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ON THE CARDINALITY SIZE OF THE HOMEOMORPHIC DENSITY TYPE TOPOLOGIES

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ABSTRACT. The paper deals with the density type topologies generated by a nondecreasing and unbounded sequence of positive reals. The cardinality of family of the homeomorphic topological spaces equipped with such density type topologies is discussed.

Let ℓ be the standard Lebesgue measure in $\mathbb{R}$ and $\mathcal{L}$ the family of all Lebesgue measurable sets. By $\mathbb{N}$ we shall denote the set of all positive integers and by $\mathcal{S}$ the family of all unbounded and nondecreasing sequences of positive numbers. If $\{s_n\}_{n\in\mathbb{N}} \in \mathcal{S}$, we shall denote $\langle s\rangle = \{s_n\}_{n\in\mathbb{N}}$.

Definition 1. We shall say that $x \in \mathbb{R}$ is a *density point* of a set $A \in \mathcal{L}$ with respect to a sequence $\langle s\rangle \in \mathcal{S}$ ($\langle s\rangle$-density point in short) if

$$\lim_{n\to\infty} \frac{\ell\big(A \cap [x - \frac{1}{s_n}, x + \frac{1}{s_n}]\big)}{\frac{2}{s_n}} = 1 .$$

Reminding the concept of an ordinary density point it is worth observing (see [CLO]) that a point $x \in \mathbb{R}$ is a density point of a set $A \in \mathcal{L}$ if

$$\lim_{n\to\infty} \frac{\ell\big(A \cap [x - \frac{1}{n}, x + \frac{1}{n}]\big)}{\frac{2}{n}} = 1 .$$

It is clear that x is $\langle s\rangle$-density point of a set $A \in \mathcal{L}$ if and only if 0 is $\langle s\rangle$-density point of the set $A - x = \{y\colon y = a - x,\ a \in A\}$. Let $\langle s\rangle \in \mathcal{S},\ A \in \mathcal{L}$. Putting

$$\Phi_{\langle s\rangle}(A) = \big\{x \in \mathbb{R}\colon x \text{ is a } \langle s\rangle\text{-density point of } A\big\}$$

we get that the operator $\Phi_{\langle s\rangle}\colon \mathcal{L} \to \mathcal{L}$ is the lower density operator (see [FH]).

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Let $\mathcal{T}_{\langle s\rangle} = \{A \in \mathcal{L}\colon A \subset \Phi_{\langle s\rangle}(A)\}$. Then the family $\mathcal{T}_{\langle s\rangle}$ forms a topology on the real line. If $\langle s\rangle = \{n\}_{n\in\mathbb{N}}$ then $\mathcal{T}_{\langle s\rangle}$ is of course the classical density topology denoted by $\mathcal{T}_d$. It is clear that $\mathcal{T}_d \subset \mathcal{T}_{\langle s\rangle}$ for every $\langle s\rangle \in \mathcal{S}$. Many interesting properties of $\langle s\rangle$-density topologies are presented in [FH], [FFH], [FHW], [L]. We shall recall some of them to continue further considerations.

Theorem 1. ([FH]) *Let $\langle s\rangle \in \mathcal{S}$. Then*

$$\mathcal{T}_{\langle s\rangle} = \mathcal{T}_d \quad \textit{iff} \quad \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0\,.$$

Let $\mathcal{S}_0 = \{\langle s\rangle \in \mathcal{S}\colon \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\}$.

Theorem 2. ([FH]) *If $\langle s\rangle, \langle t\rangle \in \mathcal{S}_0$ and $\lim_{n\to\infty} \frac{s_n}{t_n} = \alpha \in (0,\infty)$ then $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_{\langle t\rangle}$ if and only if $\alpha = 1$.*

Let us fix $\langle s\rangle, \langle t\rangle \in \mathcal{S}$, then

$$\underset{n\in\mathbb{N}}{\forall}\ \underset{k_n!}{\exists}\ t_n \in (s_{k_n}, s_{k_n+1}]\,.$$

The following theorem describes a more general situation than that in Theorem 2.

Theorem 3. ([FFH]) *The following conditions are equivalent:*

(i) *for an arbitrary increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers*

$$\liminf_{i\to\infty} \frac{t_{n_i}}{s_{k_{n_i}}} < \infty \quad \textit{or} \quad \liminf_{i\to\infty} \frac{s_{k_{n_i}+1}}{t_{n_i}} = 1\,,$$

(ii) $\mathcal{T}_{\langle s\rangle} \subset \mathcal{T}_{\langle t\rangle}$.

Let $\mathcal{T} = \{\mathcal{T}_{\langle s\rangle}\colon \langle s\rangle \in \mathcal{S}\}$. The following theorem expresses the size of the family $\mathcal{T}$. Namely, we have

Theorem 4. ([FH]) $\operatorname{card}\mathcal{T} = \mathfrak{c}$.

In [FFH] the problem of homeomorphism between topological spaces $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ and $(\mathbb{R}, \mathcal{T}_{\langle t\rangle})$ for $\langle t\rangle, \langle s\rangle \in \mathcal{S}$ was investigated. The following results were established.

Theorem 5. ([FHW]) *Let $\langle s\rangle, \langle t\rangle \in \mathcal{S}$. If $\mathcal{T}_{\langle s\rangle} \setminus \mathcal{T}_{\langle t\rangle} \neq \emptyset$ and $\mathcal{T}_{\langle t\rangle} \setminus \mathcal{T}_{\langle s\rangle} \neq \emptyset$ then the topological spaces $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ and $(\mathbb{R}, \mathcal{T}_{\langle t\rangle})$ are not homeomorphic.*

Theorem 6. ([FFH]) *Let $\langle s\rangle \in \mathcal{S}_0$. Then the topological spaces $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ and $(\mathbb{R}, \mathcal{T}_d)$ are not homeomorphic.*

In this paper we shall present more results concerning the size of the sets of topological spaces $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$, $\langle s\rangle \in \mathcal{S}$, which are homeomorphic between themselves or contrary, not homeomorphic. If the topological spaces $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ and $(\mathbb{R}, \mathcal{T}_{\langle t\rangle})$ are homeomorphic, we shall use the notation $(\mathbb{R}, \mathcal{T}_{\langle s\rangle}) \overset{\text{Hom}}{\sim} (\mathbb{R}, \mathcal{T}_{\langle t\rangle})$. If not, we shall use the notation $\neg\big((\mathbb{R}, \mathcal{T}_{\langle s\rangle}) \overset{\text{Hom}}{\sim} (\mathbb{R}, \mathcal{T}_{\langle t\rangle})\big)$.

Theorem 7. *For every sequence $\langle s\rangle \in S_0$ there exists a set $S_1 \subset S$ such that*

(i) $\operatorname{card} S_1 = \mathfrak{c}$,

(ii) $\underset{\langle u\rangle, \langle t\rangle \in S_1}{\forall} \big(\langle u\rangle \neq \langle t\rangle \implies \mathcal{T}_{\langle u\rangle} \neq \mathcal{T}_{\langle t\rangle}\big)$,

(iii) $\underset{\langle t\rangle \in S_1}{\forall} (\mathbb{R}, \mathcal{T}_{\langle s\rangle}) \overset{\text{Hom}}{\sim} (\mathbb{R}, \mathcal{T}_{\langle t\rangle})$.

Proof. Let $\langle s\rangle \in S_0$ and $m \in (0, +\infty)$. Then $\langle ms\rangle \in S_0$, where $\langle ms\rangle = \{ms_n\}_{n\in\mathbb{N}}$. Putting $S_1 = \{\langle ms\rangle : m \in (1, +\infty)\}$ we have that $S_1 \subset S_0$ and $\operatorname{card} S_1 = \mathfrak{c}$. Moreover, by Theorem 2, for every $m_1, m_2 \in (1, +\infty)$, if $m_1 \neq m_2$ then $\mathcal{T}_{\langle m_1 s\rangle} \neq \mathcal{T}_{\langle m_2 s\rangle}$. According to $\mathcal{T}_{\langle ms\rangle} = \frac{1}{m}\mathcal{T}_{\langle s\rangle}$ for $m > 1$ we get that function $h(x) = \frac{1}{m}x$ guarantees the homeomorphism of topological spaces $(\mathbb{R}, \mathcal{T}_{\langle s\rangle})$ and $(\mathbb{R}, \mathcal{T}_{\langle ms\rangle})$. $\square$

Lemma 8. *For every sequence $\langle s\rangle \in S_0$ there exists a sequence $\langle t\rangle \in S_0$ such that $\mathcal{T}_{\langle s\rangle} \setminus \mathcal{T}_{\langle t\rangle} \neq \emptyset$ and $\mathcal{T}_{\langle t\rangle} \setminus \mathcal{T}_{\langle s\rangle} \neq \emptyset$.*

Proof. Let $\langle s\rangle \in S_0$. Then $\liminf\limits_{n\to\infty} \frac{s_n}{s_{n+1}} = 0$. Let $\{s_{k_n}\}_{n\in\mathbb{N}}$ be an increasing sequence such that $\lim\limits_{n\to\infty} \frac{s_{k_n}}{s_{k_n+1}} = 0$. Putting $t_n = \sqrt{s_{k_n} \cdot s_{k_n+1}}$ we see that $\langle t\rangle = \{t_n\}_{n\in\mathbb{N}} \in S_0$. Applying Theorem 3 we show that $\mathcal{T}_{\langle s\rangle} \nsubseteq \mathcal{T}_{\langle t\rangle}$. It is clear that $s_{k_n} < t_n < s_{k_n+1}$ for $n \in \mathbb{N}$. Also, $\liminf\limits_{n\to\infty} \frac{t_n}{s_{k_n}} = \infty$ and $\liminf\limits_{n\to\infty} \frac{s_{k_n+1}}{t_n} = 0$. Thus $\mathcal{T}_{\langle s\rangle} \nsubseteq \mathcal{T}_{\langle t\rangle}$. Further, we prove that $\mathcal{T}_{\langle t\rangle} \nsubseteq \mathcal{T}_{\langle s\rangle}$. For every $s_n > t_1$ there exists $j_n \in \mathbb{N}$ such that $s_n \in (t_{j_n}, t_{j_n+1}\rangle$. As $t_n < s_{k_n+1} < t_{n+1}$ for $n \in \mathbb{N}$ we have $j_{k_n+1} = n$. Simultaneously $\liminf\limits_{n\to\infty} \frac{s_{k_n+1}}{t_{j_{k_n+1}}} = \liminf\limits_{n\to\infty} \frac{s_{k_n+1}}{t_n} = \infty$ and $\liminf\limits_{n\to\infty} \frac{t_{j_{k_n+1}+1}}{s_{k_n+1}} = \liminf\limits_{n\to\infty} \frac{t_{n+1}}{s_{k_n+1}} = \liminf\limits_{n\to\infty} \sqrt{\frac{s_{k_{n+1}} \cdot s_{k_{n+1}+1}}{s_{k_n+1} \cdot s_{k_n+1}}} \geqslant \liminf\limits_{n\to\infty} \sqrt{\frac{s_{k_{n+1}+1}}{s_{k_n+1}}} \geqslant \liminf\limits_{n\to\infty} \sqrt{\frac{s_{k_{n+1}+1}}{s_{k_{n+1}}}} = \infty$. Hence, by Theorem 3, we get $\mathcal{T}_{\langle t\rangle} \nsubseteq \mathcal{T}_{\langle s\rangle}$. $\square$

According to Theorem 4 we conclude

Corollary 9. $\neg\big((\mathbb{R}, \mathcal{T}_{\langle s\rangle}) \overset{\text{Hom}}{\sim} (\mathbb{R}, \mathcal{T}_{\langle t\rangle})\big)$.

THEOREM 10. *For every sequence $\langle s\rangle \in S$ there exists a set $S_2 \subset S$ such that*

(i) $\operatorname{card} S_2 = \mathfrak{C}$,

(ii) $\underset{\langle u\rangle,\langle t\rangle\in S_2}{\forall} \big(\langle u\rangle \neq \langle t\rangle \implies \mathcal{T}_{\langle u\rangle} \neq \mathcal{T}_{\langle t\rangle}\big)$,

(iii) $\underset{\langle t\rangle\in S_2}{\forall} \neg\big((\mathbb{R},\mathcal{T}_{\langle s\rangle}) \overset{\mathrm{Hom}}{\sim} (\mathbb{R},\mathcal{T}_{\langle t\rangle})\big)$.

P r o o f. 1^o Let us suppose that $\langle s\rangle \in S\setminus S_0$. Then, by Theorem 1, $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_d$. Since that $\operatorname{card}\{\mathcal{T}_{\langle s\rangle} : s\in S_0\} = \mathfrak{C}$ we conclude that there exists a set $S_2 \subset S_0$ such that $\operatorname{card} S_2 = \mathfrak{C}$ and for every $\langle s\rangle,\langle t\rangle \in S_2$ if $\langle s\rangle \neq \langle t\rangle$, then $\mathcal{T}_{\langle s\rangle} \neq \mathcal{T}_{\langle t\rangle}$. Applying Theorem 6, we get $\neg\big((\mathbb{R},\mathcal{T}_d) \overset{\mathrm{Hom}}{\sim} (\mathbb{R},\mathcal{T}_{\langle t\rangle})\big)$ for every $\langle t\rangle \in S_2$.

2^o Let us suppose that $\langle s\rangle \in S_0$. By Corollary 9, there exists a sequence $\langle t\rangle \in S_0$ such that $\neg\big((\mathbb{R},\mathcal{T}_{\langle s\rangle}) \overset{\mathrm{Hom}}{\sim} (\mathbb{R},\mathcal{T}_{\langle t\rangle})\big)$. Let $S_2 = \big\{\langle mt\rangle : m\in(1,+\infty)\big\}$. Then $\operatorname{card} S_2 = \mathfrak{C}$. If $\langle t_1\rangle,\langle t_2\rangle \in S_2$ and $\langle t_1\rangle \neq \langle t_2\rangle$ then, by Theorem 2, we have that $\mathcal{T}_{\langle t_1\rangle} \neq \mathcal{T}_{\langle t_2\rangle}$. Moreover, for an arbitrary sequence $\langle u\rangle \in S_2$ we conclude that $\neg\big((\mathbb{R},\mathcal{T}_{\langle s\rangle}) \overset{\mathrm{Hom}}{\sim} (\mathbb{R},\mathcal{T}_{\langle u\rangle})\big)$, otherwise we will obtain $\big((\mathbb{R},\mathcal{T}_{\langle s\rangle}) \overset{\mathrm{Hom}}{\sim} (\mathbb{R},\mathcal{T}_{\langle t\rangle})\big)$. □

Finally, we have the following theorem.

THEOREM 11. *There exists a set $S_3 \subset S$ such that*

1^o $\operatorname{card} S_3 = \mathfrak{C}$,

2^o $\underset{\langle s\rangle,\langle t\rangle\in S_3}{\forall} \big(\langle s\rangle \neq \langle t\rangle \implies \neg\big((\mathbb{R},\mathcal{T}_{\langle s\rangle}) \overset{\mathrm{Hom}}{\sim} (\mathbb{R},\mathcal{T}_{\langle t\rangle})\big)\big)$.

P r o o f. Let $\langle s^\alpha\rangle = \{a^{\alpha n!}\}_{n\in\mathbb{N}}$, for a fix $a>1$ and $\alpha>0$. We get that $\langle s^\alpha\rangle \in S_0$ for every $\alpha>0$. Let $S_3 = \{\langle s^\alpha\rangle : \alpha>0\}$. Then $\operatorname{card} S_3 = \mathfrak{C}$. We show that for every $\alpha,\beta>0$, if $\alpha\neq\beta$ then $\mathcal{T}_{\langle s^\alpha\rangle}\setminus\mathcal{T}_{\langle s^\beta\rangle} \neq \emptyset$ and $\mathcal{T}_{\langle s^\beta\rangle}\setminus\mathcal{T}_{\langle s^\alpha\rangle} \neq \emptyset$. Let $\alpha<\beta$. Then there exists $n_0\in\mathbb{N}$ such that for every $n>n_0$ we have $a^{\alpha n!} < a^{\beta n!} < a^{\alpha(n+1)!}$. It is sufficient to take $n_0\in\mathbb{N}$ such that $n_0 \geqslant \frac{\beta}{\alpha}$. Applying Theorem 3 we calculate $\lim\limits_{n\to\infty}\frac{a^{\beta n!}}{a^{\alpha n!}} = \infty$ and $\lim\limits_{n\to\infty}\frac{a^{\alpha(n+1)!}}{a^{\beta n!}} = \infty$. It implies that $\mathcal{T}_{\langle s^\alpha\rangle} \nsubseteq \mathcal{T}_{\langle s^\beta\rangle}$. Similarly, we notice that there exists $n_1\in\mathbb{N}$ such that for any $n>n_1$ we have

$$a^{\beta(n-1)!} < a^{\alpha n!} < a^{\beta n!}.$$

In the same way, as above, applying Theorem 3, we get $\mathcal{T}_{\langle s^\beta\rangle} \nsubseteq \mathcal{T}_{\langle s^\alpha\rangle}$.

Finally, Theorem 5 proves $\neg\big((\mathbb{R},\mathcal{T}_{\langle s^\alpha\rangle}) \overset{\mathrm{Hom}}{\sim} (\mathbb{R},\mathcal{T}_{\langle s^\beta\rangle})\big)$ for every $\alpha\neq\beta$. □

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