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THE MEASURABILITY OF THE PRODUCT OF ARITHMETIC PROGRESSIONS

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ABSTRACT. This paper deals with arithmetic progressions and measurability in the sense of Buck.

1. Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers. Put for $a \in \mathbb{N}$ and $A \subseteq \mathbb{N}$ $a \cdot A = \{a \cdot n \colon n \in A\}$, $a + A = \{a + n \colon n \in A\}$. Then for $a, m \in \mathbb{N}$, $m \geq 1$, the set $a + m\mathbb{N}$ is the arithmetic progression with the difference m and the first element a.

We recall the concept of an asymptotic density. If $A \subseteq \mathbb{N}$ and the limit

$$d(A) = \lim_{n \to \infty} \frac{1}{n} \operatorname{card}\bigl(A \cap \{0, \dots, n\}\bigr)$$

exists then $d(A)$ is called the *asymptotic density* of A. It is well-known that $d(a + m\mathbb{N}) = \frac{1}{m}$, $m \geq 1$ (cf. [T]).

In the paper [ST] the asymptotic density of the product $(a_1 + m_1\mathbb{N})(a_2 + m_2\mathbb{N})$, $(a_1, m_1) = 1$, $(a_2, m_2) = 1$, was studied, where for two $A_1, A_2 \subseteq \mathbb{N}$ we put $A_1 \cdot A_2 = \{n_1 \cdot n_2 \colon n_1 \in A_1 \wedge n_2 \in A_2\}$.

For a survey of the asymptotic density we refer to the monograph [T].

In 1946 in the paper [BUC] another type of density was introduced. Let $\mathscr{D}_0$ denote the system of all finite unions of arithmetic progressions. Then each set from $\mathscr{D}_0$ has the asymptotic density. Let $A \subseteq \mathbb{N}$, put

$$\mu^*(A) = \inf\bigl\{d(S) \colon a \subseteq S \wedge S \in \mathscr{D}_0\bigr\}.$$

Then the value $\mu^*(A)$ is called the *measure density* of A. Now put

$$\mathscr{D}_\mu = \bigl\{A \subseteq \mathbb{N} \colon \mu^*(A) + \mu^*(\mathbb{N} \setminus A) = 1\bigr\}.$$

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Then we have

i) $\mathscr{D}_\mu$ is an algebra of sets.

ii) The restriction $\mu = \mu^*\big|_{\mathscr{D}_\mu}$ is a finitely additive probability measure on $\mathscr{D}_\mu$.

iii) If $A \subseteq \mathbb{N}$ then $\mu^*(A) = 0 \Rightarrow A \in \mathscr{D}_\mu$.

(See [BUC]).

The algebra $\mathscr{D}_\mu$ is called an *algebra of measurable sets* and each set $A \in \mathscr{D}_\mu$ is called an *measurable set*. It is well-known that every measurable set A has the asymptotic density and $d(A) = \mu(A)$.

Remark that there exists a lot of examples of the sets having the asymptotic density which are not measurable. The aim of this paper is to improve the results from [ST] in the sense of measurability.

Measurability of the product of arithmetic progressions

Inspired by the proof of [ST] we give the first lemma:

Lemma 1. *Let $A \in \mathscr{D}_\mu$ and $\mu(A) = 1$. Let $B_1 \in \mathscr{D}_\mu$ and $B_2 \subseteq \mathbb{N}$. If $B_1 \cap A = B_2 \cap A$, then $B_2 \in \mathscr{D}_\mu$ and $\mu(B_2) = \mu(B_1)$.*

This lemma follows immediately from ii) and iii).

Now we shall need the following statement proved in [PAS]: Let $A \in \mathbb{N}$ and p be a prime. Put $A_p = \{a \in A \colon p \mid a \wedge p^2 \nmid a\}$. Then it holds

Lemma A. (Corollary 2 of [PAS]) Let $\{p_i\}$ be such a sequence of primes that $\sum p_i^{-1} = \infty$. Then for every $A \subseteq \mathbb{N}$ we have

$$\left(\forall_i \colon \mu^*(A_{p_i}) = 0\right) \implies A \in \mathscr{D}_\mu \wedge \mu(A) = 0 .$$

Lemma 2. *Let $B \in \mathscr{D}_\mu$ and $\mu(B) = 1$. Let $\{p_i\}$ be such a sequence of primes that $\sum\limits_{i=1}^{\infty} p_i^{-1} = \infty$. Put $A = \bigcup\limits_{i=1}^{\infty} p_i B$. Then $A \in \mathscr{D}_\mu$ and $\mu(A) = 1$.*

Proof. Consider $(\mathbb{N} \setminus A)_{p_i}$. Let $n \in (\mathbb{N} \setminus A)_{p_i}$. Then $n \notin A$ and $n = p_i \cdot m$, $(m, p_i) = 1$. If $m \in B$, then $n \in p_i B$ and so $n \in A$—a contradiction. Thus $m \in \mathbb{N} \setminus B$, this yields $(\mathbb{N} \setminus A)_{p_i} \subseteq p_i(\mathbb{N} \setminus B)$, therefore $\mu\big((\mathbb{N} \setminus A)_{p_i}\big) = 0$. From Lemma A now we have that $\mu(\mathbb{N} \setminus A) = 0$, and the assertion follows. $\square$

Lemma 3. *Let* $\{p_i^{(1)}\},\dots,\{p_i^{(k)}\}$ *are such sequences of primes that* $\sum\limits_{i=1}^{\infty}\frac{1}{p_i^{(l)}}=\infty$, *for* $l=1,\dots,k$. *Put* $A\subseteq\mathbb{N}$ *in the following way:*

$$n\in A \iff n=p_{i_1}^{(1)}\cdots p_{i_k}^{(k)}m\,,\quad m\in\mathbb{N}\,.$$

Then $A\in\mathscr{D}_\mu$, *and* $\mu(A)=1$.

Proof. By the induction. If $k=1$, then

$$A=\bigcup_{i=1}^{\infty}p_i^{(1)}\mathbb{N}\,,$$

and the assertion follows from Lemma 2.

Let the statement holds for $k-1$. Now, A can be represented as

$$A=\bigcup_{i=1}^{\infty}p_i^{(k)}B\,, \tag{1}$$

where $n\in B\Leftrightarrow n=p_{i_1}^{(1)}\cdots p_{i_{k-1}}^{(k-1)}\cdot m$, and from the induction assumption we have $B\in\mathscr{D}_\mu$ and $\mu(B)=1$, and so Lemma 2 and (1) imply the assertion. □

Theorem 1. *Let* $m\in\mathbb{N}$, $m\ge 1$ *and* $a_1,\dots,a_k\in\mathbb{N}$ *and* $(a_i,m)=1$, $i=1,\dots,k-1$. *Put*

$$S=(a_1+m\mathbb{N})\cdots(a_{k-1}+m\mathbb{N})\,(a_k+m\mathbb{N})\,.$$

Then $S\in\mathscr{D}_\mu$ *and* $\mu(S)=\frac{1}{m}$.

Proof. Clearly

$$S\subseteq a_1a_2\cdots a_k+m\mathbb{N}\,. \tag{2}$$

Let $\{p_i^{(j)}\}$ be the sequence of all primes from $a_j+m\mathbb{N}$, $j=1,\dots,k-1$. From the Dirichlet theorem (cf. [S–B], p. 388) we have $\sum\limits_{i=1}^{\infty}\frac{1}{p_i^{(j)}}=\infty$, $j=1,\dots,k-1$.

Let us define the set A as in Lemma 3. Assume that

$$b\in(a_1\cdots a_k+m\mathbb{N})\cap A\,,$$

then

$$b\equiv a_1\cdots a_k\pmod m$$

and

$$b=p_{i_1}^{(1)}\cdots p_{i_{k-1}}^{(k-1)}\cdot b'\,,$$

thus

$$p_{i_1}^{(1)}\cdots p_{i_{k-1}}^{(k-1)}\cdot b'\equiv a_1\cdots a_k\pmod m\,. \tag{3}$$

For every $j = 1, \dots, k-1$ we have $p_{i_j} \equiv a_j \pmod{m}$ and so (3) yields

$$a_1 \cdots a_{k-1} b' \equiv a_1 \cdots a_k \pmod{m}.$$

For $j = 1, \dots, k-1$ we have $(a_j, m) = 1$, and thus from the last congruence it follows $b' \equiv a_k \pmod{m}$, and so $b \in S$. We have proved

$$(a_1 \cdots a_k + m\mathbb{N}) \cap A \subseteq S \cap A.$$

From (2) we obtain the other inclusion, thus

$$S \cap A = (a_1 \cdots a_k + m\mathbb{N}) \cap A,$$

and the assertion follows from Lemma 1. □

THEOREM 2. *Let $m_1, \dots, m_k$ be positive integers with the same primes in the canonic representation. Let $a_1, \dots, a_k$ be such positive integers that $(a_j, m_j) = 1$, $j = 1, \dots, k$. Then the set*

$$S = (a_1 + m_1\mathbb{N}) \cdots (a_k + m_k\mathbb{N})$$

is measurable.

Proof. Put $m = [m_1, \dots, m_k]$. Then $(a_j, m) = 1$, $j = 1, \dots, k$, and every $a_j + m_j\mathbb{N}$ can be represented in the form

$$a_j + m_j\mathbb{N} = \bigcup_{k=0}^{\frac{m}{m_j}-1} b_{j,k} + m\mathbb{N},$$

where $b_{j,k} = a_j + k \cdot m_j$. Clearly $(b_{j,k}, m) = 1$. Thus the set S is a finite union of sets of the type as in Theorem 1, therefore i) yields $S \in \mathscr{D}_\mu$. □

Unfortunately, we are not able to guarantee that the "summands" from the last proof are disjoint, and so we cannot determine the value $\mu(S)$.

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