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ON THE EXTENSION OF SOME FUNCTIONS TO Q_{s_1}-FUNCTIONS AND ON THE SUMS OF TWO Q_{s_1}-FUNCTIONS

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ABSTRACT. A function $f\colon \mathbb{R}^m \to \mathbb{R}$ satisfies the condition Q_{s_1} at a point $\boldsymbol{x} \in \mathbb{R}^m$ if for each real $\varepsilon > 0$ and for each set $U \ni \boldsymbol{x}$ belonging to the density topology there is an open set V such that $\emptyset \neq U \cap V \subset f^{-1}\big((f(\boldsymbol{x}) - \varepsilon, f(\boldsymbol{x}) + \varepsilon)\big) \cap C(f)$, where $C(f)$ denotes the set of all continuity points of f. For a nonempty set $A \subset \mathbb{R}^m$ it is proved that the Lebesgue measure $\lambda\big(\mathrm{cl}(A)\big) = 0$ if and only if for each λ-almost everywhere continuous function $f\colon \mathbb{R}^m \to \mathbb{R}$ there is a function $g \in Q_{s_1}$ such that $f|_A = g|_A$. Moreover, it is proved that every function $f\colon \mathbb{R}^m \to \mathbb{R}$ satisfying the condition $\lambda\big(\mathrm{cl}(D(f))\big) = 0$, where $D(f) = \mathbb{R}^m \setminus C(f)$, is the sum of two functions $g, h : \mathbb{R}^m \to \mathbb{R}$ with the condition Q_{s_1}.

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all reals, the set of all integers and the set of all positive integers respectively. Moreover, let $\mathbb{R}^m$ denote the m-dimensional product space $\mathbb{R} \times \cdots \times \mathbb{R}$ with the standard metric and let λ (resp. λ_e) denote the Lebesgue measure (resp. the outer Lebesgue measure) in $\mathbb{R}^m$.

For each positive integer $n \geqslant 1$ and for each system of integers $k_1, \ldots, k_m$ let

$$P^n_{k_1,\ldots,k_m} = \left[\frac{k_1 - 1}{2^n}, \frac{k_1}{2^n}\right) \times \cdots \times \left[\frac{k_m - 1}{2^n}, \frac{k_m}{2^n}\right).$$

Moreover, let

$$\mathcal{P}_n = \left\{P^n_{k_1,\ldots,k_m} : k_1, \ldots, k_m \in \mathbb{Z}\right\} \quad \text{and let} \quad \mathcal{P} = \bigcup_n \mathcal{P}_n .$$

Observe that

(1) if $(k_1, \ldots, k_m) \neq (l_1, \ldots, l_m)$ then $P^n_{k_1,\ldots,k_m} \cap P^n_{l_1,\ldots,l_m} = \emptyset$;

(2) $\mathbb{R}^m = \bigcup\limits_{k_1,\ldots,k_m \in \mathbb{Z}} P^n_{k_1,\ldots,k_m}$;

(3) if $n_1 > n_2$ then for each system $(k_1, \ldots, k_m)$ there is a unique system $(l_1, \ldots, l_m)$ such that $P^{n_1}_{k_1,\ldots,k_m} \subset P^{n_2}_{l_1,\ldots,l_m}$;

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(4) for each point $\boldsymbol{x} \in \mathbb{R}^m$ and for each positive integer n there is a unique system $\big(k_1(\boldsymbol{x}), \dots, k_m(\boldsymbol{x})\big)$ such that $\boldsymbol{x} \in P^n_{k_1(\boldsymbol{x}),\dots,k_m(\boldsymbol{x})} = P^n(\boldsymbol{x})$.

For a set $A \subset \mathbb{R}^m$ and a point $\boldsymbol{x} \in \mathbb{R}^m$ denote by

$$d_u(A, \boldsymbol{x}) = \limsup_{n\to\infty} \frac{\lambda_e\big(A \cap P^n(\boldsymbol{x})\big)}{\lambda\big(P^n(\boldsymbol{x})\big)} \quad \left(\text{resp. } d_l(A, \boldsymbol{x}) = \liminf_{n\to\infty} \frac{\lambda_e\big(A \cap P^n(\boldsymbol{x})\big)}{\lambda\big(P^n(\boldsymbol{x})\big)}\right)$$

the *upper* (resp. *lower*) *density* of a set $A \subset \mathbb{R}$ at a point $\boldsymbol{x}$ (compare [1]).

A point $\boldsymbol{x} \in \mathbb{R}^m$ is called *a density point* of a set $A \subset \mathbb{R}^m$ if there exists a λ-measurable (i.e., measurable in the sense of Lebesgue) set $B \subset A$ such that $d_l(B, \boldsymbol{x}) = 1$. The family

$$\mathcal{T}_d = \big\{A \subset \mathbb{R}^m \colon A \text{ is } \lambda\text{-measurable and } d_l(A, \boldsymbol{x}) = 1 \text{ for } \boldsymbol{x} \in A\big\}$$

is a topology called *the density topology* ([1], [2], [8], [10]).

Moreover, let $\mathcal{T}_e$ be the Euclidean topology in $\mathbb{R}^m$ and let $C(f)$ denotes the set of all points at each $f \colon \mathbb{R}^m \to \mathbb{R}$ is continuous.

We will say that a function $f \colon \mathbb{R}^m \to \mathbb{R}$ satisfies the condition Q_{s_1} or that f is s_1*-strongly quasicontinuous at a point* $\boldsymbol{x} \in \mathbb{R}^m$ (or $f \in Q_{s_1}(\boldsymbol{x})$) if for every set $A \in \mathcal{T}_d$ containing $\boldsymbol{x}$ and for every real $\varepsilon > 0$ there is a nonempty open set U such that $f^{-1}\big((f(\boldsymbol{x})-\varepsilon, f(\boldsymbol{x})+\varepsilon)\big) \cap C(f) \supset U \cap A \neq \emptyset$ ([9], and compare [3]).

If $f \in Q_{s_1}(\boldsymbol{x})$ for each point $\boldsymbol{x} \in \mathbb{R}^m$ then we will write $f \in Q_{s_1}$.

We will say that a function $f \colon \mathbb{R}^m \to \mathbb{R}$ satisfies the condition S_1 ($f \in S_1$) if for every nonempty set $A \in \mathcal{T}_d$ there is an open set U such that $C(f) \supset U \cap A \neq \emptyset$ ([9], and compare [3]).

Evidently the sum of two functions $f, g \colon \mathbb{R}^m \to \mathbb{R}$ belonging to S_1 satisfies also the condition S_1 and $Q_{s_1} \subset S_1$.

Observe that if a function $f \colon \mathbb{R}^m \to \mathbb{R}$ satisfies the condition S_1, then f is λ-almost everywhere continuous ([9], and compare [4]).

Let $\mathcal{C}_{\mathrm{ae}}$ denote the family of all λ-almost everywhere continuous functions $f \colon \mathbb{R}^m \to \mathbb{R}$.

We use the following lemma in the proofs of main results.

Lemma 1. ([9]) *Let $B \subset \mathbb{R}^m$ be a nonempty closed set of λ-measure zero, let $E \subset B$ be a nonempty closed set and let U be an open set containing E. Then there exists a sequence of pairwise disjoint sets $K_n \subset U \setminus B$ ($n = 1, 2, \dots$) which are the unions of cubes from $\mathcal{P}$ and such that*

(1) *the family of closures of these cubes is $\mathcal{P}$-locally finite at each point $\boldsymbol{y} \in \mathbb{R}^m \setminus E$, i.e., for each point $\boldsymbol{y} \in \mathbb{R}^m \setminus E$ there is a cube $P \in \mathcal{P}$ such that $\boldsymbol{y} \in P$ and the family of all cubes from this family which intersect P is finite;*

(2) *the closures of these cubes are pairwise disjoint and contained in $U \setminus B$;*

(3) *for every increasing sequence* (n_k) *of positive integers and for every point* $\boldsymbol{x} \in E$ *the upper density*

$$d_u\left(\bigcup_{k=1}^{\infty} K_{n_k},\ \boldsymbol{x}\right) = 1, \quad \textit{and}$$

(4) *the sequence* $\big(\mathrm{cl}(K_n)\big)$ *of closures of sets* K_n $(n = 1, 2, \dots)$ *converges to the set* E *in the sense of Hausdorff metric.*

The conditions Q_{s_1} and S_1, which are investigated in this article, are suitable to conditions $\mathcal{A}_3$ and $\mathcal{A}_5$ considered by Z. Grande [5] for functions $f\colon \mathbb{R} \to \mathbb{R}$. In [5] the author investigates also families $\Phi_{\mathcal{A}_i}$ $(i = 3, 5)$ of all nonempty sets $A \subset \mathbb{R}$ such that for every almost everywhere continuous functions $g\colon \mathbb{R} \to \mathbb{R}$ there is a function $f\colon \mathbb{R} \to \mathbb{R}$ having property $\mathcal{A}_i$ such that $f|_A = g|_A$, and proves that a nonempty set $A \subset \mathbb{R}$ belongs to $\Phi_{\mathcal{A}_3} = \Phi_{\mathcal{A}_5}$ if and only if $\lambda\big(\mathrm{cl}(A)\big) = 0$.

Let $\Phi_{Q_{s_1}}$ (Φ_{S_1}) be the family of all nonempty sets $A \subset \mathbb{R}^m$ such that for every function $g \in \mathcal{C}_{\mathrm{ae}}$ there is a function $f \in Q_{s_1}$ $(f \in S_1$, respectively) such that $f|_A = g|_A$.

Theorem 1. *The equality* $\Phi_{Q_{s_1}} = \Phi_{S_1}$ *is true. Moreover, a nonempty set* $A \subset \mathbb{R}^m$ *belongs to* $\Phi_{Q_{s_1}}$ *if and only if* $\lambda\big(\mathrm{cl}(A)\big) = 0$.

Proof. Since $Q_{s_1} \subset S_1$, we have $\Phi_{Q_{s_1}} \subset \Phi_{S_1}$.

First we show that if $A \in \Phi_{S_1}$ then $\lambda\big(\mathrm{cl}(A)\big) = 0$. Let $A \subset \mathbb{R}^m$ be a set such that $\lambda\big(\mathrm{cl}(A)\big) > 0$. There is a countable set $A_1 \subset A$ such that $\mathrm{cl}(A_1) = \mathrm{cl}(A)$. Then the set

$$M = \big\{\boldsymbol{x} \in \mathrm{cl}(A)\colon d_l\big(\mathrm{cl}(A),\ \boldsymbol{x}\big) = 1\big\} \in \mathcal{T}_d$$

and it is nonempty. Let

$$N = \{\boldsymbol{a}_1, \boldsymbol{a}_2,\ \dots,\ \boldsymbol{a}_n,\ \dots\} \subset M \setminus A_1$$

be a countable set dense in M and let $g\colon \mathbb{R}^m \to \mathbb{R}$ be defined as follows

$$g(\boldsymbol{x}) = \begin{cases} \frac{1}{n} & \text{for } \boldsymbol{x} = \boldsymbol{a}_n,\ n \geqslant 1, \\ 0 & \text{otherwise on } \mathbb{R}^m. \end{cases}$$

Then $D(g) = N$ and $\lambda\big(D(g)\big) = 0$. Thus $g \in \mathcal{C}_{\mathrm{ae}}$.

Next, we assume that there exists a function $f\colon \mathbb{R}^m \to \mathbb{R}$ belonging to S_1 and such that $f|_A = g|_A$.

The set $A_1 \subset A$ and $f(A_1) = g(A_1) = \{0\}$, so we have $f(\boldsymbol{u}) = 0$ at almost all points $\boldsymbol{u} \in \mathrm{cl}(A)$. Consequently $N \subset D(f)$.

Since $f \in S_1$ and the nonempty set $M \in \mathcal{T}_d$ there is an open set O such that

$$\emptyset \neq O \cap M \subset C(f). \tag{$*$}$$

But the set N is dense in M, so we obtain a contradiction with $(*)$. So, for each set $A \in \Phi_{S_1}$ we have $\lambda\big(\mathrm{cl}(A)\big) = 0$.

Now, we will show that $\Phi_{S_1} \subset \Phi_{Q_{s_1}}$. Let $A \in \Phi_{S_1}$ be a nonempty set and let $g \in \mathcal{C}_{\mathrm{ae}}$.

By Lemma 1, for $B = E = \mathrm{cl}(A)$ and $U = \mathbb{R}^m$ there exists a sequence of pairwise disjoint sets $K_n \subset \mathbb{R}^m \setminus \mathrm{cl}(A)$ $(n = 1, 2, \dots)$, which are the unions of cubes from $\mathcal{P}$ and satisfies conditions (1)–(4) from Lemma 1.

Let (w_n) be a one-to-one sequence of all rationals and let (N_i) be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$ such that $\mathbb{N} = \bigcup\limits_{i=1}^{\infty} N_i$.

Put

$$f(\boldsymbol{x}) = \begin{cases} w_n & \text{for } \ \boldsymbol{x} \in K_i\,, \ \text{ where } \ i \in N_n\,,\ n \geqslant 1\,, \\ g(\boldsymbol{x}) & \text{for } \ \boldsymbol{x} \in \mathrm{cl}(A)\,, \\ 0 & \text{otherwise on } \ \mathbb{R}^m\,. \end{cases}$$

Then $f = g$ on the set A. Moreover, for each point $\boldsymbol{x} \in \mathbb{R}^m \setminus \mathrm{cl}(A)$ there exists an open set $O \ni \boldsymbol{x}$ disjoint with $\mathrm{cl}(A)$ and such that the set of all positive integers n for which $O \cap K_n \neq \emptyset$ is empty or finite. Thus $f \in Q_{s_1}$ at each point $\boldsymbol{x} \in \mathbb{R}^m \setminus \mathrm{cl}(A)$. The function f has also property Q_{s_1} at all points $\boldsymbol{x} \in \mathrm{cl}(A)$. Indeed, fix a real $\varepsilon > 0$, a point $\boldsymbol{x} \in \mathrm{cl}(A)$ and a set $G \in \mathcal{T}_d$ containing $\boldsymbol{x}$. There exists a positive integer k with

$$\big|f(\boldsymbol{x}) - w_k\big| = \big|g(\boldsymbol{x}) - w_k\big| < \varepsilon\,.$$

Since

$$d_l(G,\ \boldsymbol{x}) = 1 \qquad \text{and} \qquad d_u\Bigg(\bigcup_{j \in N_k} K_j\,,\, \boldsymbol{x}\Bigg) = 1$$

there is an open set

$$G' \subset \bigcup_{j \in N_k} K_j$$

such that $G' \cap G \neq \emptyset$. For all points $\boldsymbol{u} \in G' \cap G$ we have

$$\big|f(\boldsymbol{u}) - f(\boldsymbol{x})\big| = \big|w_k - f(\boldsymbol{x})\big| < \varepsilon$$

and thus $f \in Q_{s_1}$. Consequently $A \in \Phi_{Q_{s_1}}$ and the proof is finished. $\square$

THEOREM 2. *For a function $f\colon \mathbb{R}^m \to \mathbb{R}$ with $\lambda\bigl(\mathrm{cl}(D(f))\bigr) = 0$ there are two functions $g, h \in Q_{s_1}$ such that $f = g + h$.*

Proof. We use Lemma 1. For $B = E = \mathrm{cl}\bigl(D(f)\bigr)$ and $U = \mathbb{R}^m$ there exists a sequence of pairwise disjoint sets $K_n \subset \mathbb{R}^m \setminus \mathrm{cl}\bigl(D(f)\bigr)$ $(n = 1, 2, \dots)$ which are the unions of cubes from $\mathcal{P}$ and satisfying conditions (1)–(4) from Lemma 1.

Let (w_n) be a one-to-one sequence of all rationals and let $\mathbb{N} = \bigcup_{k=0}^{\infty} N_k$, where N_k, $k = 0, 1, 2, \dots$, are pairwise disjoint and infinite sets. Observe that for each positive integer k and for each point $\boldsymbol{x} \in \mathrm{cl}\bigl(D(f)\bigr)$ we have the upper density

$$d_u\Biggl(\bigcup_{i \in N_k} K_i, \boldsymbol{x}\Biggr) = 1 .$$

Define the functions g, $h\colon \mathbb{R}^m \to \mathbb{R}$ as follows

$$g(\boldsymbol{x}) = \begin{cases} w_n & \text{for } \boldsymbol{x} \in K_{2n-1}\ (n = 1, 2, \dots), \\ f(\boldsymbol{x}) - w_n & \text{for } \boldsymbol{x} \in K_{2n}\ (n = 1, 2, \dots), \\ f(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \mathrm{cl}\bigl(D(f)\bigr), \\ 0 & \text{otherwise on } \mathbb{R}^m, \end{cases}$$

and

$$h(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) - w_n & \text{for } \boldsymbol{x} \in K_{2n-1}\ (n = 1, 2, \dots), \\ w_n & \text{for } \boldsymbol{x} \in K_{2n}\ (n = 1, 2, \dots), \\ 0 & \text{for } \boldsymbol{x} \in \mathrm{cl}\bigl(D(f)\bigr), \\ f(\boldsymbol{x}) & \text{otherwise on } \mathbb{R}^m . \end{cases}$$

Since $g, h \in Q_{s_1}$ and $f = g + h$, the proof is finished. □

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