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ON S_i-METACOMPACT SPACES

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ABSTRACT. The primary purpose of this paper is to introduce and study new variations of metacompactness by utilizing semi-open sets.

1. Introduction and preliminaries

Let (X,τ) be a topological space. We shall denote the closure and the interior of a subset A of (X,τ) by $\operatorname{cl} A$ and $\operatorname{int} A$, respectively. A subset A of (X,τ) is called *α-open* (or an *α-set* [6]), if $A \subseteq \operatorname{int}(\operatorname{cl}(\operatorname{int} A))$, *semi-open* ([4]) if $A \subseteq \operatorname{cl}(\operatorname{int} A)$, *preopen* ([5]) if $A \subseteq \operatorname{int}(\operatorname{cl} A)$ and *regular closed* if $A = \operatorname{cl}(\operatorname{int} A)$. Clearly, α-sets and regular closed sets are semi-open. Njastad [6] pointed out that the family of all α-sets of (X,τ), denoted by τ^α, is a topology on X finer than τ. Reilly and Vamanamurthy observed in [7] that a subset A of (X,τ) is an α-set if and only if it is semi-open and preopen. We denote the family of all semi-open (resp. preopen, regular closed) sets in (X,τ) by $\mathrm{SO}(X,\tau)$ (resp. $\mathrm{PO}(X,\tau)$, $\mathrm{RC}(X,\tau)$).

Remark 1.1. [3] Let (X,τ) be a space. Then $(\tau^\alpha)^\alpha = \tau^\alpha$, $\mathrm{SO}(X,\tau^\alpha) = \mathrm{SO}(X,\tau)$, $\mathrm{PO}(X,\tau^\alpha) = \mathrm{PO}(X,\tau)$ and $\mathrm{RC}(X,\tau^\alpha) = \mathrm{RC}(X,\tau)$.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively. The Stone–Cech compactification of $\mathbb{N}$ is denoted by $\beta\mathbb{N}$. The cardinality of a set A will be denoted by $|A|$. No separation axioms are assumed unless stated explicitly. For concepts not defined in our paper we refer the reader to [2].

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2. S_i-metacompact spaces

Definition 1. A space (X,τ) is called

(i) S_1-*metacompact* if every semi-open cover has a point-finite open refinement,

(ii) S_2-*metacompact* if every semi-open cover has a point-finite semi-open refinement,

(iii) S_3-*metacompact* if every open cover has a point-finite semi-open refinement.

Definition 2. A space (X,τ) is called

(i) *globally disconnected* ([1]) if every semi-open set is open,

(ii) *nodec* ([10]) (or an α-*space*) if every nowhere dense subset is closed, or equivalently (see [6]), if $\tau = \tau^\alpha$.

Clearly, if (X,τ) is globally disconnected then all the notions in Definition 1 coincide and are equivalent to metacompactness. In general, we have the following obvious implications:

$$\begin{array}{ccc} S_1\text{-metacompact} & \Longrightarrow & S_2\text{-metacompact} \\ \Downarrow & & \Downarrow \\ \text{metacompact} & \Longrightarrow & S_3\text{-metacompact} \end{array}$$

It will be seen, however, that none of the implications above can be reversed, and that S_2-metacompactness and metacompactness are independent of each other. We shall first characterize globally disconnected spaces and S_1-metacompact spaces.

Proposition 2.1. *For a space (X,τ) the following are equivalent:*

1) *(X,τ) is globally disconnected,*

2) *every semi-open cover of (X,τ) has an open refinement.*

Proof. 1) $\Rightarrow$ 2) is clear. To show that 2) $\Rightarrow$ 1) let $A \in \mathrm{SO}(X,\tau)$ and let $U = \operatorname{int} A$. Then $\mathcal{U} = \{U \cup \{x\} \colon x \in \operatorname{cl} U\} \cup \{X \setminus \operatorname{cl} U\}$ is a semi-open cover of (X,τ). By assumption, there exists an open refinement of $\mathcal{U}$, say $\mathcal{V}$. Suppose that A is not open, i.e., there is $x \in A \setminus U$. Pick $V \in \mathcal{V}$ such that $x \in V$. Then $V \subseteq U \cup \{x\} \subseteq A$ and thus $x \in U$, a contradiction. Hence A is open. $\square$

Corollary 2.2. *A space (X,τ) is S_1-metacompact if and only if it is metacompact and globally disconnected.*

Recall that a space (X,τ) is said to be *extremally disconnected* if the closure of each open set is open, or equivalently, if each regular closed set is clopen.

Jankovic [3] observed that (X, τ) is extremally disconnected if and only if $\mathrm{SO}(X, \tau) \subseteq \mathrm{PO}(X, \tau)$. The next (known) result follows easily from the fact that $\tau^\alpha = \mathrm{SO}(X, \tau) \cap \mathrm{PO}(X, \tau)$ [7] and from Remark 1.1.

Remark 2.3. For a space (X, τ) the following are equivalent:

(i) (X, τ) is extremally disconnected,

(ii) (X, τ^α) is extremally disconnected,

(iii) $\mathrm{SO}(X, \tau) \subseteq \tau^\alpha$,

(iv) (X, τ^α) is globally disconnected.

Remark 2.4. (i) A space (X, τ) is globally disconnected if and only if it is extremally disconnected and nodec.

(ii) If (X, τ) is extremally disconnected then the concepts of (X, τ^α) being metacompact, S_1-metacompact, S_2-metacompact and S_3-metacompact coincide.

Remark 2.5. The notions of (X, τ) being metacompact and of (X, τ^α) being metacompact are, in general, independent of each other. In Corollary 2.10 we provide an example of a compact, consequently metacompact, space (X, τ) such that (X, τ^α) fails to be metacompact. On the other hand, if (X, τ) denotes the space from Theorem 2.14 (2) then (X, τ) fails to be metacompact while (X, τ^α) is metacompact.

Clearly, every discrete space is S_1-metacompact. To provide an example of a non-discrete S_1-metacompact space, we consider $X = \mathbb{N} \cup \{p\}$ where p denotes a free ultrafilter on $\mathbb{N}$. As a base for the topology τ we take all sets of the form $U \cup \{p\}$, where $U \in p$, together with all singletons $\{n\}$, $n \in \mathbb{N}$. It is very well-known (see, e.g., [9, Ex. 114]) that (X, τ) is perfectly normal, extremally disconnected and metacompact. Clearly every nowhere dense set is closed, so (X, τ) is nodec and thus globally disconnected. Hence (X, τ) is S_1-metacompact by Corollary 2.2.

Remark 2.6. Spaces which fail to be S_1-metacompact exist in profusion. The usual space $\mathbb{R}$ of reals is metacompact but not S_1-metacompact. $\beta\mathbb{N}$ is a compact T_2 space which is not S_1-metacompact as $\mathbb{N} \cup \{p\}$ with $p \in \beta\mathbb{N} \setminus \mathbb{N}$ is a semi-open subset of $\beta\mathbb{N}$ that is not open.

We shall now consider S_2-metacompact spaces and S_3-metacompact spaces. First of all, by Remark 1.1 we have that (X, τ) is S_2-metacompact if and only if (X, τ^α) is S_2-metacompact.

Example 2.7. Let $X^* = X \cup \{p\}$ denote the 1-point-compactification of an infinite discrete space X, where $p \notin X$. Let $\mathcal{U}$ be a semi-open cover of X^*. Pick $S \in \mathcal{U}$ such that $p \in S$. Then $S = \{p\} \cup A$ where A is an infinite subset of X. Clearly, $\mathcal{V} = \{S\} \cup \{\{x\}: x \in X \setminus A\}$ is a disjoint semi-open refinement of $\mathcal{U}$. Thus X^* is a compact Hausdorff S_2-metacompact space. In addition, it is easy to see that X^* fails to be globally disconnected, hence it cannot be S_1-metacompact.

Next, we recall the following set-theoretic construction that has been observed in [8].

Lemma 2.8. ([8]) *Let X be an infinite set with $|X| = \mu$ and let $X = A \cup B$ where $|A| = \mu$, $|B| = \nu < \mu$ and $A \cap B = \emptyset$. For each $x \in A$ let $C_x = B \cup \{x\}$. Then $\mathcal{C} = \{C_x: x \in A\}$ is a cover of X. Suppose that $\mathcal{D}$ is a refinement of $\mathcal{C}$. For each $x \in A$ there exists $D_x \in \mathcal{D}$ such that $x \in D_x \subseteq C_x$. If $|D_x| > 1$ for each $x \in A$ then there exists an element $z \in B$ such that z lies in at least μ elements of the refinement $\mathcal{D}$.*

Theorem 2.9. *Let (X, τ) be an infinite space possessing a dense open subset D such that $|D| < |X|$. Then (X, τ) fails to be S_2-metacompact.*

Proof. Let $A = X \setminus D$ and $B = D$. Utilizing Lemma 2.8 we obtain a semi-open cover $\mathcal{C}$ that does not have a semi-open point finite refinement. Thus (X, τ) fails to be S_2-metacompact. □

Corollary 2.10. *(i) $\beta\mathbb{N}$ fails to be S_2-metacompact.*

(ii) Let σ denote the topology of $\kappa\mathbb{N}$, the Katetov extension of $\mathbb{N}$, and let τ be the topology of $\beta\mathbb{N}$. It is well-known that $\sigma = \tau^\alpha$. Clearly $\kappa\mathbb{N}$ fails to be S_2-metacompact. Though $\beta\mathbb{N}$ is known to be extremally disconnected, by Remark 2.4, $\kappa\mathbb{N}$ even fails to be S_3-metacompact.

Theorem 2.11. *Let (X, τ) be an infinite space possessing a closed nowhere dense subset N and a dense subset B such that $|B| < |N| \geqslant \aleph_0$. Then (X, τ) fails to be S_2-metacompact.*

Proof. For each $x \in N$ let $S_x = (X \setminus N) \cup \{x\}$. Then $\mathcal{U} = \{S_x: x \in N\}$ is a semi-open cover of X. Suppose that $\mathcal{V}$ is a semi-open refinement of $\mathcal{U}$. For each $x \in N$ there exists $T_x \in \mathcal{V}$ such that $x \in T_x$. Clearly $T_x \subseteq S_x$ for $x \in N$, and $T_x \neq T_y$ for distinct points $x, y \in N$. Now, $\{\operatorname{int} T_x: x \in N\}$ is a family of $|N|$ nonempty open subsets of (X, τ). Since B is dense and $|B| < |N| \geqslant \aleph_0$, there must be a point $z \in B$ which is contained in infinitely many sets T_x. Thus $\mathcal{V}$ cannot be point-finite and (X, τ) fails to be S_2-metacompact. □

Corollary 2.12. *The usual space $\mathbb{R}$ fails to be S_2-metacompact (let N be the Cantor set, B the set of rationals and apply Theorem 2.11).*

In the following we exhibit a useful method for generating new topological spaces from the given spaces. Let (Y, σ) be a topological space, Z be an infinite set disjoint from Y and let $X = Y \cup Z$. We define a topology τ on X in the following way: every set $S \subseteq Y$ with $S \in \sigma$ is open in (X,τ) and a basic neighbourhood of $z \in Z$ is of the form $\{z\} \cup Y$. Clearly, (Y,σ) is a dense and open subspace of (X,τ). It is easily checked that for $S \neq \emptyset$ we have $S \in \mathrm{SO}(X,\tau)$ if and only if $S \cap Y$ is a nonempty semi-open subset of (Y,σ).

Proposition 2.13. *Let (Y,σ) be a topological space, Z be an infinite set and let (X,τ) be as defined above. Then*

(a) *(X,τ) fails to be metacompact,*

(b) *(X,τ) is extremally disconnected if and only if (Y,σ) is hyperconnected (i.e., every nonempty open set in (Y,σ) is dense in (Y,σ)),*

(c) *if (Y,σ) has a nonempty finite open set, then (X,τ) is not S_2-metacompact,*

(d) *if Z is countable and (Y,σ) is countably infinite, T_1 and dense-in-itself then (X,τ) is S_2-metacompact,*

(e) *let Z be countable, then (X,τ) is S_3-metacompact if and only if (Y,σ) has a countably infinite point-finite family of nonempty semi-open sets.*

Proof. (a) First observe that the open cover $\mathcal{U} = \{Y \cup \{z\} \colon z \in Z\}$ cannot have a point-finite open refinement. Thus (X,τ) fails to be metacompact.

(b) Suppose that (X,τ) is extremally disconnected and let $\emptyset \neq U \subseteq Y$ be open in (Y,σ). Then $U \in \tau$ and $Z \subseteq \mathrm{cl}\,U$. By assumption, $\mathrm{cl}\,U$ is open in (X,τ) and thus $\mathrm{cl}\,U = X$. Hence U is dense in (Y,σ), i.e., (Y,σ) is hyperconnected. To prove the converse, suppose that (Y,σ) is hyperconnected and let $U \neq \emptyset$ be open in (X,τ). Then $V = U \cap Y$ is nonempty and open in (Y,σ). Since $\mathrm{cl}\,V = \mathrm{cl}_Y V \cup Z$, we have, by assumption, that $\mathrm{cl}\,V = X$ and thus $\mathrm{cl}\,U = X$, i.e., (X,τ) is extremally disconnected. Observe that we have also shown that (X,τ) is hyperconnected if and only if (Y,σ) is hyperconnected.

(c) Now let $U \neq \emptyset$ be a finite open subset of (Y,σ). We may assume that U is minimal open, i.e., U does not properly contain a nonempty open set. Let $\mathcal{V}$ be a semi-open refinement of the semi-open cover $\mathcal{U} = \{Y\} \cup \big\{U \cup \{z\} \colon z \in Z\big\}$. For each $z \in Z$ there is $T_z \in \mathcal{V}$ such that $z \in T_z$, and $T_z \neq T_{z'}$ whenever $z \neq z'$. Since U is minimal open we have $T_z = U \cup \{z\}$. This shows that $\mathcal{V}$ cannot be point-finite, i.e., (c) holds.

(d) Let $Z = \{z_n \colon n \in \mathbb{N}\}$, $Y = \{y_n \colon n \in \mathbb{N}\}$ and suppose that (Y,σ) is T_1 and dense-in-itself. Let $C_1 = Y$ and $C_n = Y \setminus \{y_1, y_2, \ldots, y_{n-1}\}$ for each $n > 1$. By assumption, each C_n is open and dense in (Y,σ). Suppose that $\mathcal{U}$ is a semi-open cover of (X,τ). For each $n \in \mathbb{N}$ we pick $U_n, V_n \in \mathcal{U}$ such that $y_n \in U_n$ and $z_n \in V_n$. Let $S_n = U_n \cap C_n$ and $T_n = \{z_n\} \cup (V_n \cap C_n)$ for each $n \in \mathbb{N}$. Observe that S_n and $V_n \cap C_n$ are always nonempty semi-open subsets of (Y,σ).

Clearly we have $y_n \in S_n \subseteq U_n$ and $z_n \in T_n \subseteq V_n$ for each $n \in \mathbb{N}$, hence $\mathcal{V} = \{S_n : n \in \mathbb{N}\} \cup \{T_n : n \in \mathbb{N}\}$ is a semi-open refinement of $\mathcal{U}$. Moreover, it is easily checked that $\mathcal{V}$ is point-finite, thus (d) holds.

(e) Let $Z = \{z_n : n \in \mathbb{N}\}$. First suppose that (X, τ) is S_3-metacompact. Then $\mathcal{U} = \big\{\{z_n\} \cup Y : n \in \mathbb{N}\big\}$ is an open cover of (X, τ). By assumption, there exists a point-finite semi-open refinement $\mathcal{V}$ of $\mathcal{U}$. For each $n \in \mathbb{N}$ pick $V_n \in \mathcal{V}$ such that $z_n \in V_n$. Then $\{V_n \cap Y : n \in \mathbb{N}\}$ is a countably infinite point-finite family of nonempty semi-open subsets of (Y, σ). To prove the converse, let $\{S_n : n \in \mathbb{N}\}$ be a point-finite family of nonempty semi-open subsets of (Y, σ). Let $\mathcal{U}$ be an open cover of (X, τ). For each $n \in \mathbb{N}$ there exists $U_n \in \mathcal{U}$ such that $z_n \in \{z_n\} \cup Y \subseteq U_n$. If $T_n = \{z_n\} \cup S_n$ for each $n \in \mathbb{N}$, then $\mathcal{V} = \{T_n : n \in \mathbb{N}\} \cup \{Y\}$ is a point-finite semi-open refinement of $\mathcal{U}$. Hence (X, τ) is S_3-metacompact. $\square$

As an application of Proposition 2.13 we have the following result which concludes our discussion of S_i-metacompact spaces.

THEOREM 2.14. (*1*) *There is an S_3-metacompact space that is neither S_2-metacompact nor metacompact.*

(*2*) *There is an S_2-metacompact space that is not metacompact.*

Proof. (1) Let Z be a countably infinite set and let (Y, σ) be an infinite discrete space such that $Y \cap Z = \emptyset$. By applying Proposition 2.13 (e), (c) and (a), the resulting space (X, τ) is S_3-metacompact but neither S_2-metacompact nor metacompact.

(2) Let Y and Z be two disjoint countably infinite sets and let σ be the cofinite topology on Y. By applying Proposition 2.13 (d) and (a), the resulting space (X, τ) is S_2-metacompact but not metacompact. In addition, (Y, σ) is hyperconnected so (X, τ) is extremally disconnected by Proposition 2.13 (b). Since (X, τ^α) is also S_2-metacompact, (X, τ^α) is metacompact by Remark 2.4. $\square$

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