Tatra Mt. Math. Publ. **34** (2006), 71–76

Tatra Mountains
Mathematical Publications

MINIMAL MULTIFUNCTIONS AND THE CLUSTER SETS

Milan Matejdes

ABSTRACT. The paper deals with the minimal multifunctions which can be considered as the "best" selections closely related to continuous as well as quasi-continuous selections. We will give a characterization of minimal multifunction with closed graph by a cluster multifunction with respect to open sets. The conditions under which minimal multifunction is single valued except for a set of first category are given.

1. Introduction

In the sequel $F\colon X \to Y$ means a non-empty valued mapping from X to Y^2, briefly a multifunction. Let $F\colon X \to Y$ be a compact valued multifunction. If the graph $\mathrm{Gr}(F) = \{[x,y] \in X \times Y : y \in F(x)\}$ of F is closed, then by Zorn's lemma there is a non-empty compact valued multifunction F_0 with closed graph such that $F_0(x) \subset F(x)$ for any $x \in X$ (briefly $F_0 \subset F$) and F_0 is minimal. That means if G is a non-empty compact valued multifunction with closed graph and $G \subset F_0$, then $G = F_0$. Since F need not have a continuous selection, a minimal multifunction $F_0 \subset F$ can be considered as the "best" submultifunction of F which is "closed" to a continuous selection of F. If F is a compact valued and upper semi continuous, then there is a minimal multifunction $F_0 \subset F$ such that for any open sets $U \subset X$, $V \subset Y$ and any $x_0 \in U$ such that $F_0(x_0) \cap V \neq \emptyset$ there is a non-empty open set $H \subset U$ such that $F_0(x) \subset V$ for any $x \in H$ [2, Lemma 2.5]. That means F_0 has a continuity property which seems to be derived from the lower and upper quasi continuity. Intuitively for any open set $V \subset Y$ intersecting the range of F_0 there is a non-empty open set $H \subset X$ on which the values of F are "small", namely $F(x) \subset V$ for any $x \in H$. We will study the properties of minimal multifunction defined in [6] and it will be characterized by a cluster multifunction. The next definitions of

2000 Mathematics Subject Classification: 54C60.
Keywords: multifunction, generalized continuity, cluster set, minimal multifunction.

u.s.c., l.s.c., l.q.c. and u.q.c. as well as further survey information concerning continuity properties of multifunctions can be found in [7].

Definition 1. A multifunction $F\colon X \to Y$ is *upper* (*lower*) *semicontinuous* (briefly *u.s.c.* (*l.s.c.*)) *at* x_0 if for any open set G such that $F(x_0) \subset G$ $\big(F(x_0) \cap G \neq \emptyset\big)$ there is an open set $U \ni x_0$ such that $F(x) \subset G$ $\big(F(x) \cap G \neq \emptyset\big)$ for any $x \in U$. A multifunction F is *continuous at* x_0 if F is u.s.c. and l.s.c. at x_0. F is *upper* (*lower*) *semicontinuous* if it is such at any point. The set of all points in which F is not u.s.c. (l.s.c.) is denoted by $D_u(F)$ $\big(D_l(F)\big)$. A multifunction F is *upper* (*lower*) *quasicontinuous* at x_0 if for any open sets $V, U \ni x_0$ such that $F(x_0) \subset V\big(F(x_0) \cap V \neq \emptyset\big)$ there is a non-empty open set $H \subset U$ such that $F(x) \subset V\big(F(x) \cap V \neq \emptyset\big)$ for any $x \in H$. F is *upper* (*lower*) *quasicontinuous* (u.q.c. (l.q.c.)) if it is such at any point. Finally, F is *quasicontinuous* at x_0 if for any open sets $V, W, U \ni x_0$ such that $F(x_0) \subset W$ and $F(x_0) \cap V \neq \emptyset$ there is a non-empty open set $H \subset U$ such that $F(x) \subset W$ and $F(x) \cap V \neq \emptyset$ for any $x \in H$. A function $f\colon X \to Y$ is a *selection* of F, if $f(x) \in F(x)$ for any $x \in X$.

Using a combination of lower and upper quasicontinuity we define a minimality (see [6]).

Definition 2. A multifunction $F\colon X \to Y$ (with arbitrary non-empty values) is *minimal at* x_0 if for any open sets $V, U \ni x_0$ such that $F(x_0) \cap V \neq \emptyset$ there is a non-empty open set $H \subset U$ such that $F(x) \subset V$ for any $x \in H$. F is *minimal* if it is such at any point. It is clear that the minimality of a function coincides with the quasicontinuity.

2. Minimality and quasicontinuity

Lemma 1. *Let F be minimal. Then any non-empty valued multifunction $F_0 \subset F$ is quasicontinuous and minimal.*

Proof. Let $x_0 \in X$ and $V, W, U \ni x_0$ be open such that $F_0(x_0) \subset W$ and $F_0(x_0) \cap V \neq \emptyset$ (for minimality only $F_0(x_0) \cap V \neq \emptyset$). From the minimality of F there is a non-empty open set $H \subset U$ such that $F(x) \subset W \cap V$ (for minimality only $F(x) \subset V$) for any $x \in H$. Since $F_0 \subset F$, $F_0(x) \subset W$ and $F_0(x) \cap V \neq \emptyset$ (for minimality $F_0(x) \subset V$) for any $x \in H$. That means F_0 is quasicontinuous at x_0 (minimal at x_0). □

Corollary 1. *Any minimal multifunction F is quasicontinuous and any selection of F is quasicontinuous, as well.*

The minimality of F can be characterized by the quasicontinuity of any selection of F.

Theorem 1. *A multifunction F is minimal if and only if any selection of F is quasicontinuous.*

P r o o f . Implication $\Rightarrow$ follows from Corollary 1. Now, let any selection of F be quasicontinuous. Suppose, F is not minimal at x_0. Then there are open sets U, V and a set $S \subset U$ such that $x_0 \in U$, $F(x_0) \cap V \neq \emptyset$, $F(x) \cap (Y \setminus V) \neq \emptyset$ for any $x \in S$ and S is dense in U. Define a selection f of F by $f(x_0) \in F(x_0) \cap V$, $f(x) \in F(x) \cap (Y \setminus V)$ for $x \in S$ and $f(x) \in F(x)$ elsewhere. It is clear that f is not quasicontinuous at x_0. $\square$

3. Minimality, singleton and continuity points

Now we will give some conditions under which a minimal multifunction is single valued and continuous ([6]).

Theorem 2. *Let Y be Hausdorff. If f is a selection of minimal multifunction F and f is continuous at x_0, then $F(x_0) = \{f(x_0)\}$. Moreover, if Y is regular, then F is continuous at x_0.*

P r o o f . Let $y_0 \in F(x_0)$, $y_0 \neq f(x_0)$. Y is Hausdorff, then there are disjoint open sets V, W such that $y_0 \in V$, $f(x_0) \in W$. Since f is continuous at x_0, there is an open set $U \ni x_0$ such that $f(U) \subset W$. From the minimality of F there is a non-empty open set $H \subset U$ such that $F(x) \subset V$ for any $x \in H$. Since f is a selection of F, $f(x) \in V$ for any $x \in H$, a contradiction with $f(U) \subset W$.

Now we will prove that F is u.s.c. at x_0. If not, there is an open set $V \supset F(x_0) = \{f(x_0)\}$ such that for any open set $U \ni x_0$ there is $x \in U$ such that $F(x) \not\subseteq V$. As f is continuous at x_0 so there is an open $G \ni x_0$ such that $f(G) \subset V_0 \subset \overline{V_0} \subset V$ where V_0 is open, $f(x_0) \in V_0$ (such V_0 exists since Y is regular). Then there is $x \in G$ such that $F(x) \not\subseteq V$. Let $y \in F(x)$ and $y \notin \overline{V_0}$. F is minimal at x, hence there is a non-empty open set $H \subset G$ such that $F(t) \subset Y \setminus \overline{V_0}$ for any $t \in H$. Since f is a selection of F, $f(t) \in Y \setminus \overline{V_0}$, contradiction with $f(t) \in \overline{V_0}$ for $t \in H \subset G$. Since F is singleton and u.s.c. at x_0, F is l.s.c. at x_0, as well. $\square$

It is known (see [8]) that if $f \colon X \to Y$ is a quasicontinuous function and Y is second countable, then the set of discontinuity points of f is of first category. Using the above theorem we have

Corollary 2. *Let Y be second countable and F be minimal.*

a) If Y is Hausdorff, then F is single valued except for a set of first category.

b) If Y is regular, then F is single valued and continuous except for a set of first category.

Proof. Let f be a selection of F. By Corollary 1, f is quasicontinuous and by [8] f is continuous except for a set of first category. Using Theorem 2, the proof is finished. □

COROLLARY 3. *Let Y be Hausdorff and second countable. If F is minimal with compact values, then F is single valued and continuous except for a set of first category.*

Proof. From Corollary 2 a) we have that F is single valued except for a set A of first category. By [7, Th. 3.1.7] F is u.s.c. except for a set B of first category. Then F is single valued and continuous for any $x \notin A \cup B$. □

COROLLARY 4. *Let $F\colon X \to Y$ be minimal and Y be arbitrary metric. Then F is single valued and continuous except for a set of first category.*

Proof. Taking any selection f of F, f is quasicontinuous by Corollary 1. Since a quasicontinuous function from X to a metric space is continuous except for a set of first category (see [1]), it is sufficient to use Theorem 2. □

4. Cluster sets and minimality

Now, we will deal with cluster sets of a function with respect to non-empty open sets. First it was introduced (even for multifunction) in [3], [4]. Later they were investigated in [5], [9], [10].

DEFINITION 3. A point $y \in Y$ is *$\mathcal{O}$-cluster point* of a function $f\colon X \to Y$ at a point x_0 if for any open sets U, V with $x_0 \in U$, $y \in V$, there is a non-empty open set $H \subset U$ such that $f(H) \subset V$. The set of all $\mathcal{O}$-cluster points of f at x will be denoted by $\mathcal{O}_f(x)$ (*$\mathcal{O}$-cluster set* of f at x). If $\mathcal{O}_f(x)$ is non-empty for any x, then the multifunction $\mathcal{O}_f\colon x \mapsto \mathcal{O}_f(x)$ is called *$\mathcal{O}$-cluster multifunction* of f.

Remark 1. It is clear that f is quasicontinuous at x if and only if $f(x) \in \mathcal{O}_f(x)$ ([9]).

THEOREM 3. *Let Y be regular and $\mathcal{O}_f(x) \neq \emptyset$ for any $x \in X$. Then the multifunction $\mathcal{O}_f\colon x \mapsto \mathcal{O}_f(x)$ is minimal with closed graph.*

Proof. Let $x_0 \in X$ and U, V be open such that $x_0 \in U$, $V \cap \mathcal{O}_f(x_0) \neq \emptyset$. Let $y_0 \in V \cap \mathcal{O}_f(x_0)$. Since Y is regular, there is an open set V_0 such that $y_0 \in V_0 \subset \overline{V_0} \subset V$. Observe that the point y_0 is $\mathcal{O}$-cluster point then there is a non-empty set $H \subset U$ such that $f(H) \subset V_0$. We will show that $\mathcal{O}_f(t) \subset \overline{V_0}$ for any $t \in H$. Suppose $\mathcal{O}_f(t_0) \not\subset \overline{V_0}$, $t_0 \in H$. Let $z \in \mathcal{O}_f(t_0) \cap (Y \setminus \overline{V_0})$. Then there is a non-empty open set $G \subset H \subset U$ such that $f(G) \subset Y \setminus \overline{V_0}$, a contradiction with $f(H) \subset V_0$. That means $\mathcal{O}_f(t) \subset \overline{V_0} \subset V$ for any $t \in H \subset U$, hence $\mathcal{O}_f$ is minimal at x_0.

Now, we will show that $\mathrm{Gr}(\mathcal{O}_f)$ is closed. Let $[x_0, y_0] \in \overline{\mathrm{Gr}(\mathcal{O}_f)}$ and $x_0 \in U$, $y_0 \in V$ where U and V are open. Let $[a, b] \in (U \times V) \cap \mathrm{Gr}(\mathcal{O}_f)$. That means $b \in \mathcal{O}_f(a)$, $a \in U$, $b \in V$. From the definition of $\mathcal{O}$-cluster point there is a non-empty open set $H \subset U$ such that $f(H) \subset V$. Hence y_0 is $\mathcal{O}$-cluster point of f at x_0, or $[x_0, y_0] \in \mathrm{Gr}(\mathcal{O}_f)$. □

COROLLARY 5. *$\mathcal{O}_f(x)$ is closed for any $x \in X$.*

THEOREM 4. *Let Y be regular. A function f is quasicontinuous if and only if $\mathcal{O}_f$ is maximal minimal multifunction containing f (that means f is a selection of $\mathcal{O}_f$ and if f is a selection of a minimal multifunction F, then $F \subset \mathcal{O}_f$).*

P r o o f. Let f be quasicontinuous. From Remark 1 it follows that f is a selection of $\mathcal{O}_f$ and the minimality of $\mathcal{O}_f$ follows from Theorem 3. Let f is a selection of a minimal multifunction F. Suppose $F(x_0) \not\subseteq \mathcal{O}_f(x_0)$. Let $y \in F(x_0) \setminus \mathcal{O}_f(x_0)$. Then there are open sets $U \ni x_0$ and $V \ni y$ such that $f(x) \notin V$ for any $x \in S \subset U$ where S is dense in U. Since F is minimal there is a non-empty open set $H \subset U$ such that $F(x) \subset V$ for any $x \in H$. Hence $f(x) \in V$ for any $x \in H \cap S$, a contradiction with $f(x) \notin V$ for any $x \in S$.

Implication $\Leftarrow$ follows from Remark 1. □

The last theorem deals with the representation of a minimal multifunction with closed graph by $\mathcal{O}$-cluster multifunction of quasicontinuous function. The similar representation of a multifunction by cluster multifunction with respect to non-empty set is solved in [10].

THEOREM 5. *Let Y be regular. Then F is minimal with closed graph if and only if for any selection f of F the equality $F = \mathcal{O}_f$ holds.*

P r o o f. Suppose F is minimal with closed graph. Let f be a selection of F. Then f is quasicontinuous, by Corollary 1, and $f(x) \in \mathcal{O}_f(x)$ for any $x \in X$ by Remark 1. From Theorem 4 we have $F \subset \mathcal{O}_f$. Suppose there is a point $y_0 \in \mathcal{O}_f(x_0) \setminus F(x_0)$. Let V, U be arbitrary open, $y_0 \in V$, $x_0 \in U$. Then there is a non-empty open set $H \subset U$ such that $f(H) \subset V$. Since $f(x) \in F(x)$, $[x_0, y_0] \in \overline{\mathrm{Gr}(F)} = \mathrm{Gr}(F)$, a contradiction with $y_0 \notin F(x_0)$.

Implication $\Leftarrow$ follows from Theorem 3. □

REFERENCES

[1] BLEDSOE, W. W.: *Neighbourly functions*, Proc. Amer. Math. Soc. **3** (1972), 114–115.

[2] GILES, J. R.—MOORS, W. B.: *A continuity property related to Kuratovski's index of non-compactness, its relevance to the drop property and its implications for differentiability theory*, J. Math. Anal. Appl. **178** (1993), 247–268.

[3] MATEJDES, M.: *Sur les seléctors des multifonctions*, Math. Slovaca **37** (1987), 111–124.

[4] MATEJDES, M.: *On selections of multifunctions*, Math. Bohemica **118** (1993), 255–260.

[5] MATEJDES, M.: *Continuity of multifunctions*, Real Anal. Exchange **19** (1993–1994), 394–413.

[6] MOORS, W. B.—GILES, J. R.: *Generic continuity of minimal set-valued mappings*, J. Austral. Math. Soc. Ser. A **63** (1997), 238–262.

[7] NEUBRUNN, T.: *Quasi-continuity*, Real Anal. Exchange **14** (1988–89), 259–306.

[8] LEVINE, N.: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[9] RICHTER, CH.—STEPHANI, I.: *Cluster sets and approximation properties of quasi-continuous and cliquish functions*, Real Anal. Exchange **29** (2003–2004), 299–322.

[10] RICHTER, CH.: *Iterated reduced cluster functions*, Real Anal. Exchange **30** (2004–2005), 43–58.

Received October 29, 2004

Technical University
Masarykova 24
SK–960 53 Zvolen
SLOVAKIA

E-mail: matejdes@vsld.tuzvo.sk