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CONNECTIVITY PROPERTY OF MULTIVALUED MAPS

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ABSTRACT. The main purpose of the paper is to extend Fast theorem for a real function onto the multivalued maps case.

THEOREM. (Fast) *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a real function. There exists a function $u : I \to \mathbb{R}$ such that for any $y \in \mathbb{R}$ the function $f^y + u$ has Darboux property.*

1. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, I an interval contained in $\mathbb{R}$. For a set $A \subset I$, $\mathrm{cl}(A)$ denotes its closure in natural topology. For a non-empty set $A \subset \mathbb{R}$ and a number $\varepsilon > 0$ we denote $K_\varepsilon(A) = \{x\colon \text{there exists } y \in A \text{ such that } |x-y| < \varepsilon\}$. By the vector sum $A + B$ of sets A, $B \subset \mathbb{R}$ we mean the set $\{a + b\colon a \in A,\ b \in B\}$. For two sets $A, B \subset \mathbb{R}$ we define

$$A < B \ (\text{resp. } A > B) \iff x < y \ (\text{resp. } x > y) \quad \text{for any } x \in A,\ y \in B\,.$$

For $M \subset X \times Y$, where $X, Y \subset \mathbb{R}$, we write $\pi(M)$ for the projection of M on the x-axis and M_x (M^y) for a vertical (horizontal) x-section (y-section) through M.

In this paper $F\colon I \to \mathbb{R}$ denotes a multivalued map F which assigns a non-empty subset $F(x) \subset \mathbb{R}$ to each point $x \in I$. By the graph of F we mean the set $\bigcup\{(x, y)\colon x \in I,\ y \in F(x)\}$. We make no distinction between a map and its graph. For sets $A \subset I$ and $B \subset \mathbb{R}$ set

$$\begin{aligned} F(A) &= \bigcup\{F(x)\colon x \in A\}\,,\\ F^+(B) &= \{x \in I\colon F(x) \subset B\}\,,\\ F^-(B) &= \{x \in I\colon F(x) \cap B \neq \emptyset\}\,. \end{aligned}$$

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F is weakly Lebesgue measurable if for any open set V the set $F^{-}(V)$ is Lebesgue measurable. F has Darboux property if the image $F(E)$ is connected for any connected set $E \subset I$.

We say that a sequence of multivalued maps $\{F_n\}_{n\in\mathbb{N}}$ converges to F if for any x and any $\varepsilon > 0$ there exist $k \in \mathbb{N}$, such that for $n > k$, $F_n(x) \subset K_\varepsilon(F(x))$ and $F(x) \subset K_\varepsilon(F_n(x))$. If $F\colon A \to \mathbb{R}$, $A \subset I$, is any function or multivalued map, then by a multivalued extension of F we mean any multivalued map $\Phi\colon I \to \mathbb{R}$, such that $\Phi|_A = F$.

2. Generalization of Fast theorem

We introduce the following notation.

$$\mathcal{M} = \big\{M \subset I \times \mathbb{R} \colon M \text{ is a continuum with non-degenerate projection } \pi(M)\big\},$$
$$\mathcal{P} = \{P \in \mathcal{M} \colon P \text{ is a horizontal interval contained in } I \times \mathbb{R}\},$$
$$\mathcal{G} = \big\{M \subset I \times \mathbb{R} \colon M \text{ is the graph of a continuous function } f\colon [a,b] \to \mathbb{R}, \text{ where } [a,b] \subset I,\ a < b\big\}.$$

Definition 1. A multivalued map $F\colon I \to \mathbb{R}$ has (C) *property* if $F \cap M \neq \emptyset$ for any $M \in \mathcal{M}$.

Definition 2. A set $P \subset I$ is *c-dense* in I if for any open set $U \subset I$ the set $U \cap P$ has the continuum cardinality.

Lemma 1. *Let $P \subset I$ be c-dense in I. There exists a function $h\colon P \to \mathbb{R}$ such that $h \cap M \neq \emptyset$ for any $M \in \mathcal{M}$. Moreover, if $F\colon P \to \mathbb{R}$ is any multivalued map for which h is a selector, then any multivalued extension of F has (C) property.*

Proof. Since P is c-dense in I, it is a union of continuum many disjoint sets P_t dense in I, $t \in T$ (Sierpiński [7]). Since the cardinality of M is continuum, there exists a one-to-one function $p\colon T \to \mathcal{M}$. For $t \in T$ and $x \in [a_t, b_t]$ let us denote $p(t) = M_t$, $\pi(M_t) = [a_t, b_t]$, $a_t < b_t$ and $y_x^t = \inf(M_t)_x$. Let us define the function $h\colon P \to \mathbb{R}$ as follows

$$h(x) = \begin{cases} y_x^t; & x \in P_t \cap [a_t, b_t], \\ 0; & x \in P \setminus \bigcup\limits_{t\in T} P_t \cap [a_t, b_t]. \end{cases}$$

Let us see that for any $t \in T$, $\pi(h \cap M_t)$ is dense in $\pi(M_t)$. The same holds for any multivalued extension $\Phi\colon I \to \mathbb{R}$ of a multivalued map F, for which h is a selector. $\square$

The following theorem is a generalization of Fast theorem ([4]).

Theorem 1. *Let $F\colon I\times\mathbb{R}\to\mathbb{R}$ be a multivalued map. There exists a function $u\colon I\to\mathbb{R}$ such that for any $y\in\mathbb{R}$ the multivalued map F^y+u has (C) property. If F is weakly Lebesgue measurable and has closed values, then u can be chosen Lebesgue measurable.*

P r o o f. Let $f\colon I\times\mathbb{R}\to\mathbb{R}$ be a selector from F. Let $P\subset I$ be c-dense in I and of Lebesgue measure zero. Then P can be decomposed into continuum disjoint, non-empty subsets $\{P_\alpha\}_{\alpha<\Omega}$, each of which is c-dense in I. Since P is of measure zero, then each set P_α is measurable. By Lemma (1), there exist functions $h_\alpha\colon P_\alpha\to\mathbb{R}$, $\alpha<\Omega$, for which any multivalued extension on I has (C) property. Define $h\colon I\to\mathbb{R}$ by

$$h(x)=\begin{cases} h_\alpha(x); & x\in P_\alpha\,,\\ c; & x\in I\setminus P\,,\end{cases}$$

where c is any constant. Let $\{y_\alpha\}_{\alpha<\Omega}$ be an enumeration of $\mathbb{R}$ and define the function $k\colon I\to\mathbb{R}$ by

$$k(x)=\begin{cases} y_\alpha; & x\in P_\alpha\,,\\ c; & x\in I\setminus P\,.\end{cases}$$

Let us define $u\colon I\to\mathbb{R}$ as follows

$$u(x)=h(x)-f\bigl(x,k(x)\bigr)\,.$$

For $y\in\mathbb{R}$ let us consider the multivalued map $G(x)=F^y(x)+u(x)$. Then $y=y_\alpha$, for some $\alpha<\Omega$ and $G(x)=F(x,y_\alpha)+h_\alpha(x)-f(x,y_\alpha)$ for $x\in P_\alpha$. Since $0\in F(x,y_\alpha)-f(x,y_\alpha)$, for any $x\in P_\alpha$, then h_α is a selector of $G|_{P_\alpha}$ and by Lemma (1), G has (C) property.

If F is weakly Lebesgue measurable and has closed values, then f can be Lebesgue measurable (K u r a t o w s k i, N a r d z e w s k i [6]). If in definition of h we use c as the number, for which the section $f(x,c)$ is Lebesgue measurable function then u will be Lebesgue measurable, since $u(x)=c-f(x,c)$ for $x\notin P$. $\square$

Point 1. Let $\mathcal{F}$ be the subfamily of $\mathcal{M}$ defined by properties

(1) it contains family $\mathcal{P}$ of horizontal intervals,

(2) if $M\in\mathcal{F}$, $C\subset M$ and $C\in\mathcal{M}$, then $C\in\mathcal{F}$.

Throughout the paper, by $\mathcal{F}$ we mean the subfamily of $\mathcal{M}$ for which the conditions (1) and (2) hold.

Definition 3. (C z a r n o w s k a [2]) A multivalued map $F\colon I\to\mathbb{R}$ is *strongly $\mathcal{F}$-connected* if for any two distinct points $x_1,x_2\in I$ and any $y_1\in$

$F(x_1)$, there exists $y_2 \in F(x_2)$ such that for any set $M \in \mathcal{F}$, if $\pi(M) = [x_1, x_2]$ and

$$\begin{aligned} & y_1 < M_{x_1} \text{ and } y_2 > M_{x_2} \\ \text{or} \quad & y_1 > M_{x_1} \text{ and } y_2 < M_{x_2}, \end{aligned}$$

then $M \cap F|_{(x_1,x_2)} \neq \emptyset$.

Using in particular $\mathcal{F}$ as the family $\mathcal{P}$ of horizontal intervals we obtain an intermediate value property for multivalued maps, which for maps with connected values is equivalent to those given by Ceder ([1]).

If F has connected values then taking $\mathcal{F}$ as the family $\mathcal{G}$ we obtain the functional connectivity. For real functions the functional connectivity was defined by Jastrzębski and Jędrzejewski ([5]).

Remark 1. (Czarnowska, Kwiecińska [3]) Let $F\colon I \to \mathbb{R}$ be a multivalued map with connected values. If F has the intermediate value property, then it has Darboux property.

Remark 2. (Czarnowska [2]) In the case when $\mathcal{F} = \mathcal{M}$ and a map $F\colon I \to \mathbb{R}$ has connected values and it is strongly $\mathcal{F}$-connected, then its graph is connected.

Remark 3. If $F\colon I \to \mathbb{R}$ has (C) property, then it is strongly $\mathcal{F}$-connected for any family $\mathcal{F} \subset \mathcal{M}$.

From Theorem 1 and Remark 3 we get the following corollary.

Corollary 1. *Let $F\colon I \times \mathbb{R} \to \mathbb{R}$ be a multivalued map. There exists a function $u\colon I \to \mathbb{R}$ such that for any $y \in \mathbb{R}$, the map $F^y + u$ is strongly $\mathcal{F}$-connected.*

If F is weakly Lebesgue measurable and has closed values, then u can be chosen Lebesgue measurable.

Corollary 2. *Any multivalued map $G\colon I \to \mathbb{R}$ is the sum of two maps with (C) property. If G is weakly Lebesgue measurable, then it is a sum of two weakly Lebesgue measurable maps with (C) property.*

Proof. Let $G\colon I \to \mathbb{R}$ be any multivalued map and set

$$F(x,y) = \begin{cases} G(x); & y \neq 0, \\ 0; & y = 0. \end{cases}$$

By Theorem 1 there exists a function u such that $F^y + u$ has (C) property, for each $y \in \mathbb{R}$. It means, that $G + u$ and u have (C) property and G is the sum of $G + u$ and $-u$.

We shall show that if G is weakly Lebesgue measurable, then $-u$ and $G+u$ are measurable, too. □

Remark 4. Any multivalued map $F\colon I \to \mathbb{R}$ is a limit of a sequence of multivalued maps with (C) property.

If F is weakly Lebesgue measurable, then it is a limit of weakly Lebesgue measurable maps with (C) property.

P r o o f. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of disjoint c-dense in I sets whose union is of $\bigcup_{n=1}^{\infty} P_n = P$, where P has Lebesgue measure zero. By Lemma 1, there exist functions $h_i\colon P_i \to \mathbb{R}$, $i \in \mathbb{N}$, whose multivalued extensions on I have (C) property. Let us define the sequence of multivalued maps with (C) property as follows

$$F_n(x) = \begin{cases} F(x)\,; & x \in I \setminus \bigcup\limits_{i=n}^{\infty} P_i\,, \\ h_i(x)\,; & x \in P_i\,,\ i \geqslant n\,. \end{cases}$$

It is easy to see that $\{F_n\}_{n=1}^{\infty}$ converges to F. If F is weakly Lebesgue measurable then F_n are weakly Lebesgue measurable as well. □

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