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WHEN THE DERIVATIVES OF SOLUTIONS OF THE CAUCHY'S PROBLEM ARE (S)-CONTINUOUS?

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ABSTRACT. Some conditions implying that the derivatives of solutions of the Cauchy's problem $y'(x) = f\big(x, y(x)\big)$, with an initial condition $y(x_0) = y_0$, are (S)-continuous or (S)-path continuous are presented.

Let $\mathbb{R}$ be the set of all reals, let Y be a separable Banach space and let $\mathbb{R}\times Y$ be the product space.

In [2] the following theorem has been proved.

Theorem 1. *Let $D \subset \mathbb{R}\times Y$ be a nonempty open set and let $f\colon D \to Y$ be a locally bounded function such that*

(a) *for all points $x \in \mathbb{R}$ and $y \in Y$ with $(x,y) \in D$ and for each positive real η there is a real $\delta > 0$ for which*

$$I\big((x,y),\delta\big) = (x-\delta, x+\delta)\times K(y,\delta) \subset D,$$

where $K(y,\delta) = \big\{u \in Y\colon ||u-y|| < \delta\big\}$, and

$$\big\|f(t,u) - f(t,y)\big\| < \eta \qquad \text{for } (t,u) \in I\big((x,y),\delta\big)\,.$$

If the sections $f^u(t) = f(t,u)$, where $t \in \mathbb{R}$, $u \in Y$ and $(t,u) \in D$, are derivatives then for each continuous function $g\colon I \to Y$ with a nonempty open domain $I \subset \mathbb{R}$ and the graph $\mathrm{Gr}(g) \subset D$, the Carathéodory superposition

$$h(x) = f\big(x, g(x)\big)\,, \qquad x \in I\,,$$

is a derivative, i.e., for each $x \in I$ the equality

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} h(t)\,dt = h(x)$$

is true (the integral is in the sense of Bochner).

From this (compare [2, Remark 4]) it follows

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Corollary 1. *Let $D \subset \mathbb{R} \times Y$ be a nonempty open set and let $f\colon D \to Y$ be a locally bounded function satisfying condition* (a) *from Theorem 1. If the sections $f^u(t) = f(t,u)$, where $t \in \mathbb{R}$, $u \in Y$ and $(t,u) \in D$, are derivatives, and an absolutely continuous function $g\colon I \to Y$ defined on open interval I satisfies the Cauchy's problem*

$$y'(t) = f\big(t, y(t)\big), \quad y(x) = y \tag{$*$}$$

almost everywhere on I, then g satisfies $(*)$ everywhere on I. So, every Carathéodory's solution of $(*)$ is a solution (in the ordinary sense) of this equation.

From this, by Theorem 1 from [1], p. 7 (compare [2, Theorem 5] and [3, Theorem 6]), it follows

Theorem 2. *Let $D \subset \mathbb{R} \times \mathbb{R}^m$ be a nonempty open set and let $f\colon D \to \mathbb{R}^m$ be a locally bounded function satisfying condition* (a) *from Theorem 1. If the sections $f^u(t) = f(t,u)$, where $t \in \mathbb{R}$, $u \in \mathbb{R}^m$ and $(t,u) \in D$, are derivatives, then for each point $(x,y) \in D$ there is a differentiable solution $y\colon I \to \mathbb{R}^m$ of the Cauchy's problem*

$$y'(t) = f\big(t, y(t)\big), \quad y(x) = y\,. \tag{$*$}$$

Another conclusion from Corollary 1 and Pianigiani's theorem from [5] on the existence of a solution of the Cauchy's problem $(*)$ is shown in Theorem 8 of [2] or in Theorem 8 of [3].

In this article I introduce some additional conditions concerning the sections f^u which imply that derivatives of solutions of $(*)$ are (S)-continuous with respect to a local system S.

We start with the following definitions [6, p. 3 and p. 70].

Definition 1. By a *local system* we mean a family S of subsets of $\mathbb{R}$ such that at each point $x \in \mathbb{R}$ there is given a nonempty collection of sets $S(x) \subset S$ with the following properties:

(i) $\{x\} \notin S(x)$,

(ii) if $A \in S(x)$ then $x \in A$,

(iii) if $A_1 \in S(x)$ and $A_2 \supset A_1$ then $A_2 \in S(x)$,

(iv) if $A \in S(x)$ and $\delta > 0$ then $A \cap (x - \delta, x + \delta) \in S(x)$.

Definition 2. Let S be a local system, let $I \subset \mathbb{R}$ be a nonempty open set and let $\phi\colon I \to Y$ be a function. We say that ϕ is (S)-*continuous* at the point x provided that for every $\eta > 0$

$$\big\{t \in I\colon \|\phi(t) - \phi(x)\| < \eta\big\} \in S(x)\,.$$

Definition 3. Let S be a local system in $\mathbb{R}$ and let $I \subset \mathbb{R}$ be a nonempty open set. A function $\phi\colon I \to Y$ is (S)-*path continuous* at the point x if there is a set $A \in S(x)$ such that

$$\lim_{A \ni t \to x} \phi(t) = \phi(x).$$

Observe that (compare [6, p. 27]) the (S)-path continuity of ϕ at x implies the (S)-continuity of ϕ. The converse, however, need not happen (see [6, p. 27 and p. 29, Th. (14.3)]).

Theorem 3. *Let $D \subset \mathbb{R} \times Y$ be a nonempty open set and let $f\colon D \to Y$ be a locally bounded function satisfying condition* (a) *from Theorem 1. If S is a local system in $\mathbb{R}$ and if the sections $f^u(t) = f(t,u)$, where $t \in \mathbb{R}$, $u \in Y$ and $(t,u) \in D$, are (S)-continuous (resp. (S)-path continuous), then for each continuous function $g\colon I \to Y$ with a nonempty open domain $I \subset \mathbb{R}$ and the graph $\mathrm{Gr}(g) \subset D$, the Carathéodory superposition*

$$h(x) = f\big(x, g(x)\big), \qquad x \in I,$$

is (S)-continuous (resp. (S)-path continuous).

Proof. Fix a point $x \in I$ and a positive real η. Since f satisfies the condition (a), there is a real $\delta > 0$ such that $I\big(\big(x, g(x)\big), \delta\big) \subset D$ and

$$\big\|f(t,u) - f\big(t, g(x)\big)\big\| < \frac{\eta}{3} \qquad \text{for } (t,u) \in I\big(\big(x, g(x)\big), \delta\big).$$

By the continuity of the function g at x there is a real $r \in (0, \delta)$ such that $g\big((x-r,\, x+r)\big) \subset K\big(g(x), \delta\big)$. Since the section $f^{g(x)}$ is (S)-continuous, the set $A = \big\{t \in I\colon \big\|f\big(t, g(x)\big) - f\big(x, g(x)\big)\big\| < \frac{\eta}{3}\big\} \in S(x)$. So, the set $B = A \cap (x-r,\, x+r) \in S(x)$ and for $t \in B$ we have

$$\begin{aligned}\big\|h(t) - h(x)\big\| &= \big\|f\big(t, g(t)\big) - f\big(x, g(x)\big)\big\| \\ &\leqslant \big\|f\big(t, g(t)\big) - f\big(t, g(x)\big)\big\| + \big\|f\big(t, g(x)\big) - f\big(x, g(x)\big)\big\| \leqslant \frac{\eta}{3} + \frac{\eta}{3} < \eta.\end{aligned}$$

Consequently h is (S)-continuous at x and in the case of (S)-continuity of h the proof is completed.

For the proof of the alternative case observe that the section $f^{g(x)}$ is (S)-path continuous and there is a set $Z \in S(x)$ such that

$$\lim_{Z \ni t \to x} f\big(t, g(x)\big) = f\big(x, g(x)\big).$$

Let $r_1 \in (0, r)$ be such that

$$\big\|f\big(t, g(x)\big) - f\big(x, g(x)\big\| < \frac{\eta}{3} \qquad \text{for } t \in A = Z \cap (x - r_1,\, x + r_1).$$

For $t \in A$ we have

$$\begin{aligned}\|h(t)-h(x)\| &= \|f(t,g(t)) - f(x,g(x))\| \\ &\leqslant \|f(t,g(t)) - f(t,g(x))\| + \|f(t,g(x)) - f(x,g(x))\| \leqslant \frac{\eta}{3} + \frac{\eta}{3} < \eta .\end{aligned}$$

Consequently

$$\lim_{Z \ni t \to x} h(t) = h(x) ,$$

h is (S)-path continuous at x and the proof is completed. □

From Corollary 1 and Theorem 3 it follows

Theorem 4. *Let $D \subset \mathbb{R} \times Y$ be a nonempty open set and let $f\colon D \to Y$ be a locally bounded function satisfying condition* (a) *from Theorem 1. If S is a local system in $\mathbb{R}$ and if the sections $f^u(t) = f(t,u)$, where $t \in \mathbb{R}$, $u \in Y$ and $(t,u) \in D$, are (S)-continuous derivatives (resp. (S)-path continuous derivatives) then each Carathéodory's solution $g\colon I \to Y$ of the Cauchy's problem*

$$y'(t) = f(t, y(t)), \quad y(x) = y , \tag{$*$}$$

is a differentiable solution of this problem, whose derivative g' is (S)-continuous (resp. (S)-path continuous).

P r o o f. If an absolutely continuous function $g\colon I \to Y$ satisfies equation $(*)$ almost everywhere, then by Corollary 1 it is a differentiable solution of $(*)$ in the ordinary sense. From Theorem 3 it follows that the derivative $g'(t) = f(t, g(t))$ is (S)-continuous (resp. (S)-path continuous). □

In particular from Theorems 2 and 4 it follows:

Theorem 5. *Let $D \subset \mathbb{R} \times \mathbb{R}^m$ be a nonempty open set and let $f\colon D \to \mathbb{R}^m$ be a locally bounded function satisfying condition* (a) *from Theorem 1. If S is a local system in $\mathbb{R}$ and if the sections $f^u(t) = f(t,u)$, where $t \in \mathbb{R}$, $u \in \mathbb{R}^m$ and $(t,u) \in D$, are (S)-continuous derivatives (resp. (S)-path continuous derivatives) then for each point $(x,y) \in D$ there is a differentiable solution $g\colon I \to \mathbb{R}^m$ of the Cauchy's problem*

$$y'(t) = f(t, y(t)), \quad y(x) = y , \tag{$*$}$$

whose derivative g' is (S)-continuous (resp. (S)-path continuous).

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