Tatra Mt. Math. Publ. **34** (2006), 19–27

TATRA MOUNTAINS
Mathematical Publications

ON THE STRUCTURE OF SOME SUBSETS IN THE SPACE OF FUNCTIONS OF BOUNDED Λ-VARIATION

Robert Drozdowski

ABSTRACT. In the paper [D. Waterman: *On Λ-bounded variation*, Studia Math. **57** (1976), 33–45] a concept of functions of bounded Λ-variation was introduced. In this paper the structure of different subsets of the space of Λ-bounded variation is described.

Definition 1. ([5]) A sequence $\Lambda = (\lambda_i)_{i=1}^{\infty}$ is said to be a *λ-sequence* if it satisfies the following conditions:

(1) $\forall_{i\in\mathbb{N}}\,(\lambda_i > 0)$,

(2) $\forall_{i\in\mathbb{N}}\,(\lambda_i \leqslant \lambda_{i+1})$,

(3) $\sum\limits_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$.

Additionally, if $\lim\limits_{i\to\infty} \lambda_i = \infty$, then it is said to be a *proper λ-sequence*.

In this paper we will assume that E is an infinite subset of $\mathbb{R}$ and Y is a Banach space. The symbol $S(E)$ will denote the family of all finite families of non-degenerate closed and non-overlapping intervals, which end points are members of E. For any s from $S(E)$ let

$$s = \left\{ \left[a_i^{(s)}, b_i^{(s)}\right] :\ i = \{1, \ldots, m_s\} \right\}.$$

For a function $f : E \to Y$ let

$$V_\Lambda(f, E) = \sup \left\{ \sum_{i=1}^{m_s} \frac{\left\| f\left(a_i^{(s)}\right) - f\left(b_i^{(s)}\right) \right\|}{\lambda_i} : s \in S(E) \right\}.$$

2000 Mathematics Subject Classification: 26A45.
Keywords: normed spaces, functions of bounded variation, porous sets, strongly porous sets.

Definition 2. ([6]) A mapping $f\colon E \to Y$ is said to be of *bounded Λ-variation*, if

$$V_\Lambda(f,E) < \infty\,.$$

In this article we will often apply the following property:

Lemma 1. ([1]) *If $f_j\colon E \to Y$ form a sequence converging pointwise to a function $f\colon E \longrightarrow Y$, then*

$$V_\Lambda(f,E) \leqslant \liminf_{j\to\infty} V_\Lambda(f_j,E)\,.$$

In the sequel $\mathscr{V}_\Lambda(E,Y)$ (resp. $\mathscr{V}(E,Y)$) will denote the space of all mappings of bounded Λ-variation (resp. bounded variation) from E to Y. It is known from [6] that $\mathscr{V}(E,Y)$ is a subspace of the space $\mathscr{V}_\Lambda(E,Y)$ for any λ-sequence Λ and it is a proper subspace if and only if the λ-sequence Λ is proper.

In [6] D. Waterman proved that the space of real valued functions of bounded Λ-variation defined in a set E is a Banach space with the norm

$$\|f\|_\Lambda = \bigl|f(a)\bigr| + V_\Lambda(f,E)\,,$$

where a is a fixed point of E.

Since Y is a Banach space, then we can analogously prove that $\mathscr{V}_\Lambda(E,Y)$ is a Banach space equipped with the norm

$$\|f\|_\Lambda = \|f(a)\| + V_\Lambda(f,E)\,, \tag{1}$$

where a is a fixed point of E.

Additionally, let us see that for any points a,b from E the norms

$$\|f\|_{\Lambda,a} = \bigl\|f(a)\bigr\| + V_\Lambda(f,E)\,,$$
$$\|f\|_{\Lambda,b} = \bigl\|f(b)\bigr\| + V_\Lambda(f,E)$$

are equivalent.

Remark 1. In this paper we will always consider the space $\mathscr{V}_\Lambda(E,Y)$ with the norm (1).

Remark 2. For a subset A of $\mathscr{V}_\Lambda(E,Y)$ by the symbol $\overline{A}^\Lambda$ we will understand the closure of the set A in the space $\mathscr{V}_\Lambda(E,Y)$.

For any λ-sequences Λ and Γ we would consider the spaces $\mathscr{V}_\Lambda(E,Y)$ and $\mathscr{V}_\Gamma(E,Y)$. We will observe that if $\mathscr{V}_\Gamma(E,Y)$ is a proper subset of $\mathscr{V}_\Lambda(E,Y)$, then $\mathscr{V}_\Gamma(E,Y)$ is relatively „small" set in the bigger one.

For the next step in our consideration we need the well-known definitions of strongly porous and σ-strongly porous sets.

In a metric space (X,d) the symbol $K^\circ(x,\varepsilon)$ will denote the open ball with the center x and the radius ε.

Definition 3. ([3]) It is said that a subset M of a metric space (X, d) is *porous at a point* x from X if there is a positive number α such that for any positive number ε there is a point y in the open ball $K^\circ(x,\varepsilon)$ such that

$$K^\circ\big(y, \alpha d(x,y)\big) \cap M = \emptyset .$$

If α from the above condition can be chosen arbitrarily close to 1, then the set M is said to be *strongly porous at* x.

A set which is porous (strongly porous) at every point x from Y is called *porous* (*strongly porous*).

Definition 4. ([3]) A set M in a metric space X is said to be *σ-strongly porous* if it can be represented as a countable union of strongly porous sets.

It is known that if a set M in a metric space is strongly porous then it is nowhere dense, however, in [3] V. Olevskii gave the following:

Theorem 1. ([3]) *If M is a convex and nowhere dense set in a Banach space X, then it is strongly porous.*

Theorem 2. ([2, Banach's subgroup theorem])
If $\mathcal{G}$ is a subgroup of a linear topological space $\mathcal{E}$ and if $\mathcal{G}$ contains a set A that is of the second category in $\mathcal{E}$ and satisfies the condition of Baire, then $\mathcal{G} = \mathcal{E}$. Hence if $\mathcal{G}$ is any proper subgroup of $\mathcal{E}$, then either $\mathcal{G}$ is of the first category in $\mathcal{E}$ or $\mathcal{G}$ fails to satisfy the condition of Baire.

Now, we can show

Theorem 3. *If $\mathscr{V}_\Gamma(E,Y) \subsetneq \mathscr{V}_\Lambda(E,Y)$ then $\mathscr{V}_\Gamma(E,Y)$ is σ-strongly porous subset of the space $\mathscr{V}_\Lambda(E,Y)$.*

Proof. Let

$$A_n = \{f \in \mathscr{V}_\Lambda(E,Y) \colon V_\Gamma(f,E) \leqslant n\}, \quad n \in \mathbb{N}.$$

We shall show that A_n is closed in $\mathscr{V}_\Lambda(E,Y)$ for each positive integer n. Indeed. If $f \in \overline{A_n}^{\Lambda}$, then there exists a sequence $(f_j)_{j=1}^\infty$ such that

(i) $f_j \in A_n$ if $j \in \mathbb{N}$,

(ii) $\|f_j - f\|_\Lambda \to 0$ if $j \to \infty$.

The condition (ii) implies that $(f_j)_{j=1}^\infty$ converges uniformly to f on E and by Lemma 1 we get

$$V_\Gamma(f,E) \leqslant \liminf_{j\to\infty} V_\Gamma(f_j,E) \leqslant n .$$

In consequence, $f \in A_n$. We have showed the inclusion $\overline{A}_n^{\Lambda} \subset A_n$ which implies that A_n is closed in $\mathscr{V}_\Lambda(E,Y)$. It is clear that

$$\mathscr{V}_\Gamma(E,Y) = \bigcup_{n=1}^{\infty} A_n\,.$$

Since $\mathscr{V}_\Gamma(E,Y)$ is a proper subgroup of the complete space $\mathscr{V}_\Lambda(E,Y)$ and it is F_σ set, then according to Banach's subgroup theorem, $\mathscr{V}_\Gamma(E,Y)$ is the set of first category in $\mathscr{V}_\Lambda(E,Y)$. Hence the set $\mathscr{V}_\Gamma(E,Y)$ is boundary in $\mathscr{V}_\Lambda(E,Y)$. Since A_n is closed and $A_n \subset \mathscr{V}_\Gamma(E,Y)$ for all positive integers n, then the sets A_n are nowhere dense. It is obvious that A_n are also convex in $\mathscr{V}_\Lambda(E,Y)$. By Theorem 1 the set A_n is strongly porous in the space $\mathscr{V}_\Lambda(E,Y)$ for each positive integer n.

As a result we infer that the set $\mathscr{V}_\Gamma(E,Y)$ is σ-strongly porous in $\mathscr{V}_\Lambda(E,Y)$. $\square$

It is known from [4] that if Λ and Γ, where $\Lambda = (\lambda_i)_{i=1}^{\infty}$ and $\Gamma = (\gamma_i)_{i=1}^{\infty}$, are two λ-sequences, then $\mathscr{V}_\Lambda(E,Y) \cap \mathscr{V}_\Gamma(E,Y) = \mathscr{V}_\Sigma(E,Y)$, where $\Sigma = \left(\dfrac{\lambda_i\gamma_i}{\lambda_i+\gamma_i}\right)_{i=1}^{\infty}$. From that and the previous theorem we get:

COROLLARY 1. *If $\mathscr{V}_\Lambda(E,Y)\cap\mathscr{V}_\Gamma(E,Y) \subsetneq \mathscr{V}_\Lambda(E,Y)$, then $\mathscr{V}_\Lambda(E,Y)\cap\mathscr{V}_\Gamma(E,Y)$ is a σ-strongly porous set in the space $\mathscr{V}_\Lambda(E,Y)$.*

COROLLARY 2. *The set $\mathscr{V}(E,Y)$ is σ-strongly porous in $\mathscr{V}_\Lambda(E,Y)$ for any proper λ-sequence Λ.*

By the symbol $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ we will denote the set of all continuous functions of bounded Λ-variation. It is known ([6]) that $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ is a closed subset and a nontrivial linear subspace of $\mathscr{V}_\Lambda(E,Y)$. The closure of A in $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ will be denoted by $\overline{A}^{C\Lambda}$.

THEOREM 4. *$\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ is a strongly porous set in the space $\mathscr{V}_\Lambda(E,Y)$ for every λ-sequence Λ.*

P r o o f. From Banach's subgroup theorem we infer that $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ is the set of the first category or it fails to satisfy the condition of Baire. Since $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ is closed, it satisfies the condition of Baire and hence $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ is the set of first category in $\mathscr{V}_\Lambda(E,Y)$. Since $\mathscr{V}_\Lambda(E,Y)$ is a Banach space and from the theorem of Baire we infer that $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ is a boundary set in $\mathscr{V}_\Lambda(E,Y)$. In consequence it is a nowhere dense subset of $\mathscr{V}_\Lambda(E,Y)$. Since $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ is a linear subspace, it is convex in $\mathscr{V}_\Lambda(E,Y)$. From Theorem 1 follows that $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ is strongly porous in the space $\mathscr{V}_\Lambda(E,Y)$. $\square$

Theorem 5. *Let the inclusion $\mathscr{C}\mathscr{V}_\Gamma(E,Y) \subsetneq \mathscr{C}\mathscr{V}_\Lambda(E,Y)$ holds. Then $\mathscr{C}\mathscr{V}_\Gamma(E,Y)$ is σ-strongly porous subset of $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$.*

Proof. Let

$$A_n = \{f \in \mathscr{C}\mathscr{V}_\Lambda(E,Y) \colon V_\Gamma(f,E) \leqslant n\}, \quad n \in \mathbb{N}.$$

We shall show that A_n is closed in the space $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ for each positive integer n.

If $f \in \overline{A_n}^{C\Lambda}$, then there exists a sequence $(f_j)_{j=1}^\infty$ of continuous functions of the bounded Γ-variation in the set E such that

(i) $V_\Gamma(f_j,E) \leqslant n$ if $j \in \mathbb{N}$,

(ii) $\|f_j - f\|_\Lambda \to 0$ as $j \to \infty$.

Then $(f_j)_{j=1}^\infty$ converges uniformly to f on E. Thus the function f is continuous and

$$V_\Gamma(f,E) \leqslant \liminf_{j\to\infty} V_\Gamma(f_j,E) \leqslant n.$$

Hence $f \in A_n$. As a result we obtain that A_n is closed in $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ for every positive integer n.

We only have to prove that the sets A_n are boundary in $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$. We can see that

$$\mathscr{C}\mathscr{V}_\Gamma(E,Y) = \bigcup_{n=1}^\infty A_n.$$

The set $\mathscr{C}\mathscr{V}_\Gamma(E,Y)$ is a proper subgroup of the space $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$. Since $\mathscr{C}\mathscr{V}_\Gamma(E,Y)$ is F_σ set in the complete space $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$, so from Banach's subgroup theorem we infer that the set $\mathscr{C}\mathscr{V}_\Gamma(E,Y)$ is of the first category in $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$. From the theorem of Baire we have that $\mathscr{C}\mathscr{V}_\Gamma(E,Y)$ is boundary in $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$. In consequence, A_n is nowhere dense subset of $\mathscr{C}\mathscr{V}_\Gamma(E,Y)$ if $n \in \mathbb{N}$.

Let us consider numbers α, β and functions f, g from A_n such that $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\alpha + \beta = 1$. Then the function $\alpha f + \beta g$ is continuous on E and $V_\Lambda(\alpha f + \beta g, E) \leqslant n$. Hence the set A_n is convex in $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$ for each positive integer n. It follows from Olevskii's theorem that the set $\mathscr{C}\mathscr{V}_\Gamma(E,Y)$ is σ-strongly porous in the space $\mathscr{C}\mathscr{V}_\Lambda(E,Y)$. $\square$

It is easy to see that the following theorem is a consequence of Theorem 5.

Theorem 6. ([5]) *If $\mathscr{V}_\Gamma(E,Y) \subsetneq \mathscr{V}_\Lambda(E,Y)$, then the set $\mathscr{C}\mathscr{V}_\Gamma(E,Y)$ is of first Baire category in $(\mathscr{C}\mathscr{V}_\Lambda(E,Y), \|\cdot\|_\Lambda)$.*

In the sequel, the symbol $\mathscr{U}\mathscr{C}\mathscr{V}_\Lambda(E,\mathbb{R})$ (resp. $\mathscr{L}\mathscr{C}\mathscr{V}_\Lambda(E,\mathbb{R})$) will denote the set of all real upper-continuous (resp. lower-continuous) functions of bounded Λ-variation defined on the set E. We will apply the symbol E^d to denote the set of all accumulation points of E.

THEOREM 7. *If $E \subset E^d$, then for any λ-sequence Λ the set $\mathscr{UCV}_\Lambda(E,\mathbb{R})$ is strongly porous in $\mathscr{V}_\Lambda(E,\mathbb{R})$.*

P r o o f . First we shall show that the set $\mathscr{UCV}_\Lambda(E,\mathbb{R})$ is closed in $\mathscr{V}_\Lambda(E,\mathbb{R})$. Let us consider a function f from $\overline{\mathscr{UCV}_\Lambda(E,\mathbb{R})}^\Lambda$. There exists a sequence $(f_j)_{j=1}^\infty$ of upper-continuous functions of bounded Λ-variation such that $\|f_j - f\|_\Lambda \to 0$ as $j \to \infty$. It is clear that $f \in \mathscr{V}_\Lambda(E,\mathbb{R})$ and since f is a limit of uniformly convergent sequence of upper-continuous functions, then f is upper-continuous on E. As a result $f \in \mathscr{UCV}_\Lambda(E,\mathbb{R})$. Consequently, we infer that the set $\mathscr{UCV}_\Lambda(E,\mathbb{R})$ is closed in $\mathscr{V}_\Lambda(E,\mathbb{R})$.

Let us denote

$$B_n = \{f \in \mathscr{UCV}_\Lambda(E,\mathbb{R}) \colon V_\Lambda(f,E) \leqslant n\}, \qquad n \in \mathbb{N}.$$

Thus

$$\mathscr{UCV}_\Lambda(E,\mathbb{R}) = \bigcup_{n=1}^\infty B_n .$$

Notice that if $f \in \overline{B}_n^\Lambda$, then there exists a sequence $(f_j)_{j=1}^\infty$ such that $\|f - f_j\|_\Lambda \to 0$ as $j \to \infty$ and $f_j \in \mathscr{UCV}_\Lambda(E,\mathbb{R})$ for each positive integer j. Then f is upper-continuous on the set E and $V_\Lambda(f,E) \leqslant n$, i.e., $f \in B_n$. Hence B_n is closed in $\mathscr{V}_\Lambda(E,\mathbb{R})$ for each positive integer n.

Now, we shall prove that B_n is a boundary set in $\mathscr{V}_\Lambda(E,\mathbb{R})$ for all $n \in \mathbb{N}$. Choose $\varepsilon > 0$ and $f \in B_n$. We have to show that there exists a function $h \in \mathscr{V}_\Lambda(E,\mathbb{R}) \setminus B_n$ such that $\|f - h\|_\Lambda < \varepsilon$. Let us consider any point x_0 from E and a neighbourhood U_0 of the point x_0, such that $a \notin U_0$.

Set

$$m = \inf\{f(x) \colon x \in U_0\} .$$

There exists a point z_0 in U_0 for which the inequality $|m - f(z_0)| < \frac{\varepsilon\lambda_1}{4}$ holds. Let $h \colon E \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} m - \frac{\lambda_1 \varepsilon}{4} & \text{if } x = z_0 , \\ f(x) & \text{if } x \neq z_0 . \end{cases}$$

Then

$$\limsup_{x \to z_0} h(x) = \limsup_{x \to z_0} f(x) \geqslant m > h(z_0) ,$$

and it means that the function h is not upper-continuous.

Of course, $h \in \mathscr{V}_\Lambda(E,\mathbb{R})$, since

$$V_\Lambda(h,E) \leqslant V_\Lambda(f,E) + 2 \cdot \left| m - f(z_0) - \frac{\lambda_1 \varepsilon}{4} \right| < \infty .$$

Hence $h \in \mathscr{V}_\Lambda(E,\mathbb{R}) \setminus B_n$.

Moreover,

$$(f-h)(x) = \begin{cases} f(z_0) - m + \frac{\lambda_1 \varepsilon}{4} & \text{if } x = z_0\,, \\ 0 & \text{if } x \neq z_0\,. \end{cases}$$

Then

$$\begin{aligned} \|f-h\|_\Lambda = \big|(f-h)(a)\big| + V_\Lambda(f-h,E) &\leqslant 2\cdot \frac{\big|f(z_0)-m+\frac{\lambda_1\varepsilon}{4}\big|}{\lambda_1} \\ &\leqslant 2\cdot \frac{\big|f(z_0)-m\big| + \frac{\lambda_1\varepsilon}{4}}{\lambda_1} < \varepsilon\,. \end{aligned}$$

In consequence, the set B_n is boundary for each positive integer n.

Previous facts imply that the set $\mathscr{UCV}_\Lambda(E,\mathbb{R})$ is nowhere dense in $\mathscr{V}_\Lambda(E,\mathbb{R})$.

We see that for nonnegative α, β such that $\alpha+\beta=1$ and for any functions f, g from $\mathscr{UCV}_\Lambda(E,\mathbb{R})$, the function $\alpha f+\beta g$ belongs to $\mathscr{UCV}_\Lambda(E,\mathbb{R})$, i.e., the set $\mathscr{UCV}_\Lambda(E,\mathbb{R})$ is convex in $\mathscr{V}_\Lambda(E,\mathbb{R})$.

At last the set $\mathscr{UCV}_\Lambda(E,\mathbb{R})$ is strongly porous in the space $\mathscr{V}_\Lambda(E,\mathbb{R})$ and the proof is complete. □

Corollary 3. *If $E \subset E^d$, then the set $\mathscr{LCV}_\Lambda(E,\mathbb{R})$ is strongly porous in $\mathscr{V}_\Lambda(E,\mathbb{R})$ for any λ-sequence Λ.*

Now, we consider the space $\mathscr{B}(E,Y)$ of bounded mappings on a subset E of the real line to a Banach space Y with the norm

$$\|f\|_B = \sup\big\{\|f(t)\| : t\in E\big\}\,.$$

It is known that $\mathscr{B}(E,Y)$ is a Banach space. We will use the symbol $\overline{A}^B$ to denote the closure of a set A in the space $\mathscr{B}(E,Y)$.

Lemma 2. *The set $\mathscr{V}_\Lambda(E,Y)$ is not dense in $\mathscr{B}(E,Y)$ for each λ-sequence Λ.*

P r o o f. Suppose that $\mathscr{V}_\Lambda(E,Y)$ is a dense subset of $\mathscr{B}(E,Y)$. Let us take a sequence $(\alpha_i)_{i=1}^\infty$ such that

(i) $\alpha_i \in E$,

(ii) $\alpha_i < \alpha_{i+1}$, $\quad i\in\{1,2,\dots\}$.

Let $f\colon E\to Y$ be given by

$$f(x) = \begin{cases} \boldsymbol{u} & \text{if } x = \alpha_{2i}\,, \\ \Theta & \text{if } x \neq \alpha_{2i}\,, \end{cases}$$

where $\boldsymbol{u}$ is a fixed point from Y, such that $\|\boldsymbol{u}\| = 1$.

Since $\mathscr{V}_\Lambda(E,Y)$ is dense, then there exists a sequence $(f_n)_{n=1}^\infty$ of Λ-bounded mappings such that

$$\|f_n - f\|_B \to 0 \qquad \text{if } n\to\infty\,.$$

Hence for ε such that $0 < \varepsilon < \frac{1}{3}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and every $x \in E$ we have

$$\left\|(f_n - f)(x)\right\| < \varepsilon ,$$

and then

$$\Big| \left\|f_n(x)\right\| - \left\|f(x)\right\| \Big| < \varepsilon .$$

Observe that for $x = \alpha_{2i}$, $i \in \mathbb{N}$, we obtain

$$\left\|f_n(\alpha_{2i})\right\| \geqslant \left\|f(\alpha_{2i})\right\| - \varepsilon = \|\boldsymbol{u}\| - \varepsilon = 1 - \varepsilon ,$$

and for α_{2i-1}, $i \in \mathbb{N}$,

$$\left\|f_n(\alpha_{2i-1})\right\| \leqslant \left\|f(\alpha_{2i-1})\right\| + \varepsilon = \|\Theta\| + \varepsilon = \varepsilon .$$

Thus we obtain

$$\left\|f_n(\alpha_{2i}) - f_n(\alpha_{2i-1})\right\| \geqslant 1 - 2\varepsilon$$

for all $n \in \mathbb{N}$. Let us consider the partitions

$$s_k = \left\{[\alpha_{2i-1}, \alpha_{2i}] \colon i \in \{1, \dots, k\}\right\} , \quad k \in \mathbb{N} .$$

Then for all $k \in \mathbb{N}$ we have

$$\sum_{i=1}^{k} \frac{\left\|f_n(\alpha_{2i}) - f_n(\alpha_{2i-1})\right\|}{\lambda_i} \geqslant \sum_{i=1}^{k} \frac{1 - 2\varepsilon}{\lambda_i} > \varepsilon \sum_{i=1}^{k} \frac{1}{\lambda_i} .$$

It is clear that if we take the sup for all positive integers k and partitions s_k, we will get the equality $V_\Lambda(f_n, E) = \infty$, which contradicts the fact that f_n is of bounded Λ-variation. $\square$

For the subsets A and B of a linear space X the symbol $A - B$ will mean the set

$$\{x - y \colon x \in A \wedge y \in B\} .$$

Now, we can prove the following

THEOREM 8. *The set $\mathscr{V}_\Lambda(E,Y)$ is strongly porous in the space $\mathscr{B}(E,Y)$ for each λ-sequence Λ.*

P r o o f. Observe that the inclusion $\overline{\mathscr{V}_\Lambda(E,Y)}^B - \overline{\mathcal{V}_\Lambda(E,Y)}^B \subset \overline{\mathcal{V}_\Lambda(E,Y)}^B$ holds. Let us consider a function f from $\overline{\mathcal{V}_\Lambda(E,Y)}^B - \overline{\mathcal{V}_\Lambda(E,Y)}^B$. Then $f = g - h$, where f, g belong to $\overline{\mathcal{V}_\Lambda(E,Y)}^B$. There exist sequences $(g_n)_{n=1}^\infty$ and $(h_n)_{n=1}^\infty$ of functions of bounded Λ-variation such that

$$\|g_n - g\|_B \to 0 \quad \text{and} \quad \|h_n - h\|_B \to 0 \qquad \text{if } n \to \infty .$$

Let $f_n = g_n - h_n$. It follows from previous facts, that $f_n \in \mathscr{V}_\Lambda(E,Y)$ and

$$\|f_n - f\|_B \leqslant \|g_n - g\|_B + \|h_n - h\|_B \to 0\,, \qquad \text{where } n \to \infty\,,$$

which implies that $f \in \overline{\mathcal{V}_\Lambda(E,Y)}^B$.

According to Banach's subgroup theorem, the set $\overline{\mathcal{V}_\Lambda(E,Y)}^B$ is of the first category in $\mathscr{B}(E,Y)$. Since $\mathscr{B}(E,Y)$ is a Banach space, then $\overline{\mathcal{V}_\Lambda(E,Y)}^B$ is a boundary set in $\mathscr{B}(E,Y)$. As a result, we infer that $\mathscr{V}_\Lambda(E,Y)$ is nowhere dense. Since $\mathscr{V}_\Lambda(E,Y)$ is a linear subspace of the space $\mathscr{B}(E,Y)$, then $\mathscr{V}_\Lambda(E,Y)$ is convex in $\mathscr{B}(E,Y)$. In consequence, the set $\mathscr{V}_\Lambda(E,Y)$ is strongly porous in the space $\mathscr{B}(E,Y)$. $\square$

REFERENCES

[1] CHISTYAKOV, V. V.—SOLYCHEVA, O. M.: *Lipschitzian operators of substitution in the algebra ΛBV*, J. Differ. Equations Appl. **29** (2002), 404–416.

[2] KELLEY, J. L.—NAMIOKA, J.: *Linear Topological Spaces*, D. van Nostrand Company, Inc., Princeton, New Jersey, 1963.

[3] OLEVSKII, J.: *A note on the Banach-Stainhaus theorem*, Real Anal. Exchange **17** (1991–1992), 399-401.

[4] PERLMAN, P. S.: *Functions of generalized variation*, Fund. Math. **105** (1980), 199–211.

[5] PRUS-WIŚNIOWSKI, F.: *Λ variation and Baire category*, Real Anal. Exchange **20** (1994–1995), 134–139.

[6] WATERMAN, D.: *On Λ-bounded variation*, Studia Math. **57** (1976), 33–45.

Received October 11, 2004

Institute of Mathematics
Academia Pomeraniensis
Arciszewskiego 22 b
PL–76-200 Słupsk
POLAND
E-mail: r.drozdowski@wp.pl