Tatra Mt. Math. Publ. **34** (2006), 29–36

TATRA MOUNTAINS
Mathematical Publications

ON CONTINUITY CONCERNED WITH ψ-DENSITY TOPOLOGIES

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ABSTRACT. This paper is concerned with three kinds of topologies: the natural topology, density topology and ψ-density topology on the real line. We can consider different classes of continuous functions using these topologies on the domain and the range of functions. They can be compared with the classes of Baire 1, Baire*1 and Darboux functions.

Throughout the paper we will use the following notations: $\mathbb{R}$ will denote the set of real numbers, S—the σ-algebra of Lebesgue measurable subsets on $\mathbb{R}$, m—the Lebesgue measure on $\mathbb{R}$, $\mathcal{I}$—the σ-ideal of sets of Lebesgue measure zero. If a set A is open (closed, nowhere dense, etc) in topology $\mathcal{T}$, we will say that A is $\mathcal{T}$-open ($\mathcal{T}$-closed, $\mathcal{T}$-nowhere dense, etc). For $A \subset \mathbb{R}$ let A' be its complement. Let C be a family of nondecreasing continuous functions $\psi\colon (0,\infty) \to (0,\infty)$ such that $\lim_{t\to 0+} \psi(t) = 0$.

We say that $x \in \mathbb{R}$ is a *ψ-density point* of a measurable set $A \subset \mathbb{R}$ if

$$\lim_{h\to 0+} \frac{m\big(A' \cap [x-h, x+h]\big)}{2h\,\psi(2h)} = 0\,.$$

A point $x \in \mathbb{R}$ is a *right-hand ψ-density point* of $A \in S$ if and only if

$$\lim_{h\to 0+} \frac{m\big(A' \cap [x, x+h]\big)}{2h\,\psi(2h)} = 0\,.$$

The left-hand case is treated in an analogous way.

For any $A \in S$ we denote

$$\Phi_\psi(A) = \{x \in \mathbb{R} \colon x \text{ is a } \psi\text{-density point of A}\}$$

and

$$\Phi(A) = \{x \in \mathbb{R} \colon x \text{ is a density point of } A\}\,.$$

2000 Mathematics Subject Classification: 54A10, 28A05.
Keywords: density point, ψ-density point, density topology, ψ-density topology, approximately continuous functions, ψ-approximately continuous function.

From [TW-B, Theorem 1.4] we obtain that the family $\mathcal{T}_\psi = \{A \in S\colon A \subset \Phi_\psi(A)\}$ is a topology stronger than the natural topology and weaker than the density topology.

For any topologies $\mathcal{T}_a, \mathcal{T}_b \subset 2^{\mathbb{R}}$ let $\mathcal{C}_{ab}$ be the family of all continuous functions $f\colon (\mathbb{R}, \mathcal{T}_a) \to (\mathbb{R}, \mathcal{T}_b)$. First observe two easy properties.

Property 1. *For any topologies $\mathcal{T}_a$, $\mathcal{T}_b$ and $\mathcal{T}_c$ if $\mathcal{T}_b \subset \mathcal{T}_c$, then $\mathcal{C}_{ab} \supset \mathcal{C}_{ac}$ and $\mathcal{C}_{ba} \subset \mathcal{C}_{ca}$.*

Property 2. *For any pair of topologies $\mathcal{T}_a, \mathcal{T}_b \subset 2^{\mathbb{R}}$ if $\mathcal{T}_a \subset \mathcal{T}_b$, then $\mathcal{C}_{ab} \subset \mathcal{C}_{aa} \cap \mathcal{C}_{bb}$ and $\mathcal{C}_{ba} \supset \mathcal{C}_{aa} \cup \mathcal{C}_{bb}$.*

Indeed, if $f \in \mathcal{C}_{ab}$ and $V \in \mathcal{T}_a$, then $V \in \mathcal{T}_b$ and from the continuity of f we obtain $f^{-1}(V) \in \mathcal{T}_a$. So $f \in \mathcal{C}_{aa}$. If $f \in \mathcal{C}_{ab}$ and $V \in \mathcal{T}_b$, then from the continuity of f we obtain $f^{-1}(V) \in \mathcal{T}_a \subset \mathcal{T}_b$. Hence $f \in \mathcal{C}_{bb}$. The second inclusion is obvious.

In this way we obtain nine classes of continuous functions: $\mathcal{C}_{oo}$, $\mathcal{C}_{od}$, $\mathcal{C}_{o\psi}$, $\mathcal{C}_{do}$, $\mathcal{C}_{dd}$, $\mathcal{C}_{d\psi}$, $\mathcal{C}_{\psi o}$, $\mathcal{C}_{\psi d}$ and $\mathcal{C}_{\psi\psi}$. Some of them have been examined:

1. $\mathcal{C}_{oo}$ is the class of all ordinary continuous functions;
2. $\mathcal{C}_{dd}$ is the class of density continuous functions (for the results see [CL1], [CL2], [B], [O], [CLO1], [CLO2]);
3. $\mathcal{C}_{do}$ is the class of approximately continuous functions (see [GNN], [CLO2]);
4. $\mathcal{C}_{\psi o}$ is the class of ψ-approximately continuous functions (see [F1], [TW-B], [W-B]).

We will consider some properties of functions from classes $\mathcal{C}_{o\psi}$, $\mathcal{C}_{\psi d}$ and $\mathcal{C}_{\psi\psi}$.

1. Classes $\mathcal{C}_{o\psi}$ and $\mathcal{C}_{\psi o}$

Let Const stand for the class of all constant functions.

Theorem 3. $\mathcal{C}_{o\psi} = \mathcal{C}_{od} = \mathrm{Const}$.

P r o o f. The equality $\mathcal{C}_{od} = \mathrm{Const}$ is proved in [CL2, Theorem 3.1]. We will prove that $\mathcal{C}_{o\psi} = \mathrm{Const}$. Obviously $\mathrm{Const} \subset \mathcal{C}_{o\psi}$.

Let $f \in \mathcal{C}_{o\psi}$ and suppose that there exists a closed bounded interval I such that f is not constant on I. Put

$$M = \sup_{x\in I} f(x) \quad \text{and} \quad m = \inf_{x\in I} f(x).$$

From Property 1 we know that $f \in \mathcal{C}_{oo}$, so $(m, M) \subset f(I)$. There exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} y_n = M$ and $y_n \in (m, M)$ for any $n \in \mathbb{N}$. Since $Y = \{y_n : n \in \mathbb{N}\}$ is countable, it is $\mathcal{T}_\psi$-closed. For any $n \in \mathbb{N}$ there exists $w_n \in f^{-1}(\{y_n\}) \cap I$. The sequence $\{w_n\}_{n\in\mathbb{N}}$ is bounded, so there exists a convergent subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $\{w_n\}_{n\in\mathbb{N}}$. By w_0 denote a limit $\lim_{k\to\infty} w_{n_k}$. Obviously $\lim_{k\to\infty} f(w_{n_k}) = M$. The set $f^{-1}(Y)$, as an inverse image of $\mathcal{T}_\psi$-closed set Y, is closed in natural topology on $\mathbb{R}$. Therefore $w_0 \in f^{-1}(Y)$. On the other hand, $M \notin Y$, so $w_0 \notin f^{-1}(Y)$ what gives a contradiction. □

We would like to remind some information on ψ-approximately continuous functions. Let $\psi \in \mathcal{C}$ and $f : (\mathbb{R}, \mathcal{T}_\psi) \to (\mathbb{R}, \mathcal{T}_o)$ be continuous. As $\mathcal{T}_o \subset \mathcal{T}_\psi \subset \mathcal{T}_d$ we obtain

$$\mathcal{C}_{oo} \subset \mathcal{C}_{\psi o} \subset \mathcal{C}_{do}\,.$$

From [F1, Property 5] we know that there are functions $\psi_1, \psi_2 \in \mathcal{C}$ such that

$$\mathcal{C}_{\psi_1 o} \subsetneq \mathcal{C}_{\psi o} \subsetneq \mathcal{C}_{\psi_2 o}\,.$$

Hence

$$\mathcal{C}_{oo} \subsetneq \mathcal{C}_{\psi o} \subsetneq \mathcal{C}_{do}\,.$$

In [W, Theorem 3,6] it is shown that every approximately continuous function is Darboux-Baire one, so

$$\mathcal{C}_{\psi o} \subsetneq \mathcal{DB}_1\,.$$

Moreover, in [W-B, Theorem 9] it is proved that the family $\mathcal{C}_{\psi o}$ is not contained in the class of Baire* 1 functions.

It is easy to check (compare Theorem 1.4.7 in [E]) that

Theorem 4. *Let $\psi \in \mathcal{C}$. If f, g are ψ-approximately continuous functions, then $f+g$, $f-g$, $f\cdot g$, $\max(f,g)$ and $\min(f,g)$ are ψ-approximately continuous. If f is a ψ-approximately continuous function and $f(x) \neq 0$ for $x \in X$, then $1/f$ is ψ-approximately continuous.*

Theorem 5. *Let $\psi \in \mathcal{C}$. If $f_n \in \mathcal{C}_{\psi o}$ for any $n \in \mathbb{N}$ and a sequence $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to f, then $f \in \mathcal{C}_{\psi o}$.*

Theorem 6. *If $f \in \mathcal{C}_{\psi o}$ and $g \in \mathcal{C}_{oo}$, then $g \circ f \in \mathcal{C}_{\psi o}$.*

2. Classes $\mathcal{C}_{\psi\psi}$

Fix a function ψ from the family $\mathcal{C}$. We will consider continuous functions $f : (\mathbb{R}, \mathcal{T}_\psi) \to (\mathbb{R}, \mathcal{T}_\psi)$. Such functions will be called ψ-continuous. Of course,

$$\text{Const} \subsetneq \mathcal{C}_{\psi\psi} \subset \mathcal{C}_{\psi o} \subsetneq \mathcal{DB}_1\,.$$

Hence any function from $\mathcal{C}_{\psi\psi}$ is measurable.

EXAMPLE 7. (Compare [CLO3, Example 1].)
There exists a function $f \in \mathcal{C}_{oo} \setminus \mathcal{C}_{\psi\psi}$.

There exists an interval set $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$, $b_{n+1} < a_n < b_n$, $\lim_{n\to\infty} b_n = 0$, such that $\lim_{h\to 0+} \dfrac{m(A' \cap (0,h))}{2h\psi(2h)} = 0$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence strictly decreasing to zero. We put

$$f(x) = \begin{cases} y_n & \text{for } x \in [a_n, b_n],\ n \in \mathbb{N}, \\ 0 & \text{for } x \leqslant 0, \\ \text{linear} & \text{for } x \in [b_{n+1}, a_n],\ n \in \mathbb{N}. \end{cases}$$

The function f is continuous at every point $x \neq 0$ as piecewise linear function. f is continuous at $x = 0$ because $\lim_{n\to\infty} f(y_n) = 0 = f(0)$. We will show that $f \notin \mathcal{C}_{\psi\psi}$. From the construction of A we have that $x = 0$ is its ψ-density point and $0 \notin A$, so A is not a $\mathcal{T}_\psi$-closed set. The set $\{y_n\}_{n\in\mathbb{N}}$ is $\mathcal{T}_\psi$-closed and $f^{-1}(\{y_n\}) = A$. Therefore $f \notin \mathcal{C}_{\psi\psi}$.

Since $\mathcal{C}_{oo} \subset \mathcal{C}_{\psi o}$, it follows that $\mathcal{C}_{\psi\psi} \subsetneq \mathcal{C}_{\psi o}$.

LEMMA 8. ([F2, Proposition 7]) *Let $\psi \in \mathcal{C}$ and*

$$\liminf_{x\to 0^+} \frac{\psi(\alpha x)}{\psi(x)} > 0 \qquad \textit{for any } \alpha \in \mathbb{R}_+. \tag{1}$$

If $A \in \mathcal{T}_\psi$, then $\alpha A \in \mathcal{T}_\psi$.

COROLLARY 9. *Suppose that the condition* (1) *is satisfied.*

a) *If f is a linear function, then $f \in \mathcal{C}_{\psi\psi}$.*

b) *If f is linear on intervals $(a,b]$ and $[b,c)$, then f is ψ-continuous at any point of the interval (a,c).*

Proof. Suppose that $f(x) = mx + n$ and $m \neq 0$. Let $V \in \mathcal{T}_\psi$. From Lemma 8 and the fact that $\mathcal{T}_\psi$ is invariant under translation we obtain that $f^{-1}(V) = \frac{1}{m}(V - n) \in \mathcal{T}_\psi$ and $f \in \mathcal{C}_{\psi\psi}$. If $m = 0$, then $f \in \mathrm{Const} \subset \mathcal{C}_{\psi\psi}$.

Now suppose that $f_1(x) = m_1 x + n_1$, $f_2(x) = m_2 x + n_2$ and $f_1\big|_{(a,b]} = f\big|_{(a,b]}$ and $f_2\big|_{[b,c)} = f\big|_{[b,c)}$. We can assume that $m_1 \neq 0$ and $m_2 \neq 0$. To show that f is ψ-continuous at a point $x_0 \in (a,b)$ we should check that for any $V \in \mathcal{T}_\psi$ including $f(x_0)$ there exists a $\mathcal{T}_\psi$-open neighbourhood U of a point x_0 such that $f(U) \subset V$. Let $V \in \mathcal{T}_\psi$ and $f(x_0) \in V$. It is apparent that

$x_0 \in \frac{1}{m_1}(V-n_1)\cap(a,b) \in \mathcal{T}_\psi$. In the same way we obtain that f is ψ-continuous at any point x_0 from (b,c).

Now, let $V \in \mathcal{T}_\psi$ and $f(b) \in V$. The set $U_1 = f_1^{-1}(V) \in \mathcal{T}_\psi$ and b is a left-hand ψ-density point of U_1. Moreover, $U_2 = f_2^{-1}(V) \in \mathcal{T}_\psi$ and b is a right-hand ψ-density point of U_2. Hence $U_1 \cap (a,b) \in \mathcal{T}_\psi$ and $U_2 \cap (b,c) \in \mathcal{T}_\psi$ and the set

$$U = \big(U_1 \cap (a,b)\big) \cup \big(U_2 \cap (b,c)\big) \cup \{b\}$$

is open in $\mathcal{T}_\psi$. □

COROLLARY 10. *Suppose that condition* (1) *is satisfied. If $f \in \mathcal{C}_{\psi\psi}$, then $kf \in \mathcal{C}_{\psi\psi}$ for any $k \in \mathbb{R}$.*

Let us notice that if $\liminf\limits_{x\to 0+} \dfrac{\psi(\alpha x)}{\psi(x)} = 0$, then a function $f(x) = \frac{1}{\alpha}x$ is not ψ-continuous (see [F2, Proposition 5]).

EXAMPLE 11. (Compare [CLO3, Example 2].)
There exists a function $f \in \mathcal{C}_{\psi\psi} \setminus \mathcal{C}_{oo}$.

Let $\psi \in \mathcal{C}$ be a function fulfilling condition (1) and $A = \bigcup\limits_{n=1}^{\infty} (a_n, b_n) \subset (0,\infty)$ be an interval set such that $\lim\limits_{h\to 0+} \dfrac{m\big(A\cap(0,h)\big)}{2h\,\psi(2h)} = 0$. We put

$$f(x) = \begin{cases} 0 & \text{for } x \notin A, \\ 1 & \text{for } x = \frac{a_n+b_n}{2},\ n \in \mathbb{N}, \\ \text{linear} & \text{for } x \in \big[a_n, \frac{a_n+b_n}{2}\big] \cup \big[\frac{a_n+b_n}{2}, b_n\big],\ n \in \mathbb{N}. \end{cases}$$

Such a function does not belong to the class $\mathcal{C}_{oo}$, because it is not continuous at $x_0 = 0$. Observe that $f \in \mathcal{C}_{\psi\psi}$. For $x \in (-\infty,0)$ the function f is ψ-continuous at x because it is constant. For $x \in (0,\infty)$ it is ψ-continuous at x according to Corollary 9. We shall show that f is ψ-continuous at $x_0 = 0$. Let V be any set such that $f(x_0) = 0 \in V$. We take $U = A'$. Then $U \in \mathcal{T}_\psi$ and $f(U) = \{0\} \in V$. Therefore f is ψ-continuous at $x_0 = 0$. Hence $f \in \mathcal{C}_{\psi\psi} \setminus \mathcal{C}_{oo}$.

EXAMPLE 12. There exists a function f which is ψ-approximately continuous, however it is neither continuous nor ψ-continuous.

Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers strictly decreasing to zero. Let us take an interval set $A = \bigcup\limits_{n=1}^{\infty} (a_n, b_n)$ such that we have $\lim\limits_{h\to 0+} \dfrac{m\big(A' \cap [-h,h]\big)}{2h\psi(2h)} = 0$ and $[a_n, b_n] \subset (t_{n+1}, t_n)$. Let B be an interval set $B = \bigcup\limits_{n=1}^{\infty} (c_n, d_n)$ such that $\lim\limits_{h\to 0+} \dfrac{m\big(B \cap [-h,h]\big)}{2h\psi(2h)} = 0$ and $[c_n, d_n] \subset (a_n, b_n)$

(compare [F1, Proposition 1]). We define a function f:

$$f(x) = \begin{cases} 0 & \text{for } x \leqslant 0 \quad \text{and} \quad x = a_n,\ n \in \mathbb{N}, \\ \frac{1}{n} & \text{for } x \in (a_n, b_n) \setminus (c_n, d_n),\ n \in \mathbb{N}, \\ 1 & \text{for } x = \frac{c_n + d_n}{2},\ n \in \mathbb{N}, \\ \text{linear} & \text{for the remaining } x. \end{cases}$$

The function f is continuous (so ψ-approximately continuous) at any point $x \neq 0$. It is not continuous at $x = 0$ as $\lim\limits_{n\to\infty} f\bigl(\frac{c_n+d_n}{2}\bigr) = 1 \neq f(0)$. It is ψ-approximately continuous at $x = 0$ because there exists a measurable set $E = (-\infty, t_1) \setminus B$ such that $x = 0$ is its ψ-density point and $f\big|_E$ is continuous at $x = 0$ ([W-B, Theorem 10]).

The function f is ψ-continuous at any $x \neq 0$ but it is not ψ-continuous at $x = 0$. The set $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is $\mathcal{T}_\psi$-closed as a set of measure zero. The inverse image of this set is not $\mathcal{T}_\psi$-closed, because $x = 0$ is its ψ-density point and $0 \notin f^{-1}(\{\frac{1}{n}\})$.

3. Classes $\mathcal{C}_{\psi d}$

Let $\psi \in \mathcal{C}$. By Property 2, $\mathcal{C}_{\psi d} \subset \mathcal{C}_{\psi\psi} \cap \mathcal{C}_{dd}$. Then, because functions from $\mathcal{C}_{dd}$ are measurable and $\mathcal{DB}_1^*$ ([CL2, Theorem 4.1]), every function from the family $\mathcal{C}_{\psi d}$ is measurable and $\mathcal{DB}_1^*$. On the other hand there does not exists an invertible function from that class.

Theorem 13. *If $f \in \mathcal{C}_{\psi d}$ then f is not a bijection.*

P r o o f. Let $f\colon (\mathbb{R}, \mathcal{T}_\psi) \to (\mathbb{R}, \mathcal{T}_d)$ be a continuous function. Observe that an image of any set, open in natural topology $\mathcal{T}_o$, is a measurable set. Let $U \in \mathcal{T}_o$. The set U can be represented as a sum of denumerable family of open intervals I_n. Since f is a Darboux function, $f(I_n)$ is an interval for any $n \in \mathbb{N}$ and

$$f(U) = f\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f(I_n)$$

is a measurable set.

Suppose that there exists the inverse function $f^{-1}\colon \mathbb{R} \to \mathbb{R}$. In [T, Theorem 4] there is constructed a set A, nowhere dense and perfect in natural topology $\mathcal{T}_o$, such that $m(A) > 0$ and $\mathrm{Int}_\psi(A) = \emptyset$. We will prove that $m\bigl(f(A)\bigr) = 0$. The set A is closed in $\mathcal{T}_o$, so it is closed in $\mathcal{T}_\psi$. Since $\mathrm{Int}_\psi(A) = \emptyset$, A is nowhere dense in $\mathcal{T}_\psi$. Therefore A' is dense in $\mathcal{T}_\psi$. Moreover, $A' \in \mathcal{T}_o$, so $f(A')$ is a measurable set. Hence the set $\mathbb{R} \setminus f(A') = f(A)$ is measurable. Suppose that

$m(f(A)) > 0$. Then there is a nonempty set $U \subset f(A)$ open in topology $\mathcal{T}_d$. Since f is continuous, $f^{-1}(U) \in \mathcal{T}_\psi$. As A' is dense in $\mathcal{T}_\psi$, then $f^{-1}(U) \cap A' \neq \emptyset$ and, consequently, $U \cap f(A') \neq \emptyset$. It is a contradiction, because $U \subset f(A)$. This proves that $m(f(A)) = 0$.

It follows that the set $f(A)$ and any subset B of $f(A)$ is closed in $\mathcal{T}_d$. Since $f \in \mathcal{C}_{\psi d}$, $f^{-1}(B)$ is closed in $\mathcal{T}_\psi$ for any $B \subset f(A)$. Any $\mathcal{T}_\psi$-closed set is measurable. It is easy to see that for any $C \subset A$ there exists $B \subset f(A)$ such that $f^{-1}(B) = C$. But A contains a nonmeasurable set since $m(A) > 0$. This contradiction completes the proof. □

Modifying the last proof we can prove the following

THEOREM 14. *If $f \in \mathcal{C}_{\psi d}$ and $a < b$ then $f\big|_{(a,b)}$ is not an injection.*

Proof. Suppose that $f \in \mathcal{C}_{\psi d}$, $a < b$ and $f\big|_{(a,b)}$ is an injection. Since $f \in \mathcal{C}_{\psi d}$ is a Darboux function, the image $f\big((a,b)\big)$ is an interval. By g denote the function $f\big|_{(a,b)}$. Thus the function $g\colon (a,b) \to f\big((a,b)\big)$ is a bijection.

Repeating a construction presented in [T, Theorem 4] we obtain a set $A \subset (a,b)$ perfect and nowhere dense in natural topology $\mathcal{T}_o$, such that $m(A) > 0$ and $\mathrm{Int}_\psi(A) = \emptyset$. We will show that $m\big(g(A)\big) = 0$.

The complement A' of a set A is an open set in $\mathcal{T}_o$, Hence $A' \cap (a,b) \in \mathcal{T}_o$ and as g is a Darboux function, we obtain (like in the previous proof) that $g(A')$ is a measurable set. Let us suppose that the set

$$g(A) = f\big((a,b)\big) \setminus g(A')$$

has positive measure. Therefore there is a nonempty set $U \subset g(A)$ open in topology $\mathcal{T}_d$. Since $f \in \mathcal{C}_{\psi d}$, $f^{-1}(U)$ is a nonempty $\mathcal{T}_\psi$-open set and $\emptyset \neq f^{-1}(U) \cap (a,b) \in \mathcal{T}_\psi$.

The set A' is dense in the space $(\mathbb{R}, \mathcal{T}_\psi)$, so there exists a point $x \in A' \cap f^{-1}(U) \cap (a,b)$. It follows that $g(x) \in g(A')$ and $g(x) \in U$ what contradicts our assumption that $U \subset g(A)$. This proves that $m\big(g(A)\big) = 0$. Therefore every subset of $g(A)$ is measurable meanwhile A contains a nonmeasurable set as a set of positive measure. This contradiction completes the proof. □

We do not know if there exists a nonconstant function in $\mathcal{C}_{\psi d}$.

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Received March 24, 2004

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