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THE CONSTRUCTION AND SOME PROPERTIES OF CLUSTER MULTIFUNCTION

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ABSTRACT. We present the construction and some properties of cluster multifunction $\mathcal{C}_F$ concerning various types of generalized continuities depending on the properties of multifunction F and the relation between sets $F(x)$ and $\mathcal{C}_F(x)$ at point x. Next we introduce some theorems concerning a set of all semi-continuity points of $\mathcal{C}_F$.

Introduction

The notions of cluster set and cluster function were first used by Bettazzi in 1892 and Brodén in 1897. Only Hahn in his book gave the first explicit definition of a cluster set 40 years later. The general theory of cluster sets was finally continued by Aumann in 1954 who considered maps between arbitrary topological spaces. However, the Aumann's notation did not survive. In 1958 Collingwood and Weston were more successful by it. The standard symbol $\mathcal{C}_f(x_0)$ for the cluster set of a function at a point $x_0 \in X$ used since that time stands for

$$\mathcal{C}_f(x_0) = \bigcap_{U \in \mathcal{B}(x_0)} \mathrm{Cl}\big(f(U)\big) ,$$

where the neighbourhoods U of x_0 runs through an arbitrary basis $\mathcal{B}(x_0)$ of the neighbourhood system $\mathcal{U}(x_0)$ of x_0. Obviously, one always has $f(x_0) \in \mathcal{C}_f(x_0)$. In recent years, the notion of cluster points of functions and multifunctions has been investigated in [3], [5]. The general results of this paper shall be devoted to a set of cluster points and a mapping which assigns the cluster set to a point. We shall start with the cliquishness of multifunction and introduction of this notion for multifunction between arbitrary topological spaces. The proofs of some results are usually stated in cases when according to our opinion the results are not known. All the known results contain further information from where the results were acquired.

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Notations

In what follows, X, Y denote topological spaces. Let A be a subset of a topological space X, the symbols $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$ denote interior and closure of A in a topological space X, and $\mathcal{U}(x_0)$ be the system of all neighbourhoods of a point x_0 respectively. The word multifunction has the same meaning as multi-valued mapping. If the values of a multifunction are non-empty closed (compact) sets we shall use the term closed (compact) valued multifunction. If $F\colon X \to Y$ is a multifunction then for $A \subset Y$ we denote

$$F^+(A) = \{x \in X : F(x) \subset A\},$$
$$F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}.$$

A multifunction $F\colon X \to Y$ *is upper* (*lower*) *semi-continuous* at a point $x_0 \in X$ (*u.s.c.* (*l.s.c.*)) if for any open set V, $F(x_0) \subset V$, ($F(x_0) \cap V \neq \emptyset$) there exists a neighbourhood U of x_0 such that $F(x) \subset V$, ($F(x) \cap V \neq \emptyset$) for any $x \in U$. F is said to be u.s.c. (l.s.c.) if it is u.s.c. (l.s.c.) at any $x \in X$.

Definition 1. A multifunction $F\colon X \to Y$ is *upper* (*lower*) *topologically cliquish* (*upper* (*lower*) *t-cliquish*) at a point $x_0 \in X$ if for any open covering $\mathcal{P}$ of Y and for any neighbourhood U of x_0 there is a non-empty open set $H \subset U$ and there is a set $P \in \mathcal{P}$ such that $F(x) \subset P$ ($F(x) \cap P \neq \emptyset$) for any $x \in H$.

The set of all points where F is upper (lower) t-cliquish will be denoted by K_F^u (K_F^l). From the definition of t-cliquishness the inclusion $K_F^u \subset K_F^l$ is evident. On the analogy of functions we can prove the following theorem for multifunctions.

Theorem 2. *Let $F\colon X \to Y$ be a multifunction. The sets K_F^u and K_F^l are closed.*

Proof. We will give a proof for K_F^l. The proof for K_F^u is similar. Let $x_0 \in \mathrm{Cl}(K_F^l)$, U be a neighbourhood of x_0 and $\mathcal{P}$ be an open cover of Y. Then there exists a point $x_0' \in K_F^l \cap U$, there is a non-empty open set $H \subset U$ and there is a set $P \in \mathcal{P}$ such that $F(x) \cap P \neq \emptyset$ for any $x \in H$. Hence F is lower t-cliquish at x_0 and $x_0 \in K_F^l$. □

Whereas the t-cliquishness we do not know an adequate accurate characterization of accumulation values of a multifunction at a neighbourhood of a point x (the accumulation depends on selection of covering sets and selection of the sets H and P), the introducing of $\mathcal{C}$-cluster points gives us a better localization of values of the multifunction F at a point x.

Definition 3. Let $\mathcal{C}$ be a non-empty family of non-empty subsets of X. A point $y \in Y$ is *$\mathcal{C}$-cluster point* of multifunction $F\colon X \to Y$ at x_0 if for any open sets U, V with $x_0 \in U$, $y \in V$, there is a set $C \in \mathcal{C}$, $C \subset U$, such that for any $x \in C$ we have $F(x) \cap V \neq \emptyset$.

A mapping $x \mapsto \mathcal{C}_F(x) = \{y \in Y : y$ is $\mathcal{C}$-cluster point of F in $x\}$ will be called a cluster multifunction of F and we shall denote it by $\mathcal{C}_F$. It can be empty valued mapping.

EXAMPLE 4. For a function

$$f(x) = \begin{cases} 0 & \text{if } x = 0\,, \\ \frac{1}{x} & \text{otherwise}\,, \end{cases}$$

we have the set of cluster points $\mathcal{C}_f(x) = \{f(x)\}$ for $x \neq 0$ and $\mathcal{C}_f(0) = \emptyset$.

Next, we introduce some properties of this mapping. The following lemma is known [4, Lemma 4.3] and we shall need it to prove some further theorems.

LEMMA 5. *If $\{x_t\}_{t\in T}$ is a net in X convergent to x_0 and $\{y_t\}_{t\in T}$ is a net in Y convergent to y_0 such that y_t is the $\mathcal{C}$-cluster point of F at x_t, then y_0 is the $\mathcal{C}$-cluster point of the multifunction F at x_0.*

THEOREM 6. *Let $F\colon X \to Y$ be a multifunction and Y be a compact space. Then $X_0 = \{x\colon \mathcal{C}_F(x) \neq \emptyset\}$ is closed in X.*

P r o o f. Let $x_0 \in \mathrm{Cl}(X_0)$ and U is any neighbourhood of x_0. Then there exist points $x_u, y_u \in \mathcal{C}_F(x_u)$. Since Y is compact there is a subnet T' of $T = \mathcal{U}(x_0)$ such that $\{x_{t'}\}_{t'\in T'}$ converges to x_0 and $\{y_{t'}\}_{t'\in T'}$ converges to y_0. By Lemma 5, $y_0 \in \mathcal{C}_F(x_0)$. So $\mathcal{C}_F(x_0)$ is non-empty, hence $x_0 \in X_0$. ☐

As a corollary we have

COROLLARY 7. *If X_0 is dense then $X_0 = X$.*

We shall use the c-continuity for multifunctions which were introduced in [2]. We say that a multifunction $F\colon X \to Y$ is *upper* (*lower*) *c-continuous* (*u.c.c.* (*l.c.c.*)) at $x_0 \in X$ if for any open set V such that $Y \backslash V$ is compact and $F(x_0) \subset V$, ($F(x_0) \cap V \neq \emptyset$) there exists a neighbourhood U of x_0 such that $F(x) \subset V$, ($F(x) \cap V \neq \emptyset$) for any $x \in U$. F is said to be u.c.c. (l.c.c.) if it is u.c.c. (l.c.c.) at any $x \in X$.

THEOREM 8. *A multifunction $\mathcal{C}_F\colon X_0 \to Y$ is u.c.c.*

P r o o f. Suppose $\mathcal{C}_F$ is not u.c.c at x_0. Then there is a non-empty open set $G \supset \mathcal{C}_F(x_0)$, $Y \setminus G$ is compact such that for any neighbourhood $U \in \mathcal{U}(x_0)$, there exists a point $x_u \in U \cap X_0$ with $\mathcal{C}_F(x_u) \setminus G \neq \emptyset$. Hence there is a point $y_u \in \mathcal{C}_F(x_u) \setminus G$. Since $Y \setminus G$ is compact there is $y_0 \in Y \setminus G$ which is an accumulation point of the net $\{y_u\}_U$. We can choose subnets $\{x_{u'}\}_{U'}$ and $\{y_{u'}\}_{U'}$ convergent to x_0 and y_0, respectively. By Lemma 5 y_0 is $\mathcal{C}$-cluster point of $\mathcal{C}_F$ at x_0. So $y_0 \in \mathcal{C}_F(x_0) \subset G$, a contradiction with $y_0 \in Y \setminus G$. ☐

Corollary 9. *Let Y be a compact space. A multifunction $\mathcal{C}_F\colon X_0 \to Y$ is* u.s.c.

From [4] (see Lemma 4.1) we have

Theorem 10. *Let $F\colon X \to Y$ be a multifunction. Then a set*

$$\mathcal{C}_F(x) = \{y \in Y\colon y \text{ is a } \mathcal{C}\text{-cluster point } F \text{ at } x\}$$

is closed.

Using Theorem 8 and [2, Theorem 4] we can qualify a set of all points at which the multifunction $\mathcal{C}_F$ is not u.s.c.

Theorem 11. *Let X be a Baire space and Y be a Hausdorff σ-compact space (i.e., $Y = \bigcup_{n=1}^{\infty} C_n$ where C_n are compact sets). Let $\mathcal{C}_F\colon X_0 \to Y$ be a multifunction such that $\mathcal{C}_F^+(C_n) = \mathcal{C}_F^-(C_n)$ for $n = 1, 2, 3\ldots$ Then the set of all points at which $\mathcal{C}_F$ is not u.s.c. is nowhere dense in X.*

The next reflections will be addressed to the relation $F(x) \subset \mathcal{C}_F(x)$ for each point $x \in X$. We say that a multifunction $F\colon X \to Y$ is *lower-$\mathcal{C}$* (l-$\mathcal{C}$) *continuous* at x_0 if for any non-empty open set G with $G \cap F(x_0) \neq \emptyset$ and for any non-empty open set U with $U \ni x_0$ there is a set $C \in \mathcal{C}$, $C \subset U\colon F(x) \cap G \neq \emptyset$ for any $x \in C$. From [4, Lemma 4.2] we have that a multifunction $F\colon X \to Y$ is l-$\mathcal{C}$ continuous at x_0 if and only if $F(x_0) \subset \mathcal{C}_F(x_0)$. If $\mathcal{C}$ is a family of non-empty open sets it coincides with the lower quasi-continuity (l.q.s.c.). Let

$$\mathcal{C}(\mathcal{I}) = \{A \subset X\colon A \notin \mathcal{I}\},$$

where $\mathcal{I}$ is σ-ideal. It is known that a set of points where F is not l-$\mathcal{C}(\mathcal{I})$ continuous belongs to $\mathcal{I}$ (see [4, Lemma 1.4]). For a multifunction $\mathcal{C}_F$ we can prove next theorem.

Theorem 12. *Let X, Y be second countable spaces. A multifunction $\mathcal{C}_F\colon X_0 \to Y$ is l-$\mathcal{C}(\mathcal{I})$ continuous.*

Proof. Let $x_0 \in X_0$, an open set $G \subset Y$ such that $\mathcal{C}_F(x_0) \cap G \neq \emptyset$, and $U \in \mathcal{U}(x_0)$ be given. Then there exists a point $y \in \mathcal{C}_F(x_0) \cap G$ which is $\mathcal{C}(\mathcal{I})$-cluster point of F at x_0. Then for open sets U, G there is $C \in \mathcal{C}(\mathcal{I})$, $C \subset U$, with $F(x) \cap G \neq \emptyset$ for any $x \in C$. Since F is l-$\mathcal{C}(\mathcal{I})$ continuous except for a set $A \in \mathcal{I}$ (by [4, Lemma 1.4]), so for any $x \in C \backslash A$ we have $F(x) \subset \mathcal{C}_F(x)$ by [4, Lemma 4.2]. The set $C \setminus A \in \mathcal{C}(\mathcal{I})$ and $\mathcal{C}_F(x) \cap G \neq \emptyset$ for any $x \in C \setminus A$. □

The next two theorems deal with the structure of the set of all points where a multifunction $\mathcal{C}_F$ is l.s.c. and l.q.s.c. Since a u.s.c. multifunction is l.s.c. except for a set of first category [1, Theorem 2], from Corollary 9 we have the following theorem.

Theorem 13. *Let Y be a compact metric space. Then a set of all points where a multifunction $\mathcal{C}_F\colon X_0 \to Y$ is not l.s.c. is of first category in X_0.*

Using [4, Theorem 1.1], if F is l.s.c. on the residual set and F is l-$\mathcal{C}(\mathcal{I})$ continuous where $\mathcal{I}$ is the system of sets of first category then F is l.q.s.c. and by Theorems 13 and 12 we obtain the next theorem.

Theorem 14. *Let Y be a compact metric space. Then a multifunction $\mathcal{C}_F\colon X_0 \to Y$ is l.q.s.c.*

The case when the intersection of sets $F(x)$ and $\mathcal{C}_F(x)$ is empty or non-empty we is described in the following theorem.

Theorem 15. *Let $\mathcal{C}$ be a non-empty family of non-empty open subsets of X, Y be a second countable normal space. Suppose that $F\colon X \to Y$ is a compact valued multifunction and $\mathcal{C}_F(x)$ is compact non-empty for any $x \in X$. Then a set $A = \{x \in X \colon \mathcal{C}_F(x) \cap F(x) = \emptyset\}$ is of first category in X.*

Proof. If $x \in A$ then there exist disjoint sets G_n, G_m such that $x \in F^+(G_n)$ and $x \in \mathcal{C}_F^+(G_m)$ where $G_n, G_m \in \mathcal{B}$ and $\mathcal{B} = \{G_n\}_{n=1}^{\infty}$ are systems of all finite unions of members from a countable base such that $G_n \cap G_m = \emptyset$ for $n \neq m$. For any n, m $(n \neq m)$ let $A_{n,m} = \{x \in X \colon x \in F^+(G_n)$ and $x \in \mathcal{C}_F^+(G_m)\}$. Then

$$A = \bigcup_{n \neq m} A_{n,m}\,.$$

We shall show that a set $A_{n,m}$ is nowhere dense in X. Let U be open, $x_0 \in U \cap A_{n,m}$. Take a point $y_0 \in \mathcal{C}_F(x_0)$ which is $\mathcal{C}$-cluster point of F at x_0. Then there is an open set $C \in \mathcal{C}$, $C \subset U$, such that $F(x) \cap G_m \neq \emptyset$ for any $x \in C$. It is clear that $C \cap A_{n,m} = \emptyset$. If not, for $x \in C \cap A_{n,m}$ we have $F(x) \cap G_m \neq \emptyset$ and $F(x) \subset G_n$ a contradiction with $G_n \cap G_m = \emptyset$. So the set $A_{n,m}$ is nowhere dense. □

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