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APPROXIMATE AND GENERALIZED APPROACHES TO CONFIDENCE INTERVALS ON A VARIANCE COMPONENT

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ABSTRACT. We consider a construction of confidence intervals on a variance component in two-component mixed linear models. Firstly, some existing approximate intervals and their properties are discussed, then the problem is considered from the viewpoint of generalized inference and the possibility of using generalized intervals as confidence intervals (in our setting) is examined. In particular, we focus on intervals obtained by a generalized pivot, firstly proposed by [D. J. Park, R. K. Burdick: *Performance of confidence intervals in regression models with unbalanced one-fold nested error structures*, Comm. Statist. Simulation Comput. **32** (2003), 717–732].

1. Model

In a mixed linear model with two variance components one assumes that the n-dimensional vector of observations y comes from a multivariate normal distribution $N_n(X\beta, \sigma_1^2 ZZ^T + \sigma^2 I)$, where X, Z are known matrices and β and $(\sigma_1^2, \sigma^2)^T$ are vectors of unknown parameters, $\sigma_1^2 \geq 0, \sigma^2 > 0$. We also suppose that $\mathcal{M}(Z) \nsubseteq \mathcal{M}(X)$, where $\mathcal{M}(A)$ denotes a linear subspace generated by columns of the matrix A. Inference on variance components σ_1^2, σ^2 is usually based on a set of statistics U_i that is minimal sufficient for the family of distributions of By, where B is of full row rank and $B^T B$ is the orthogonal projector on $\mathcal{M}^{\perp}(X)$, $\mathcal{M}^{\perp}(X)$ denotes the subspace orthogonal to $\mathcal{M}(X)$. The U_is are quadratic forms in y with the following properties:

$$U_i \sim \left(\lambda_i \sigma_1^2 + \sigma^2\right)\chi^2_{\nu_i}, \qquad i = 1, \dots, r,$$
$$U_i,\ i = 1, \dots, r \quad \text{are mutually independent,}$$

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ν_is and λ_is are known constants, $\lambda_1 > \ldots > \lambda_{r-1} > 0$ and typically $\lambda_r = 0$, so that $U_r \sim \sigma^2\chi^2_{\nu_r}$ (this is what we will suppose). We will be interested in constructing two-sided confidence intervals on the variance component σ_1^2. This task is complicated by the presence of a nuisance parameter σ^2, due to which exact solutions are not known.

In the following we put $s = \sum_{i=1}^{r-1} \nu_i$ and $\chi^2_{m;\alpha}$, $F_{m,n;\alpha}$ denote α quantiles of the corresponding χ^2 and F distributions.

2. Some approximate solutions

Probably the simplest special case of our model is a balanced one-way random-effects model ($r = 2$). A well-known approximate confidence interval on σ_1^2 in this situation is the Williams-Tukey(WT) interval, see [16], [12]. Williams derived it by combining two two-sided exact confidence intervals on σ_1^2 constructed for a known σ^2. Namely, an interval on σ_1^2 derived from an exact interval on $\lambda_1\sigma_1^2 + \sigma^2$(based on U_1) and an interval on σ_1^2 derived from an exact interval on σ_1^2/σ^2(based on U_1/U_2). The WT interval was recommended for general use by Boardman [3] based on the results of his comparative simulation study. Besides performing well in this study, it can be shown that the WT interval fulfills three reasonable requirements we may pose on approximate confidence intervals on σ_1^2. These ensure a good performance of the intervals in some special situations and we may then hope that the intervals will perform well also overall. The requirements are (they were considered already by Bulmer [4]):

An approximate $(1-\alpha)$ two-sided confidence interval on σ_1^2 should

(r. 1) be exact(for $\sigma_1^2 > 0$) when ν_r tends to ∞.
(r. 2) be exact when σ_1^2/σ^2 tends to ∞.
(r. 3) cover the true zero value of σ_1^2 as if its bounds were exact.

(r. 2) and (r. 3) ensure exactness at both ends of the range of possible values of σ_1^2/σ^2, (r. 1) is connected with a situation in which the nuisance parameter σ^2 (that complicates the situation) becomes known (its unbiased estimator U_r/ν_r becomes precise) and so we may want the interval to be exact then. Zero is excluded from (r. 1) because it has a special position as σ_1^2 is a nonnegative parameter and so the bounds of a confidence interval are forced to be nonnegative, too (common practice is to put any negative bounds resulting from an approximate procedure equal to zero). Then the true zero value of σ_1^2 is covered whenever the lower bound is zero (no matter what the upper bound is), so that even for an exact confidence interval this value would be covered with probability higher than $1-\alpha$, e.g., for an equal-tailed two-sided confidence interval with probability $1-\alpha/2$. That the approximate interval will behave similarly is ensured by (r. 3). Moreover, (r. 3) also ensures that when $\sigma_1^2 = 0$ the probability of obtaining an interval degenerate at 0 equals only the value by which the probability of covering 0 exceeds $1-\alpha$.

In the apparently more complicated general case of our model ($r > 2$, no restrictions on the form of $X,\ Z$) a solution following Williams's approach exists. We can construct an approximate two-sided confidence interval on σ_1^2 by combining two two-sided exact confidence intervals on σ_1^2 obtained for a known value of σ^2. Namely, an interval derived from an exact interval on σ_1^2/σ^2 that was proposed by Wald [13] and an interval constructed using the fact that for a known σ^2 the following expression is a pivot for σ_1^2 : $\sum_{i=1}^{r-1} U_i/(\lambda_i\sigma_1^2+\sigma^2) \sim \chi_s^2$. Bounds of the final interval are formed by intersections of the lower (upper) bounds of the two mentioned exact intervals, which eliminates the, in reality, unknown σ^2. Namely, the lower and the upper bound are nonnegative solutions to the following equation in B (or zeros if nonnegative solutions do not exist):

$$\sum_{i=1}^{r-1} \frac{U_i}{\lambda_i \mathrm{B}\dfrac{F_{s,\infty;x}}{F_{s,\nu_r;x}} + \dfrac{U_r}{\nu_r}} = sF_{s,\nu_r;x},$$

where $x = 1-\alpha/2$ for the lower bound and $x = \alpha/2$ for the upper bound (if all mentioned intervals are taken equal-tailed). This generalization of the WT interval was firstly proposed by El-Bassiouni [5]. It can be thought of as a modification of an interval proposed by Hartung and Knapp [8] (or Anderson and Bancroft (1952) for $r = 2$, cited in [5]) (by multiplying the lower and the upper bound of this by $sF_{s,\nu_r;1-\alpha/2}/\chi^2_{s;1-\alpha/2}$, $sF_{s,\nu_r;\alpha/2}/\chi^2_{s;\alpha/2}$ respectively). This multiplication ensures fulfillment of (r. 2) in addition to the requirements (r. 1) and (r. 3) already satisfied by the Hartung-Knapp (HK) interval. The HK interval combines the exact two-sided confidence interval on σ_1^2 derived for a known σ^2 from the Wald exact interval on σ_1^2/σ^2 and the unbiased estimator U_r/ν_r of σ^2. We just note that solutions proposed by Thomas and Hultquist [10] or Park and Burdick [9] fulfill only requirement (r. 2) (if $r > 2$) and do not maintain the desired confidence level for smaller values of σ_1^2/σ^2, see also [9], [2].

It seems that the problem of constructing a confidence interval on σ_1^2 is quite satisfactorily solved by approximate solutions. The main aim of this section was to briefly outline the ideas these intervals are based on and to show what supportive properties these solutions have. Anyway, in the literature there are reported also intervals based on the generalized inference approach introduced by Tsui and Weerahandi [11]. We will examine them in the following section.

3. Fiducial generalized inference approach

The notion of generalized inference was firstly introduced by Tsui and Weerahandi in [11], generalized confidence intervals (GCI) were developed by Weerahandi in [14]. Recently, the definition of a generalized pivot was reformulated by Hannig *et al.* [7], who also singled out a subset of generalized pivots which they called fiducial generalized pivots (FGP), showed how these pivots are related to Fisher's fiducial inference, suggested some recipes for construction of FGPs and investigated their asymptotic properties (e.g., they showed that the FGP suggested by Weerahandi (details will be given further) for a balanced one-way random effects model yields GCIs that are asymptotically exact for increasing number of classes and fixed number of observations per class).

The generalized inference approach can be thought of as consisting of two steps.

(st. 1) At first the observations are used to determine the exact form of the pivot to be used.

(st. 2) Then the pivot and the same observations again are used in a "classical" way to construct a confidence region for the parameter of interest.

Thus, the sufficient statistics on which our construction is based play two different roles in the procedure. To distinguish between these two roles Hannig *et al.* introduced an independent copy of the vector of sufficient statistics into their considerations. In our case this means that an FGP for σ_1^2 is a function $R_{\sigma_1^2}\big(U,\ U^*,\ \sigma_1^2,\ \sigma^2\big)$, where $U = (U_1, ..., U_r)$ and U^* denotes its independent copy, with the following properties:

(p. 1) The conditional distribution of $R_{\sigma_1^2}\big(U,\ U^*,\ \sigma_1^2,\ \sigma^2\big)$, conditional on $U = u$, does not depend on σ_1^2, σ^2.

(p. 2) For every observable value u of U, $R_{\sigma_1^2}\big(u,\ u,\ \sigma_1^2,\ \sigma^2\big) = \sigma_1^2$.

(Small letters denote observed values of random quantities denoted by corresponding capital letters.) (p. 1) ensures that in (st. 1) we will obtain a pivot $R_{\sigma_1^2}\big(u, U^*,\ \sigma_1^2,\ \sigma^2\big)$ and will be able to proceed to (st. 2), while (p. 2) ensures that the resulting confidence region will be an interval. This interval is formed by the lower and upper quantile of the conditional distribution of $R_{\sigma_1^2}$ conditional on $U = u$ (since we use u as the observed value of U^*). As mentioned in [7], if we were able to observe U^* after observing U and if (p. 2) held for $R_{\sigma_1^2}\big(u, u^*,\ \sigma_1^2,\ \sigma^2\big)$ for every possible $u,\ u^*$, resulting intervals would be exact. In fact, $R_{\sigma_1^2}$ would be a pivotal quantity for σ_1^2. However, in practice U^* is unobservable and because we use the same realization of U twice, the confidence level of the resulting interval is not guaranteed to be the intended $1 - \alpha$. Moreover, its actual confidence coverage is difficult to compute directly. (This is common for GCIs in general.) In order to show that in concrete problems GCIs can be really used as confidence intervals, one must resort to simulations. And indeed, in this manner GCIs have proved to be practically useful in many problems. However, relying only on simulation results may be considered a weak point of the method (as long as we are concerned with using GCIs as confidence intervals and frequentist properties are important). Our present goal will be to show that at least in our problem, besides performing well in simulations, intervals yielded by an FGP proposed in [9], have exact coverage in situations from (r. 1)–(r. 3) in section 2 and therefore may be considered equivalent counterparts to the existing approximate solutions.

Let $U_i^*/\big(\lambda_i\sigma_1^2+\sigma^2\big)=Q_i,\ i=1,\dots,r-1$ and $U_r^*/\sigma^2=V$. An FGP for σ_1^2 in a balanced one-way random effects model or in a situation with $r=2$ can be found in [15] and as it was pointed out, it coincides with the fiducial solution proposed by R. A. Fisher [6]. Namely, the FGP is

$$R_{\sigma_1^2}=\max\left(0,\ \frac{1}{\lambda_1}(\mathrm{U}_1/\mathrm{Q}_1-\mathrm{U}_2/\mathrm{V})\right) \tag{1}$$

(reflecting the fact that the bounds should be nonnegative). For a general case of our model Park and Burdick [9] proposed an FGP defined as a nonnegative solution to the following equation (or zero if such a solution does not exist):

$$\sum_{i=1}^{r-1}U_i/(\lambda_iR+U_r/V)=\sum_{i=1}^{r-1}Q_i$$

thus

$$R_{\sigma_1^2}=\max(0,\mathrm{R}). \tag{2}$$

This FGP reduces to (1) if $r=2$. It performed well in a simulation study in [9] as well as in simulated examples in [1]. Now we will show that in addition to the favourable simulation results, two-sided GCIs constructed by (2) fulfill the three reasonable requirements (r. 1)–(r. 3) from section 2. We just recall that the considered GCI (denote it $I(U)$) is, for each observed u, formed by the lower and upper quantiles of the conditional distribution of $R_{\sigma_1^2}$ conditional on $U=u$, i.e., $I(u)=\big[q_{l_1}(u),q_{l_2}(u)\big]$, where $l_1,\ l_2\in(0,1),\ l_1<l_2$ and $l_2-l_1=1-\alpha$.

Fulfillment of (r. 3), i.e., if $\sigma_1^2=0$, $P\big(0\in I(U)\big)=1-l_1$ and $I(U)$ is degenerate at 0 with probability $1-l_2$:

Proof. For an observed u, $0\in I(u)\Leftrightarrow q_{l_1}(u)=0\Leftrightarrow p=P\big(R_{\sigma_1^2}\big(u,U^*,\ \sigma_1^2,\ \sigma^2\big)=0\big)\geq l_1$. This may be rewritten as (using the definition of $R_{\sigma_1^2}$ and the fact that the left side of (2) is decreasing in σ_1^2):

$$p = P\left(\sum_{i=1}^{r-1} \frac{u_i}{u_r/V} \le \sum_{i=1}^{r-1} Q_i\right) = P\left(\sum_{i=1}^{r-1} \frac{u_i}{u_r} \le \sum_{i=1}^{r-1} \frac{Q_i}{V}\right) \ge l_1.$$

From the last expression we see that zero is included in the interval if and only if $\sum_{i=1}^{r-1} u_i\nu_r/(u_r s) \le F_{s,\nu_r;1-l_1}$, i.e., the true zero value of σ_1^2 is covered with probability $1-l_1$ (as $\sum_{i=1}^{r-1} U_i\nu_r/(U_r s) \sim F_{s,\nu_r}$) and similarly we obtain that when $\sigma_1^2 = 0$, the upper bound is zero with probability $1 - l_2$.

□

The probability $P\big(q_{l_1}(U) \le \sigma_1^2 \le q_{l_2}(U)\big)$ of covering the true nonzero value of σ_1^2 by the considered GCI can be rewritten as

$$P\left(l_1 \le P\left(\sum_{i=1}^{r-1} \frac{U_i}{\lambda_i\sigma_1^2 + U_r/V} \le \sum_{i=1}^{r-1} Q_i \middle| \, U\right) \le l_2\right). \tag{3}$$

Denote

$$X(U) = P\left(\sum_{i=1}^{r-1} \frac{U_i}{\lambda_i\sigma_1^2 + U_r/V} \le \sum_{i=1}^{r-1} Q_i \middle| \, U\right).$$

($\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.)

Fulfillment of (r. 2), i.e., (3) $\to$ $l_2 - l_1 = 1 - \alpha$ as $\sigma_1^2/\sigma^2 \to \infty$:

P r o o f. Let $Z_i \sim \chi^2_{\nu_i}$, $i = 1, \dots, r-1$ be mutually independent. When $\sigma_1^2/\sigma^2 \to \infty$, then $\left(\frac{U_1}{\lambda_1\sigma_1^2}, \dots, \frac{U_{r-1}}{\lambda_{r-1}\sigma_1^2}, \frac{U_r}{\sigma_1^2}\right) \xrightarrow{\mathcal{D}} (Z_1, \dots, Z_{r-1}, 0)$ and by Skorohod's representation theorem we can construct $\tilde{U}$ such that $\tilde{U} \sim U$, $\tilde{U}$ independent of U^* and $\left(\frac{\tilde{U}_1}{\lambda_1\sigma_1^2}, \dots, \frac{\tilde{U}_{r-1}}{\lambda_{r-1}\sigma_1^2}, \frac{\tilde{U}_r}{\sigma_1^2}\right) \xrightarrow{a.e.} (Z_1, \dots, Z_{r-1}, 0)$. Therefore, without loss of generality, we may suppose that

$$\left(\frac{U_1}{\lambda_1\sigma_1^2}, \dots, \frac{U_{r-1}}{\lambda_{r-1}\sigma_1^2}, \frac{U_r}{\sigma_1^2}\right) \xrightarrow{a.e.} (Z_1, \dots, Z_{r-1}, 0). \tag{4}$$

We will show that $X(U) \xrightarrow{a.e.} 1 - F_{\chi^2_s}\left(\sum_{i=1}^{r-1} Z_i\right)$, from which it follows that the probability in (3) is $l_2 - l_1$ under the limit. If $(\Omega, \mathcal{S}, P)$ is the underlying probability space, for each $\omega \in \Omega$ for which the convergence in (4) holds

$$\sum_{i=1}^{r-1} \frac{U_i(\omega)}{\lambda_i\sigma_1^2 + U_r(\omega)/V} - \sum_{i=1}^{r-1} Q_i \xrightarrow{\mathcal{D}} \sum_{i=1}^{r-1} Z_i(\omega) - \sum_{i=1}^{r-1} Q_i \tag{5}$$

by Slutsky's theorem. As the limiting distribution function in (5) is continuous, $X\big(U(\omega)\big) \longrightarrow 1 - F_{\sum\limits_{i=1}^{r-1} Q_i}\left(\sum\limits_{i=1}^{r-1} Z_i(\omega)\right)$ and the result follows. □

Fulfillment of (r. 1), i.e., (3) → $l_2 - l_1 = 1 - \alpha$ as $\nu_r \to \infty$:

P r o o f. By exploiting the fact that by Skorohod's representation we may assume $U_r/\nu_r \xrightarrow{a.e.} \sigma^2$ it can be proved that when $\nu_r \to \infty$, then $X(U) \xrightarrow{a.e.} 1 - F_{\chi^2_s}\left(\sum\limits_{i=1}^{r-1} U_i/(\lambda_i\sigma_1^2 + \sigma^2)\right)$ similarly as in the preceding.

□

Remark 1. The FGP $R_{\sigma_1^2}$ can be derived by a two-stage construction method proposed by H a n n i g *et al.* in [7]. When applied to our problem, this method requires to find two pivotal quantities, one for σ_1^2, assuming that σ^2 is known and one for σ^2 alone, both invertible as functions of σ_1^2, σ^2, respectively. These pivotal quantities are $E_1 = \sum\limits_{i=1}^{r-1} U_i/(\lambda_i\sigma_1^2 + \sigma^2)$, $E_2 = U_r/\sigma^2$. The system of the two equations is solved in σ_1^2, σ^2 and the FGP is the solution for σ_1^2 with E_1, E_2 replaced by the corresponding expressions with U^*, e.g., E_1 is replaced by $E_1^* = \sum\limits_{i=1}^{r-1} U_i^*/(\lambda_i\sigma_1^2 + \sigma^2)$.

Remark 2. Other generalized confidence intervals on σ_1^2 were proposed in [1]. They can be constructed using FGPs of the form:

$$\tilde{R}_{\sigma_1^2} = \max\left(0, \frac{\sum\limits_{\mathrm{i=1}}^{\mathrm{r-1}} \mathrm{c_i U_i} - \mathrm{U_r} \sum\limits_{\mathrm{i=1}}^{\mathrm{r-1}} \mathrm{c_i Q_i/V}}{\sum\limits_{\mathrm{i=1}}^{\mathrm{r-1}} \mathrm{c_i}\lambda_{\mathrm{i}}\mathrm{Q_i}}\right),$$

where c_is are positive constants. Using the fact that in this case the probability of covering the true nonzero value of σ_1^2 can be rewritten as

$$P\left(l_1 \le P\left(\frac{\sum\limits_{i=1}^{r-1} c_i U_i - U_r \sum\limits_{i=1}^{r-1} c_i Q_i/V}{\sum\limits_{i=1}^{r-1} c_i\lambda_i Q_i} \le \sigma_1^2 \middle|\, U\right) \le l_2\right)$$

it can be shown, similarly as above, that GCIs based on $\tilde{R}_{\sigma_1^2}$ fulfill the requirements (r. 1)–(r. 3). $\big(X(U) = P\big(\sum\limits_{i=1}^{r-1} c_i U_i \le \sum\limits_{i=1}^{r-1} c_i Q_i(\lambda_i\sigma_1^2 + U_r/V) \,|\, U\big).\big)$ However, their coverage(for particular choices of c_i) in other than these three special situations have been studied by simulation only in two illustrative examples of our model. Like $R_{\sigma_1^2}$, also $\tilde{R}_{\sigma_1^2}$ can be obtained by the two-stage construction method, taking $E_1 = \tilde{F}\Big(\sum_{i=1}^{r-1} c_i U_i\Big)$, $E_2 = U_r/\sigma^2$, where $\tilde{F}(\cdot)$ is the distribution function of $\sum_{i=1}^{r-1} c_i\big(\lambda_i\sigma_1^2 + \sigma^2\big)Q_i$.

4. Conclusions

The construction of confidence intervals on σ_1^2 in a mixed linear model with two variance components is complicated by the presence of a nuisance parameter. Satisfactory solutions can be obtained by approximate or generalized inference methods. The use of the latter for obtaining confidence intervals has been justified only by simulation results so far. Here we provided additional justification for fiducial generalized confidence intervals constructed by a particular fiducial generalized pivot by showing that they fulfill 3 reasonable requirements we may pose on approximate intervals on σ_1^2 and so they may be considered more equivalent counterparts to the approximate solutions.

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