membership function $\mu_A\colon \mathbb{R} \to [0,1]$.
The family of all fuzzy intervals will be denoted by $\mathcal{F}$. Fuzzy intervals are often used to represent linguistic variables [Wer 90]. Following [Hel 90] we use the φ-function for the representation of linguistic terms

$$\varphi(x;a,b,c,d) = \begin{cases} 1 & \text{if } b \le x \le c, \\ \varphi_1(\frac{x-a}{b-c}) & \text{if } a \le x \le b,\ a < b, \\ \varphi_2(\frac{x-c}{d-c}) & \text{if } c \le x \le d,\ c < d, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

DEFINITION 2.2. Let A be a fuzzy interval, then for any $\theta \le 0$ we define $\omega_A(\theta)$, the *modulus of continuity* of A, by

$$\omega_A(\theta) = \max_{|u-v| \le \theta} |\mu_A(u) - \mu_A(v)|.$$

DEFINITION 2.3. We metricize $\mathcal{F}$ by the metric [Kal 87],

$$D(A,B) = \sup_{\alpha \in [0,1]} d([A]^\alpha, [B]^\alpha),$$

where d denotes the classical Hausdorff metric in the family of compact subsets of $\mathbb{R}^2$, i.e.

$$d([A]^\alpha,[B]^\alpha) = \max\{|a_1(\alpha) - b_1(\alpha)|, |a_2(\alpha) - b_2(\alpha)|\},$$

$[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$, $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$.

Lemma 2.1. [Fed92]. *Let $\delta \geq 0$ be a real number and let A, B be fuzzy intervals. If*

$$D(A,B) \leq \delta,$$

then

$$\sup_{t\in\mathbb{R}} |\mu_A(t) - \mu_B(t)| \leq \max\{\omega_A(\delta), \omega_B(\delta)\}.$$

3. The compositional rule of inference with several relations

In this section we show two very important features of the compositional rule of inference with several relations and under triangular norms. Namely, we prove that, (i) if the t-norm defining the composition and the membership function of the observation are continuous, then the conclusion depends continuously on the observation; (ii) if the t-norm and the membership function of the relation are continuous, then the observation has a continuous membership function.

We consider the compositional rule of inference with different observations P and P':

Observation:	X has property P
Relation 1:	X and Y are in relation W_1
	...
Relation m:	X and Y are in relation W_m
Conclusion:	Y has property Q

Observation:	X has property P'
Relation 1:	X and Y are in relation W_1
	...
Relation m:	X and Y are in relation W_m
Conclusion:	Y has property Q'

According to Zadeh's compositional rule of inference, Q and Q' are computed as

$$Q = \bigcap_{i=1}^{m} P \circ W_i, \qquad Q' = \bigcap_{i=1}^{m} P' \circ W_i.$$

Generalizing the results of [Zim 91] concerning the case of single relation, we show that when the observations are close to each other in the metric D, then there can be only a small deviation in the membership function of the conclusions.

THEOREM 3.1. *Let $\delta \geq 0$ and T be a continuous triangular norm, and let P, P' be continuous fuzzy intervals. If*

$$D(P, P') \leq \delta,$$

then

$$\sup_{y \in \mathbb{R}} |\mu_Q(y) - \mu_{Q'}(y)| \leq \omega_T(\max\{\omega_P(\delta), \omega_{P'}(\delta)\}).$$

In the following theorem we establish the continuity property of the conclusion under continuous fuzzy relations and continuous t-*norms*.

THEOREM 3.2. *Let W_i be a continuous fuzzy relation, $i = 1, \ldots, m$ and let T be a continuous t-norm. Then Q is continuous and*

$$\omega_Q(\delta) \leq \omega_T(\omega(\delta)) \quad \text{for each } \delta \geq 0,$$

where $\omega(\delta) = \max_{i=1,\ldots,m} \omega_{W_i}(\delta)$.

R e m a r k 3.1. From $\lim_{\delta \to 0} \omega(\delta) = 0$ and Theorem 3.1. it follows that

$$\sup_{y \in \mathbb{R}} |\mu_Q(y) - \mu_{Q'}(y)| \to 0 \quad \text{as} \quad D(P, P') \to 0,$$

which means the stability of the conclusion under small changes of the observation. The stability property of the conclusion under small changes of the membership function of the observation guarantees that small rounding errors of digital computation and small errors of measurement of the input data can cause only a small deviation in the conclusion, i.e. every successive approximation method can be applied to the computation of the linguistic approximation of the exact conclusion.

4. The compositional rule of inference: Computation formulae

An important problem is the (approximate) computation of the membership function of the conclusion [Dub 88, Hel 90, Mcl 89, Zim 87]. H e l l e n d o o r n

[Hel 90] showed the closure property of the compositional rule of inference under $sup - min$ composition and presented exact calculation formulas for the membership function of the conclusion when both the observation and relation parts are given by S-, π-, or φ-function. Our results in this section are connected with those presented by [Hel 90] and we generalize them. We shall determine the exact membership function of the conclusion, when the observation and the relations are given by concave φ-function, and the t-norm is Archimedean with a strictly convex additive generator function. The efficiency of our method stems from the fact that the distributions, involved in the relation and observation, are represented by a parametrized φ-function. The deduction process then consists of some simple computations performed on the parameters. The following theorem, which generalizes the result of [Ful 91] concerning the single relation case, presents an efficient method concerning the exact computation of the compositional rule of inference with several relations and under Archimedean t-norms.

THEOREM 4.1. *Let T be an Archimedean t-norm with additive generator function f, and let P and W_i be of the form* (2)

$$\mu_P(x) = \varphi(x; a, b, c, d), \quad \mu_{W_i}(x, y) = \varphi(y - x; a_i + u_i, b_i + u_i, c_i + v_i, d_i + v_i),$$

$i = 1, \dots, m$. *If φ_1 and φ_2 are twice differentiable, concave functions, and f is a twice differentiable, strictly convex function, then the conclusion of the inference scheme (1) has the following membership function*

$$\mu_Q(y) = \min_{i=1,\dots,m} \mu_{Q_i}(y),$$

where

$$\mu_{Q_i}(y) = \begin{cases} 1 & \text{if } 2b_i + u_i \le y \le 2c_i + v_i, \\ f^{[-1]}\left(2f\left(\varphi_1\left(\frac{y-2a_i-u_i}{2(b_i-a_i)}\right)\right)\right) & \text{if } 2a_i + u_i \le y \le 2b_i + u_i, \\ f^{[-1]}\left(2f\left(\varphi_2\left(\frac{y-2c_i-v_i}{2(d_i-c_i)}\right)\right)\right) & \text{if } 2c_i + v_i \le y \le 2d_i + v_i, \\ 0 & \text{otherwise}. \end{cases}$$

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