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ON THE CONVERGENCE OF SEQUENCES OF $(\mathcal{A}, \mathcal{B})$-QUASICONTINUOUS FUNCTIONS

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ABSTRACT. A set-theoretical generalization of well-known theorem on the cliquishness of the limit of a pointwise convergent sequence of quasicontinuous functions from Baire topological space to a metric space is formulated and proved in this article.

If (X, T_X) is a topological space and (Y, ρ) is a metric space then a function $f\colon X \to Y$ is said to be *quasicontinuous* (resp. *cliquish*) at a point $x \in X$ if for every set $U \in T_X$ containing x and for each positive real η there is a nonempty set $V \in T_X$ contained in U such that $f(V) \subset K(f(x), \eta) = \{t \in Y : \rho(t, f(x)) < \eta\}$ (resp. $\operatorname{diam}(f(V)) = \sup\{\rho(f(t), f(u)) : t, u \in V\} < \eta$) ([4]).

It is well known ([4], [1]) that if (X, T_X) is a Baire space, then the limits of sequences of quasicontinuous functions from X to Y are cliquish. On the other hand, if $X = \mathbb{Q}$ is the set of all rationals, $Y = \mathbb{R}$, $\rho(x, y) = |x-y|$ and T_X is the topology generated by the metric ρ, then for arbitrary disjoint decomposition $\mathbb{Q} = A \cup B$ with dense sets $A, B \subset X$ the function

$$f(x) = \begin{cases} 0 & \text{for } x \in A, \\ 1 & \text{for } x \in B, \end{cases}$$

is the limit of a sequence of continuous functions from X to Y, but it is not cliquish.

In this article I formulate and prove a generalized set-theoretical version of the above theorem.

Let Z be a nonempty set and let $\mathcal{A}$ and $\mathcal{B}$ be nonempty families of nonempty subsets of Z. A function $f\colon Z \to Y$ is said to be $(\mathcal{A}, \mathcal{B})$-*quasicontinuous* (resp. $(\mathcal{A}, \mathcal{B})$-*cliquish*) at a point $z \in Z$ if for each set $A \in \mathcal{A}$ containing z and for every real $\eta > 0$ there is a set $B \in \mathcal{B}$ contained in A such that $f(B) \subset K(f(z), \eta)$ (resp. the diameter $\operatorname{diam}(f(B)) < \eta$).

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Suppose that the families $\mathcal{A}$ and $\mathcal{B}$ satisfy the following condition:

(a) for each set $A \in \mathcal{A}$ and for each sequence of sets (A_n) with $A = \bigcup_{n=1}^{\infty} A_n$ there is an index k and a set $E \in \mathcal{A}$ such that $E \subset A$ and for each set $H \in \mathcal{B}$ contained in E and for each $(\mathcal{A},\mathcal{B})$-quasicontinuous function $f\colon Z \to Y$, if $f(A_k \cap H)$ is contained in a closed ball $K \subset Y$, then $f(H) \subset K$.

Theorem 1. *Suppose that a nonempty set Z and families $\mathcal{A}$ and $\mathcal{B}$ satisfy the condition* (a). *If a sequence (g_n) of $(\mathcal{A},\mathcal{B})$-quasicontinuous functions $g_n\colon Z \to Y$ pointwise converges to a function g, then g is $(\mathcal{A},\mathcal{B})$-cliquish.*

Proof. Fix a point x_0, a set $A \in \mathcal{A}$ containing x_0 and a positive real η. Since $\lim_{n\to\infty} g_n = g$, for each point $z \in A$ there is an index $k(z)$ such that

$$\rho\big(g_n(z), g(z)\big) < \frac{\eta}{8} \qquad \text{for } n \geqslant k(z).$$

For $k \geqslant 1$ let

$$A_k = \Big\{z \in A\colon \rho\big(g_n(z), g(z)\big) < \frac{\eta}{8} \text{ for } n \geqslant k\Big\}.$$

Since

$$A = \bigcup_{k=1}^{\infty} A_k\,,$$

by condition (a) there is an index m and a set $E \subset A$ belonging to $\mathcal{A}$ such that for each set $H \in \mathcal{B}$ contained in E and for each $(\mathcal{A},\mathcal{B})$-quasicontinuous function $f\colon Z \to Y$, if $f(H \cap A_m)$ is contained in a closed ball K, then $f(H) \subset K$.

Fix a point $x \in E$. From the $(\mathcal{A},\mathcal{B})$-quasicontinuity of the function g_m at x it follows the existence of a set $M \subset E$ belonging to $\mathcal{B}$ such that $g_m(M) \subset K\big(g_m(x), \frac{\eta}{8}\big)$. For all points $z \in M \cap A_m$ and integers $n \geqslant m$ we have

$$\begin{aligned}\rho\big(g_n(z), g_m(x)\big) &\leqslant \rho\big(g_n(z), g(z)\big) + \rho\big(g(z), g_m(z)\big) + \rho\big(g_m(z), g_m(x)\big)\\ &< \frac{\eta}{8} + \frac{\eta}{8} + \frac{\eta}{8} = \frac{3\eta}{8}.\end{aligned}$$

So, by (a) for $n \geqslant m$ we obtain

$$g_n(M) \subset K = \Big\{y \in Y\colon \rho\big(y, g_m(x)\big) \leqslant \frac{3\eta}{8}\Big\}.$$

Consequently,

$$g(M) \subset K \quad \text{and} \quad \operatorname{diam}\big(g(M)\big) \leqslant \frac{6\eta}{8} < \eta.$$

This finishes the proof. $\square$

Observe that if (Z, T_Z) is a Baire topological space and $\mathcal{A} = \mathcal{B} = T_Z \setminus \{\emptyset\}$, then the $(\mathcal{A},\mathcal{B})$-quasicontinuity is the usual quasicontinuity, the $(\mathcal{A},\mathcal{B})$-cliquishness is the usual cliquishness and in this case the condition (a) is satisfied.

Remark 1. Suppose that Z is a nonempty set, $\mathcal{I}$ is a proper σ-ideal of subsets of Z, and let $\mathcal{M} \supset \mathcal{I}$ is a σ-field of subsets of Z. Let $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}$ be families of subsets from $\mathcal{M} \setminus \mathcal{I}$ such that $Z = \bigcup \mathcal{A}$ and for each set $C \notin \mathcal{I}$ and each set $A \in \mathcal{A}$ containing C there is a set $B \in \mathcal{B}$ contained in A such that each set $D \subset B \setminus C$ belonging to $\mathcal{M}$ is an element of $\mathcal{I}$. Then condition (a) is satisfied.

Proof. Fix a set $A \in \mathcal{A}$ and a sequence (A_n) of subsets of A with $A = \bigcup_{n=1}^{\infty} A_n$. Since the set $A \in \mathcal{M} \setminus \mathcal{I}$, there is a positive integer k such that $A_k \notin \mathcal{I}$. Consequently, by our hypothesis there is a set $B \in \mathcal{B}$ such that $B \subset A$ and each set $D \subset B \setminus A_k$ belonging to $\mathcal{M}$ is an element of $\mathcal{I}$. Now, let $f\colon Z \to Y$ be a $(\mathcal{A},\mathcal{B})$-quasicontinuous function and let $H \in \mathcal{B}$ and a closed ball $K \subset Y$ be such that $H \subset B$ and $f(H \cap A_k) \subset K$. Assume for a contradiction that there is a point $w \in H$ with $f(w) \notin K$. There is a real $\eta > 0$ such that $K(f(w),\eta) \cap K = \emptyset$. Since $w \in H \in \mathcal{A}$, from the $(\mathcal{A},\mathcal{B})$-quasicontinuity of f at w it follows the existence of a set $H_1 \in \mathcal{B}$ contained in H and such that $f(H_1) \subset K(f(w),\eta) \subset Y \setminus K$. But $\mathcal{M} \ni H_1 \subset B \setminus A_k$, so $H_1 \in \mathcal{I}$. This implies a contradiction with the relation $\mathcal{B} \cap \mathcal{I} = \emptyset$. This contradiction shows that $f(H) \subset K$ and finishes the proof. ☐

As a corollary from Remark 1 we obtain:

COROLLARY 1. *Let (Z,T_Z) be a Baire topological space and let $\mathcal{B} = T_Z \setminus \{\emptyset\}$. Suppose that $\mathcal{A} \supset T_Z \setminus \{\emptyset\}$ is a family of nonempty subsets of Z with nonempty interiors. Then the condition* (a) *is satisfied.*

Proof. It suffices to denote by $\mathcal{I}$ the σ-ideal of the first category subsets of Z and by $\mathcal{M}$—the σ-field of all subsets of Z with the Baire property, and to apply Remark 1. ☐

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