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REMARKS ON f-DENSITY AND ψ-DENSITY

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ABSTRACT. In [M. Filipczak, T. Filipczak: *A generalization of the density topology*, to appear] a notion of f-density point of a Lebesgue measurable set on the real line was introduced. It is a generalization of a well-known notion of density point and simultaneously a notion of the ψ-density point, introduced by [S. J. Taylor: *On strengthening the Lebesgue density theorem*, Fund. Math. **XLVI**, (1959), 305–315]. We examine properties of topologies generated by f-density operators, for functions satisfying $\liminf_{x\to 0^+} \frac{f(x)}{x} = 0$, and their similarities to ψ-density topologies.

Throughout the paper we shall use a standard notation: $\mathbb{R}$ will denote the set of real numbers, $\mathcal{L}$ the family of Lebesgue measurable subsets of $\mathbb{R}$ and $|E|$ the Lebesgue measure of a measurable set E. We shall also write $A \sim B$ if $|A \triangle B| = 0$.

Let $\mathcal{A}$ be a family of all functions $f\colon (0,\infty) \to (0,\infty)$ such that

(A1) $\lim_{x\to 0^+} f(x) = 0$,

(A2) $\liminf_{x\to 0^+} \frac{f(x)}{x} < \infty$,

(A3) f is nondecreasing.

Let $f \in \mathcal{A}$, $A \in \mathcal{L}$ and $x \in \mathbb{R}$. We say that x is a *right-hand f-density point* of A if

$$\lim_{h\to 0^+} \frac{\bigl|(x, x+h) \setminus A\bigr|}{f(h)} = 0 .$$

By $\Phi_f^+(A)$ we denote the set of all right-hand f-density points of A. In the same way one may define a left-hand f-density point of A and the set $\Phi_f^-(A)$. We say that x is an *f-density point of A* if x is a right-hand and a left-hand f-density point of A. By $\Phi_f(A)$ we denote the set of all f-density points of A, which means that $\Phi_f(A) = \Phi_f^+(A) \cap \Phi_f^-(A)$. Putting $f(x) = x$ for $x > 0$ we obtain a classical Lebesgue density point and the set of all Lebesgue density points $\Phi_d(A)$.

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Observe that condition (A2) is essential. If we take a function f fulfilling (A1) and (A3) and such that $\liminf_{x\to 0+} \frac{f(x)}{x} = \infty$, then for every point x and any measurable set A we have

$$\lim_{h\to 0+} \frac{\left|(x, x+h) \setminus A\right|}{f(h)} \leqslant \limsup_{h\to 0+} \frac{\left|(x, x+h) \setminus A\right|}{h} \cdot \lim_{h\to 0+} \frac{h}{f(h)} = 0\,.$$

It is easy to verify that if f satisfies (A1)–(A3) and x is an f-density point of A, then x is not a dispersion point of A. From this fact and according to Lebesgue Density Theorem it follows that $\left|\Phi_f(A) \setminus A\right| = 0$ for any function $f \in \mathcal{A}$ and any measurable set A (cf. [FF1, Corollary 1]).

For any $f \in \mathcal{A}$ we denote

$$\mathcal{T}_f = \left\{A \in \mathcal{L}\colon A \subset \Phi_f(A)\right\}.$$

In [FF1] operators Φ_f and families $\mathcal{T}_f$ for continuous functions satisfying (A1)–(A3) were studied. It was shown that the families $\mathcal{T}_f$ are topologies stronger than the natural topology on the real line. In [FF2] it was proved that for any $f \in \mathcal{A}$ there is a continuous function $g \in \mathcal{A}$ such that $\mathcal{T}_f = \mathcal{T}_g$. Therefore the family $\mathcal{T}_f$ forms a topology for any $f \in \mathcal{A}$. Moreover, if $f \in \mathcal{A}$ and $|A| = 0$ then A is closed and nowhere dense in topology $\mathcal{T}_f$.

Let us divide the family $\mathcal{A}$ in two subfamilies

$$\mathcal{A}^0 = \left\{f \in \mathcal{A}\colon \liminf_{x\to 0+} \frac{f(x)}{x} = 0\right\},$$

$$\mathcal{A}^1 = \left\{f \in \mathcal{A}\colon \liminf_{x\to 0+} \frac{f(x)}{x} > 0\right\}.$$

It is not difficult to check that $\Phi_d(A) \subset \Phi_f(A)$ for any $f \in \mathcal{A}^1$. Indeed, suppose that $x \in \Phi_d^+(A)$. Thus $\lim_{h\to 0+} \frac{|(x,x+h)\setminus A|}{h} = 0$ and from the inequality $\limsup_{h\to 0+} \frac{h}{f(h)} < \infty$ we obtain

$$\lim_{h\to 0+} \frac{\left|(x, x+h) \setminus A\right|}{f(h)} \leqslant \lim_{h\to 0+} \frac{\left|(x, x+h) \setminus A\right|}{h} \cdot \lim_{h\to 0+} \sup \frac{h}{f(h)} = 0$$

and $x \in \Phi_f^+(A)$. This easy observation and the equality $\left|\Phi_f(A) \setminus A\right| = 0$ gives us

Proposition 1. *For any $f \in \mathcal{A}^1$ and any set $A \in \mathcal{L}$*

(a) $\Phi_f(A) \sim A$;

(b) $\mathcal{T}_d \subset \mathcal{T}_f$;

(c) $\operatorname{int}_{\mathcal{T}_f}(A) = \emptyset$ *if and only if* $|A| = 0$;

(*d*) *A is a set of the first category in topology $\mathcal{T}_f$ if and only if $|A| = 0$.*

We will show that none of these properties is valid for $f \in \mathcal{A}^0$.

A convenient example of a function belonging to family $\mathcal{A}^0$ is a function of the form $f(x) = x\psi(x)$ where ψ is a continuous nondecreasing function such that $\lim_{x \to 0^+} \psi(x) = 0$. It leads us to a notion of ψ-density introduced by S. J. Taylor in [T]. A point x is a *ψ-density point* of a measurable set A if

$$\lim_{\substack{h \to 0, k \to 0 \\ h \geqslant 0, k \geqslant 0, h+k>0}} \frac{\big|(x-h, x+k) \setminus E\big|}{(h+k)\psi(h+k)} = 0\,,$$

where ψ is a continuous nondecreasing function such that $\lim_{x \to 0^+} \psi(x) = 0$. Fix a function ψ satisfying the above conditions. We can denote by $\Phi_\psi(A)$ the set of all ψ-density points of a set A, and by T_ψ the family consisting of measurable sets such that $A \subset \Phi_\psi(A)$. Topologies T_ψ, called ψ-density topologies, were examined by M. Terepeta, E. Wagner-Bojakowska, W. Wilczyński and the others (in fact, they considered a "symmetric version" of ψ-density, but it could be simply repeated with the Taylor's ψ-density).

Clearly, any ψ-density topology is included in the density topology $\mathcal{T}_d$. It is known that ψ-density operators Φ_ψ and ψ-density topologies T_ψ do not fulfil any property from Proposition 1 (compare [TW-B, Theorem 1.3], and [W--B, Theorem 5]). We will show that there exists a function $f \in \mathcal{A}^0$ such that $\mathcal{T}_f \subset \mathcal{T}_d$ and $\mathcal{T}_f$ is different from any ψ-density topology.

Theorem 1. *There exists a function f belonging to $\mathcal{A}^0$ such that $\mathcal{T}_f \subset \mathcal{T}_d$ and $\mathcal{T}_f \neq \mathcal{T}_g$ for any function g of the form $g(x) = x\psi(x)$ where ψ is a continuous nondecreasing function with $\lim_{x \to 0^+} \psi(x) = 0$.*

P r o o f. Let $x_n = \frac{1}{n!}$ and $f(x) = \frac{1}{n!}$ for $x \in \big[\frac{1}{n!}, \frac{1}{(n-1)!}\big)$, $n = 2, 3 \dots$ Suppose that $g(x) = x\,\psi(x)$ for some continuous nondecreasing function $\psi\colon (0,\infty) \to (0,\infty)$ with $\lim_{x \to 0^+} \psi(x) = 0$. Let $y_n = x_n + x_n\psi(x_{n-1})$. From $y_n = x_n\big(1 + \psi(x_{n-1})\big)$ and $x_{n-1} = n x_n$ it follows that $y_n < x_{n-1}$ for $n \geqslant n_0$. Put $A = \bigcup_{n=n_0}^{\infty} [x_n, y_n]$. We have

$$\begin{aligned} \frac{\big|A \cap (0, y_n)\big|}{g(y_n)} &\geqslant \frac{x_n\psi(x_{n-1})}{\big(x_n + x_n\psi(x_{n-1})\big)\psi\big(x_n + x_n\psi(x_{n-1})\big)} \\ &\geqslant \frac{\psi(x_{n-1})}{\big(1+\psi(x_{n-1})\big)\psi(x_{n-1})} = \frac{1}{\big(1+\psi(x_{n-1})\big)} \xrightarrow[n\to\infty]{} 1\,. \end{aligned}$$

Therefore $0 \notin \Phi_g(A')$. On the other hand, for any $x \in \left[\frac{1}{n!}, \frac{1}{(n-1)!}\right)$,

$$\begin{aligned}\frac{|A \cap (0,x)|}{f(x)} &\leqslant \frac{(y_n - x_n) + y_{n+1}}{x_n} = \frac{x_n \psi(x_{n-1}) + x_{n+1} + x_{n+1}\psi(x_n)}{x_n} \\ &= \psi(x_{n-1}) + \frac{x_{n+1}}{x_n}\big(1 + \psi(x_n)\big) = \psi\left(\frac{1}{(n-1)!}\right) + \frac{1 + \psi(\frac{1}{n!})}{n+1} \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

It follows that $0 \in \Phi_f(A')$ and $A' \in \mathcal{T}_f \setminus \mathcal{T}_g$. □

The main theorem in this paper is a generalization of [T, Theorem 4] proved by S. J. Taylor for ψ-density points.

Theorem 2. *For any function $f \in \mathcal{A}^0$ there exists a nowhere dense closed set $F \subset [0,1]$ of positive measure such that no point of F is an f-density point of F.*

Proof. First, we define a few auxiliary sequences. Since $f \in \mathcal{A}^0$, there is a decreasing sequence (x_n), tending to zero, such that

$$\lim_{n\to\infty} \frac{f(x_n)}{x_n} = 0\,, \tag{1}$$

and that the sequence $\left(\frac{f(x_n)}{x_n}\right)$ is decreasing. Let $w_n = \sqrt{\frac{f(x_n)}{x_n}}$ for $n = 1, 2, \ldots$ We can assume that $w_1 < \frac{1}{4}$. Put $y_1 = \frac{1}{4} - w_1$ and $y_n = w_{n-1} - w_n$ for $n = 2, 3 \ldots$ Obviously, $y_n > 0$ and $\sum_{n=1}^{\infty} y_n = \frac{1}{4}$.

We define two sequences of positive integers, (k_i) and increasing (n_i), satisfying the following conditions

$$\left(1 - 2\sum_{j=1}^{i} y_j\right)\frac{1}{k_1 \cdots k_i} \leqslant x_{n_i} < \frac{1}{k_1 \cdots k_i}\,, \tag{2}$$

$$x_{n_i} - \frac{2y_{i+1}}{k_1 \cdots k_i} \leqslant k_{i+1} x_{n_{i+1}} < x_{n_i} - \frac{y_{i+1}}{k_1 \cdots k_i} \tag{3}$$

for any $i \in \mathbb{N}$.

As (x_n) tends to zero, there is $n_1 \in \mathbb{N}$ such that

$$x_{n_1} < y_1\,.$$

Hence there is $k_1 \in \mathbb{N}$ satisfying

$$y_1 \leqslant 1 - k_1 x_{n_1} \leqslant 2y_1\,.$$

It is clear that (2) is fulfilled for $i = 1$.

Suppose that we have defined numbers $k_1, \dots, k_i$ and $n_1, \dots, n_i$ satisfying (2). Let n_{i+1} be an arbitrary natural number greater than n_i, satisfying

$$x_{n_{i+1}} < \frac{y_{i+1}}{k_1 \cdots k_i}. \tag{4}$$

By the assumption we have $x_{n_i} > \frac{1}{2}\frac{1}{k_1 \cdots k_i}$ and $2y_{i+1} < \frac{1}{2}$. Consequently,

$$x_{n_i} - \frac{2y_{i+1}}{k_1 \cdots k_i} > 0.$$

Therefore there exists $k_{i+1} \in \mathbb{N}$ satisfying

$$x_{n_i} - \frac{2y_{i+1}}{k_1 \cdots k_i} \leqslant k_{i+1} x_{n_{i+1}} < x_{n_i} - \frac{y_{i+1}}{k_1 \cdots k_i} \tag{5}$$

and (3) is fulfilled. We will check that (2) is fulfilled, too. From (5) and the induction hypothesis it follows

$$k_{i+1} x_{n_{i+1}} < x_{n_i} < \frac{1}{k_1 \cdots k_i}$$

and, consequently,

$$x_{n_{i+1}} < \frac{1}{k_1 \cdots k_{i+1}}.$$

On the other hand,

$$\begin{aligned} k_{i+1} x_{n_{i+1}} \geqslant x_{n_i} - \frac{2y_{i+1}}{k_1 \cdots k_i} &\geqslant \left(1 - 2\sum_{j=1}^{i} y_j\right) \frac{1}{k_1 \cdots k_i} - \frac{2y_{i+1}}{k_1 \cdots k_i} \\ &= \left(1 - 2\sum_{j=1}^{i+1} y_j\right) \frac{1}{k_1 \cdots k_i}, \end{aligned}$$

which ends the construction of sequences (k_i) and (n_i).

Now, we will construct a decreasing sequence (F_i) of closed sets with the property that F_i consists of $k_1 \cdots k_i$ disjoint closed intervals of the length x_{n_i}. Put

$$F_1 = \bigcup_{j=0}^{k_1 - 1} \left[\frac{j}{k_1}, \frac{j}{k_1} + x_{n_1}\right].$$

Since $x_{n_1} < \frac{1}{k_1}$, intervals $\left[\frac{j}{k_1}, \frac{j}{k_1} + x_{n_1}\right]$, $j = 0, \dots, k_1 - 1$, are disjoint.

Suppose that we have just defined a set F_i for some $i \geqslant 1$. Denote components of F_i by $J_p^i = \left[a_p^i, a_p^i + x_{n_i}\right]$, $p = 1, \dots, k_1 \cdots k_i$. Since $x_{n_{i+1}} < \frac{x_{n_i}}{k_{i+1}}$, a set

$$F_{i+1} = \bigcup_{p=1}^{k_1 \cdots k_i} \left(\bigcup_{j=0}^{k_{i+1}-1} \left[a_p^i + x_{n_i}\frac{j}{k_{i+1}}, a_p^i + x_{n_i}\frac{j}{k_{i+1}} + x_{n_{i+1}}\right]\right)$$

consists of $k_1 \cdots k_{i+1}$ disjoint intervals of the length $x_{n_{i+1}}$, which ends the definition of (F_i). Evidently, $|F_i| = k_1 \cdots k_i \cdot x_{n_i}$ for each i.

We will show that for any $i \in \mathbb{N}$ we have

$$\big|[0,1] \setminus F_i\big| \leqslant 2(y_1 + \cdots + y_i)\,. \tag{6}$$

For $i = 1$ we have

$$\big|[0,1] \setminus F_1\big| = 1 - k_1 x_{n_1} \leqslant 2y_1\,.$$

Suppose that (6) is true for some $i \geqslant 1$. From (3) it follows that

$$\begin{aligned}\big|[0,1] \setminus F_{i+1}\big| &= \big|[0,1] \setminus F_i\big| + \big|F_i \setminus F_{i+1}\big| \\ &\leqslant 2(y_1 + \cdots + y_i) + k_1 \cdots k_i \cdot x_{n_i} - k_1 \cdots k_{i+1} \cdot x_{n_{i+1}} \\ &\leqslant 2(y_1 + \cdots + y_i) + 2y_{i+1} = 2(y_1 + \cdots + y_{i+1})\end{aligned}$$

because we have obtained the set F_{i+1} by construction on every component J_p^i of F_i, k_{i+1} intervals with the length $x_{n_{i+1}}$. It proves (6).

Let i be an arbitrary positive integer. Let J^i be a component of F_i. From (3) and (2) we conclude that

$$\big|J^i \setminus F_{i+1}\big| = x_{n_i} - k_{i+1} \cdot x_{n_{i+1}} > \frac{y_{i+1}}{k_1 \cdots k_i} > y_{i+1} \cdot x_{n_i} = y_{i+1}\big|J^i\big|\,. \tag{7}$$

We will prove that for any $p > i$,

$$\big|J^i \setminus F_p\big| \geqslant (y_{i+1} + \cdots + y_p)\big|J^i\big|\,. \tag{8}$$

For $p = i+1$ it follows from (7). Suppose that (8) is true for some $p > i$. We will prove it for $p+1$. Note that the set $J^i \cap F_p$ consists of $k_{i+1} \cdots k_p$ components. For any component J^p of this set we have, by (7),

$$\big|J^p \setminus F_{p+1}\big| > \frac{y_{p+1}}{k_1 \cdots k_p}\,.$$

Hence, using (2), we obtain

$$\big|(J^i \cap F_p) \setminus F_{p+1}\big| \geqslant k_{i+1} \cdots k_p \frac{y_{p+1}}{k_1 \cdots k_p} = \frac{y_{p+1}}{k_1 \cdots k_i} > y_{p+1} x_{n_i} = y_{p+1}\big|J^i\big|\,.$$

Since (8) is true for p, we conclude that

$$\begin{aligned}\big|J^i \setminus F_{p+1}\big| &= \big|J^i \setminus F_p\big| + \big|(J^i \cap F_p) \setminus F_{p+1}\big| \\ &\geqslant (y_{i+1} + \cdots + y_p)\big|J^i\big| + y_{p+1}\big|J^i\big| = (y_{i+1} + \cdots + y_{p+1})\big|J^i\big|\,.\end{aligned}$$

Now, we are in position to define the set F. Let

$$F = \bigcap_{i=1}^{\infty} F_i\,.$$

Clearly, F is closed and, as $\frac{1}{k_1\cdots k_i} \underset{i\to\infty}{\longrightarrow} 0$, F is nowhere dense. From (6) we see that

$$\big|[0,1]\setminus F\big| \leqslant 2\sum_{i=1}^{\infty} y_i = \frac{1}{2}\,,$$

and so $|F| \geqslant \frac{1}{2}$. We will show that no point of F is its f-density point.

Let $x \in F$. By the definition of F it follows that for any $i \in \mathbb{N}$ there is a component J^i of F_i such that $x \in J^i$. Therefore

$$\{x\} = \bigcap_{i=1}^{\infty} J^i\,.$$

From (8) we conclude that

$$\big|J^i \setminus F\big| \geqslant \left(\sum_{p=i+1}^{\infty} y_p\right)\big|J^i\big| = w_i\big|J^i\big|\,.$$

Consequently,

$$\begin{aligned}\frac{\big|J^i\setminus F\big|}{f\big(\big|J^i\big|\big)} &= \frac{\big|J^i\setminus F\big|}{\big|J^i\big|}\cdot\frac{\big|J^i\big|}{f\big(\big|J^i\big|\big)}\\ &\geqslant w_i\frac{x_{n_i}}{f(x_{n_i})} > w_{n_i}\frac{x_{n_i}}{f(x_{n_i})} = \sqrt{\frac{x_{n_i}}{f(x_{n_i})}} \underset{i\to\infty}{\longrightarrow} \infty\,.\end{aligned}$$

It means that x is not an f-density point of F, which ends the proof. □

Now, we can go back to properties defined in Proposition 1.

Proposition 2. *Let $f \in \mathcal{A}^0$ and F be the set constructed in Theorem 2. Then*

(a) $\big|F \,\Delta\, \Phi_f(F)\big| > 0$;

(b) $\mathcal{T}_d \setminus \mathcal{T}_f \neq \emptyset$;

(c) $|F| > 0$ *and* $\operatorname{int}_{\mathcal{T}_f}(F) = \emptyset$;

(d) *the space* $(\mathbb{R}, \mathcal{T}_f)$ *is of the first category.*

P r o o f . Conditions (a) and (c) are obvious. To prove (b) note that the set $F \cap \Phi_d(F)$ is nonempty and open in the density topology. But no point from $F \cap \Phi_d(F)$ is an f-density point of this set and, consequently, $F \cap \Phi_d(F) \notin \mathcal{T}_f$.

Finally, let

$$D = \bigcup_{r\in\mathbb{Q}} (F + r)\,.$$

Of course, the set D is of the first category in $(\mathbb{R}, \mathcal{T}_f)$. Moreover, $\mathbb{R}\setminus D$ has measure zero (compare [W-B, Theorem 5]) and $\mathbb{R}\setminus D$ is nowhere dense in $(\mathbb{R}, \mathcal{T}_f)$, which proves (d). □

Proposition 3. (Compare [FF1, Proposition 9].) *There exists a function* $f \in \mathcal{A}^0$ *such that* $\mathcal{T}_d \nsubseteq \mathcal{T}_f$ *and* $\mathcal{T}_f \nsubseteq \mathcal{T}_d$.

Proof. Let

$$f(x) = \frac{1}{(2n)!} \quad \text{for } x \in \left(\frac{1}{(2n+1)!}, \frac{1}{(2n-1)!}\right].$$

Obviously, $f \in \mathcal{A}$. Since

$$\lim_{n\to\infty} \frac{f\left(\frac{1}{(2n-1)!}\right)}{\frac{1}{(2n-1)!}} = 0,$$

$f \in \mathcal{A}^0$ and by Proposition 2 (b), $\mathcal{T}_d \setminus \mathcal{T}_f \neq \emptyset$.

Put

$$A = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} \left(\frac{\sqrt{2n+1}}{(2n+1)!}, \frac{1}{(2n-1)!}\right).$$

Observe that

$$\frac{\left|A \cap \left(0, \frac{\sqrt{2n+1}}{(2n+1)!}\right)\right|}{\frac{\sqrt{2n+1}}{(2n+1)!}} \leqslant \frac{\frac{1}{(2n+1)!}}{\frac{\sqrt{2n+1}}{(2n+1)!}} = \frac{1}{\sqrt{2n+1}} \xrightarrow[n\to\infty]{} 0,$$

and so 0 is not a density point of A. Thus $A \notin \mathcal{T}_d$. On the other hand, for $h \in \left(\frac{1}{(2n+1)!}, \frac{1}{(2n-1)!}\right]$ we have

$$\frac{\left|[0,h] \setminus A\right|}{f(h)} \leqslant \frac{\left|\left(0, \frac{\sqrt{2n+1}}{(2n+1)!}\right)\right|}{\frac{1}{(2n)!}} = \frac{1}{\sqrt{2n+1}}.$$

Hence $A \in \mathcal{T}_f$, so $\mathcal{T}_f \nsubseteq \mathcal{T}_d$. □

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