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ON GRAPH QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. In the paper [Z. Grande: *Sur les fonctions A-continues*, Demonstratio Math. **11** (1978), 519–526] a concept of graph continuous functions was introduced. An idea of graph quasi-continuous functions and their properties was investigated in [A. Mikucka: *Graph quasi-continuity*, Demonstratio Math. **36** (2003), 483–494]. In this paper different properties of these functions are studied.

Let X be a topological space, $(Y,\mathcal{P})$ a uniform one and $\mathcal{P}$ a family of pseudometrics on Y inducing the uniformity. In this paper by $G(f)$ we denote the graph of a function $f\colon X \to (Y,\mathcal{P})$ and the symbol $\overline{G(f)}$ denotes the closure of $G(f)$ in a topological space $(X \times Y, \mathcal{T}_X \times \mathcal{T}_{\mathcal{P}})$. The letters $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ stand for the set of natural, rational and real numbers, respectively.

A set $A \subset X$ is said to be:

- *semi-open* if $A \subset \overline{\operatorname{Int} A}$, [11];
- *semi-closed* if $X \setminus A$ is semi-open, [4];
- *pre-open* if $A \subset \operatorname{Int} \overline{A}$, [2];
- *semi-neighbourhood of* x_0 if A is semi-open and $x_0 \in A$.

Lemma 1. ([12])

(a) *Each nowhere dense set is semi-closed.*

(b) *If a set W is semi-open and dense, a set A is dense, then the set $W \cap A$ is dense.*

A function $f\colon X \to Y$ is said to be:

- *graph continuous* if there exists a continuous function g such that $G(g) \subset \overline{G(f)}$, [8];
- *graph quasi-continuous* if there exists a quasi-continuous function $g\colon X \to Y$ such that $G(g) \subset \overline{G(f)}$, [12];
- *quasi-continuous* (*almost quasi-continuous*) at $x_0 \in X$ if for each $\varrho \in \mathcal{P}$, $\varepsilon > 0$, and for each neighbourhood U of x_0 there exists a non-empty

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open (pre-open) set $U_1 \subset U$ such that $\varrho\big(f(x), f(x_0)\big) < \varepsilon$ for each $x \in U_1$, [14] ([15]);

- *cliquish* at x_0 if for each $\varrho \in \mathcal{P}$, $\varepsilon > 0$, and for each neighbourhood $U \subset X$ of x_0 there exists a non-empty open set $U_1 \subset U$ such that $\varrho\big(f(x_1), f(x_2)\big) < \varepsilon$ for each $x_1, x_2 \in U_1$, [6].

A function f is called *quasi-continuous* (*almost quasi-continuous, cliquish*) if it has this property at each point.

If $\varrho \in \mathcal{P}$ and $\varepsilon > 0$, we denote:

$$\begin{aligned} A(f,g,\varrho,\varepsilon) &= \big\{x \in X \colon \varrho\big(f(x), g(x)\big) < \varepsilon\big\},\\ B(f,g,\varrho,\varepsilon) &= \big\{x \in X \colon \varrho\big(f(x), g(x)\big) \geqslant \varepsilon\big\},\\ K(y,\varrho,\varepsilon) &= \big\{z \in Y \colon \varrho(y,z) < \varepsilon\big\}. \end{aligned}$$

Now, we give a characterization of the graph quasi-continuity:

Theorem 1. *Let X be a topological space, $(Y,\mathcal{P})$ a uniform one and $f,g \colon X \to Y$. If g is quasi-continuous, then the following conditions are equivalent:*

(a) *$G(g) \subset \overline{G(f)}$;*

(b) *the set $A(f,g,\varrho,\varepsilon)$ is dense in X for each $\varrho \in \mathcal{P}$ and for each $\varepsilon > 0$.*

Proof. (a)$\Rightarrow$(b). Let us fix $\varrho \in \mathcal{P}$, $\varepsilon > 0$, a non-empty open set $U \subset X$, and $x_0 \in U$. By quasi-continuity of g at x_0 there exists a non-empty open set $U_1 \subset U$ and $\varrho\big(g(x_0), g(x)\big) < \frac{\varepsilon}{3}$ for each $x \in U_1$. We choose $\big(x_1, g(x_1)\big) \in G(g)$ such that $x_1 \in U_1$. As $G(g) \subset \overline{G(f)}$, we have $\big(U_1 \times K\big(g(x_1), \varrho, \frac{\varepsilon}{3}\big)\big) \cap G(f) \neq \emptyset$. So there exists $x_2 \in U_1$ satisfying the inequality $\varrho\big(g(x_1), f(x_2)\big) < \frac{\varepsilon}{3}$. From above it follows that $\varrho\big(f(x_2), g(x_2)\big) \leqslant \varrho\big(f(x_2), g(x_1)\big) + \varrho\big(g(x_1), g(x_0)\big) + \varrho\big(g(x_0), g(x_2)\big) < \varepsilon$. We conclude that $x_2 \in A(f,g,\varrho,\varepsilon) \cap U$, and finally the set $A(f,g,\varrho,\varepsilon)$ is dense in X.

(b)$\Rightarrow$(a). Let us take $\varrho \in \mathcal{P}$, $\varepsilon > 0$, $\big(x_0, g(x_0)\big) \in G(g)$ and a neighbourhood $U \times K\big(g(x_0), \varrho, \frac{\varepsilon}{2}\big)$ of $\big(x_0, g(x_0)\big)$. The quasi-continuity of g at x_0 gives that there exists a non-empty open set $U_1 \subset U$ such that $\varrho\big(g(x_0), g(x)\big) < \frac{\varepsilon}{2}$ for each $x \in U_1$. According to the definition of the set $A\big(f,g,\varrho,\frac{\varepsilon}{2}\big)$, we can find $x_1 \in U_1$ with $\varrho\big(f(x_1), g(x_1)\big) < \frac{\varepsilon}{2}$. According to the above we have $\varrho\big(g(x_0), f(x_1)\big) \leqslant \varrho\big(g(x_0), g(x_1)\big) + \varrho\big(g(x_1), f(x_1)\big) < \varepsilon$. Hence $\big(x_0, g(x_0)\big) \in \overline{G(f)}$. This gives an inclusion $G(g) \subset \overline{G(f)}$ and the proof is completed. □

Theorem 2. *Let X be a topological space, $(Y,\mathcal{P})$ a uniform one and $f,g \colon X \to Y$. If g is quasi-continuous, then the following conditions are equivalent:*

(a) *$G(g) \subset \overline{G(f)}$ and f is cliquish;*

(b) *the set* $B(f,g,\varrho,\varepsilon)$ *is nowhere dense in* X *for each* $\varrho \in \mathcal{P}$ *and for each* $\varepsilon > 0$.

Proof. (a)$\Rightarrow$(b). Let $\varrho \in \mathcal{P}$, $\varepsilon > 0$, a non-empty open set $U \subset X$, and $x_0 \in U$. As f is cliquish, there exists a non-empty open set $U_1 \subset U$ such that $\varrho\bigl(f(x_1), f(x_2)\bigr) < \frac{\varepsilon}{3}$ for each $x_1, x_2 \in U_1$. Suppose that $U_1 \cap B(f,g,\varrho,\varepsilon) \neq \emptyset$. So there exists $x_1 \in U_1$ satisfying the condition

$$\varrho\bigl(f(x_1), g(x_1)\bigr) \geqslant \varepsilon . \tag{1}$$

By a quasi-continuity of g at x_1 there exists a non-empty open set $U_2 \subset U_1$ such that $\varrho\bigl(g(x_1), g(x)\bigr) < \frac{\varepsilon}{3}$ for each $x \in U_2$. Theorem 1 now yields that the set $A\bigl(f,g,\varrho,\frac{\varepsilon}{3}\bigr)$ is dense in X. So we can find $x_2 \in U_2$ with $\varrho\bigl(f(x_2), g(x_2)\bigr) < \frac{\varepsilon}{3}$. Then we obtain $\varrho\bigl(g(x_1), f(x_1)\bigr) \leqslant \varrho\bigl(g(x_1), g(x_2)\bigr) + \varrho\bigl(g(x_2), f(x_2)\bigr) + \varrho\bigl(f(x_2), f(x_1)\bigr) < \varepsilon$. This contradicts (1), so the set $B(f,g,\varrho,\varepsilon)$ is nowhere dense in X.

(b)$\Rightarrow$(a). Let us fix $\varrho \in \mathcal{P}$, $\varepsilon > 0$, a point $x_0 \in U$ and a neighbourhood U of x_0. Because g is quasi-continuous at x_0 there exists a non-empty open set $U_1 \subset U$ such that $\varrho\bigl(g(x), g(x_0)\bigr) < \frac{\varepsilon}{4}$ for each $x \in U_1$. As $B\bigl(f,g,\varrho,\frac{\varepsilon}{4}\bigr)$ is nowhere dense, we can find a non-empty open set $U_2 \subset U_1$ such that $\varrho\bigl(f(x), g(x)\bigr) < \frac{\varepsilon}{4}$ for each $x \in U_2$. Finally, we take $x_1, x_2 \in U_2$. This leads to the condition $\varrho\bigl(f(x_1), f(x_2)\bigr) \leqslant \varrho\bigl(f(x_1), g(x_1)\bigr) + \varrho\bigl(g(x_1), g(x_0)\bigr) + \varrho\bigl(g(x_0), g(x_2)\bigr) + \varrho\bigl(g(x_2), f(x_2)\bigr) < \varepsilon$. It means that f is cliquish. Because $B(f,g,\varrho,\varepsilon)$ is nowhere dense, the set $X \setminus B(f,g,\varrho,\varepsilon) = A(f,g,\varrho,\varepsilon)$ is dense in X. In virtue of Theorem 1 we obtain $G(g) \subset \overline{G(f)}$ and the proof is completed. $\square$

The following example shows that the assumption of the cliquishness of f in Theorem 2 cannot be omitted.

EXAMPLE 1. Let $\mathbb{R}$ be the space of real numbers with the natural topology and $f, g\colon \mathbb{R} \to \mathbb{R}$ be defined by: $f(x) = 1$ for $x \in \mathbb{Q}$, $f(x) = 2$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ and $g(x) = 1$ for each $x \in \mathbb{R}$. Then $G(g) \subset \overline{G(f)}$ holds but the set $B(f,g,\varrho,\varepsilon) = \mathbb{R} \setminus \mathbb{Q}$ is not nowhere dense in $\mathbb{R}$.

THEOREM 3. *Let* X *be a topological space and* $(Y,\mathcal{P})$ *a Hausdorff uniform one. If* $f\colon X \to Y$ *is cliquish, then there exists a unique continuous function* $g\colon X \to Y$ *such that* $G(g) \subset \overline{G(f)}$.

The proof is exactly as in case if Y is a metric space, [18].

THEOREM 4. *Let* X *be a topological space and* $(Y,\mathcal{P})$ *a Hausdorff uniform one. If* $f\colon X \to Y$ *is cliquish,* $g\colon X \to Y$ *is continuous,* $h\colon X \to Y$ *is almost quasi-continuous and* $G(g) \cup G(h) \subset \overline{G(f)}$, *then* $g = h$.

Proof. Suppose that $g \neq h$. Then there exists $x_0 \in X$ such that $g(x_0) \neq h(x_0)$. We can find a pseudometric $\varrho \in \mathcal{P}$ and $\varepsilon > 0$ with the property

$$\varrho\bigl(g(x_0), h(x_0)\bigr) \geqslant \varepsilon . \tag{2}$$

By the continuity of g at x_0 there exists a neighbourhood U of x_0 such that $\varrho\big(g(x_0), g(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U$. Because h is almost quasi-continuous at x_0 there exists a non-empty pre-open set $U_1 \subset U$ such that $\varrho\big(h(x_0), h(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U_1$. In virtue of Theorem 2 the set $B\big(f, g, \varrho, \frac{\varepsilon}{4}\big)$ is nowhere dense in X. So we can find a non-empty open set $U_2 \subset U \cap \mathrm{Int}\,\overline{U_1}$ with $\varrho\big(f(x), g(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U_2$. The set $U_1 \cap U_2$ is non-empty, so let us take $x_1 \in U_1 \cap U_2$. Since $G(h) \subset \overline{G(f)}$ we can find $x_2 \in U_2$ such that $\varrho\big(h(x_1), f(x_2)\big) < \frac{\varepsilon}{4}$. Finally we have

$$\begin{aligned}\varrho\big(g(x_0), h(x_0)\big) &\leqslant \varrho\big(g(x_0), g(x_2)\big) + \varrho\big(g(x_2), f(x_2)\big)\\ &\quad + \varrho\big(f(x_2), h(x_1)\big) + \varrho\big(h(x_1), h(x_0)\big) < \varepsilon\,.\end{aligned}$$

This contradicts (2), which means that $g(x_0) = h(x_0)$, and in consequence, $g = h$. The proof is completed. $\square$

In Theorem 3, the continuity of g cannot be replaced with the quasi-continuity, as it is shown in [12]. In the next theorem we will show that if we have additional assumptions then in $\overline{G(f)}$ there exists a unique graph of quasi-continuous function.

Definition 1. ([5]) A topological space X is said to be *extremally disconnected* if the closure of each open set $U \subset X$ is open.

Lemma 2. *A topological space X is extremally disconnected if and only if for each non-empty open set $U, V \subset X$ we have: $\overline{U} \cap \overline{V} = \overline{U \cap V}$.*

The simple proof is omitted.

Lemma 3. *A topological space X is extremally disconnected if and only if a set $A \cap B$ is semi-open for each semi-open sets $A, B \subset X$.*

P r o o f. Assume that X is extremally disconnected. We take semi-open sets $A, B \subset X$. So, $A \subset \overline{\mathrm{Int}\,A}$ and $B \subset \overline{\mathrm{Int}\,B}$. By Lemma 2 we have $\overline{\mathrm{Int}(A \cap B)} = \overline{\mathrm{Int}\,A \cap \mathrm{Int}\,B} = \overline{\mathrm{Int}\,A} \cap \overline{\mathrm{Int}\,B} \supset A \cap B$. It means that $A \cap B$ is semi-open. Now, assume that $A \cap B$ is semi-open for semi-open sets $A, B \subset X$. We take open sets $A, B \subset X$. The sets $\overline{A}, \overline{B}$ are semi-open, so by assumption, the set $\overline{A} \cap \overline{B}$ is semi-open. Thus we have $\overline{A} \cap \overline{B} \subset \overline{\mathrm{Int}\,(\overline{A} \cap \overline{B})}$. So we have $\overline{A} \cap \overline{B} \subset \overline{\mathrm{Int}(\overline{A} \cap \overline{B})} = \overline{\mathrm{Int}\,\overline{A} \cap \mathrm{Int}\,\overline{B}} = \overline{\mathrm{Int}(\overline{A} \cap \mathrm{Int}\,\overline{B})} \subset \overline{\mathrm{Int}(\overline{\overline{A} \cap \mathrm{Int}\,\overline{B}})} = \overline{\mathrm{Int}(\overline{A \cap \mathrm{Int}\,\overline{B}})} \subset \overline{\mathrm{Int}(\overline{A \cap \overline{B}})} = \overline{\mathrm{Int}\,(\overline{A \cap B})} \subset \overline{A \cap B}$. The inverse inclusion is evident, so we have $\overline{A} \cap \overline{B} = \overline{A \cap B}$. It means that X is extremally disconnected. $\square$

Theorem 5. *Let X be a topological space. Then the following conditions are equivalent:*

(a) *X is an extremally disconnected space;*

(b) *for each Hausdorff uniform space $(Y,\mathcal{P})$ and for each graph quasi-continuous and cliquish function $f\colon X \to Y$ there exists a unique quasi-continuous function $g\colon X \to Y$ such that $G(g) \subset \overline{G(f)}$;*

(c) *for each graph quasi-continuous and cliquish function $f\colon X \to [0,1]$ there exists a unique quasi-continuous function $g\colon X \to [0,1]$ such that $G(g) \subset \overline{G(f)}$.*

Proof. (a)$\Rightarrow$(b). Suppose that there exist different functions g_1, g_2 satisfying inclusion $G(g_1) \cup G(g_2) \subset \overline{G(f)}$. Because g_1, g_2 are different there exists $x_0 \in X$ for which $g_1(x_0) \neq g_2(x_0)$. If Y is a Hausdorff space we can find a pseudometric $\varrho \in \mathcal{P}$ and $\varepsilon > 0$ such that

$$\varrho\big(g_1(x_0), g_2(x_0)\big) \geqslant \varepsilon\,. \tag{3}$$

Let us take a neighbourhood U of x_0. The quasi-continuity of g_1 implies the existence of a semi-neighbourhood $U_1 \subset U$ of x_0 such that $\varrho\big(g_1(x_0), g_1(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U_1$. Moreover, by quasi-continuity of g_2 we can find a semi-neighbourhood $U_2 \subset U$ such that $\varrho\big(g_2(x_0), g_2(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U_2$. According to Lemma 3, a set $U_1 \cap U_2$ is semi-open. We put $A = \operatorname{Int}(U_1 \cap U_2)$. It is evident that A is non-empty and open. By Theorem 2, the sets $B\big(f, g_1, \varrho, \frac{\varepsilon}{4}\big)$ and $B\big(f, g_2, \varrho, \frac{\varepsilon}{4}\big)$ are nowhere dense in X. So there exist non-empty open sets $U_3 \subset A$ and $U_4 \subset U_3$ such that $\varrho\big(g_1(x), f(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U_3$ and $\varrho\big(g_2(x), f(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U_4$. Hence for each $x \in U_4$ we have $\varrho\big(g_1(x_0), g_2(x_0)\big) \leqslant \varrho\big(g_1(x_0), g_1(x)\big) + \varrho\big(g_1(x), f(x)\big) + \varrho\big(f(x), g_2(x)\big) + \varrho\big(g_2(x), g_2(x_0)\big) < \varepsilon$. This contradicts (3), which means that $g_1(x_0) = g_2(x_0)$, and in consequence, functions g_1 and g_2 are equal.

The implication (b) $\Rightarrow$(c) is evident.

(c)$\Rightarrow$(a). Suppose that X is not extremally disconnected. So there exists a non-empty open set $U \subset X$ such that $\overline{U}$ is not open. We define $f\colon X \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{for } \ x \in \operatorname{Int}\overline{U}, \\ \frac{1}{2} & \text{for } \ x \in \overline{U} \setminus \operatorname{Int}\overline{U}, \\ 1 & \text{for } \ x \in X \setminus \overline{U}. \end{cases}$$

Let us remark that f is cliquish. Moreover, $\overline{G(f)} = \big(\overline{\operatorname{Int}\overline{U}} \times \{0\}\big) \cup \big((\overline{U} \setminus \operatorname{Int}\overline{U}) \times \{\frac{1}{2}\}\big) \cup \big(\overline{X \setminus \overline{U}} \times \{1\}\big)$.

Now, we define functions $g_1, g_2 \colon X \to [0,1]$ as follows:

$g_1(x) = 0$ for $x \in \operatorname{Int}\overline{U}$, $g_1(x) = 1$ otherwise and $g_2(x) = 0$ for $x \in \overline{U}$, $g_2(x) = 1$ otherwise. Hence $G(g_1) = \big(\operatorname{Int}\overline{U} \times \{0\}\big) \cup \big((X \setminus \operatorname{Int}\overline{U}) \times \{1\}\big)$ and $G(g_2) = (\overline{U} \times \{0\}) \cup \big((X \setminus \overline{U}) \times \{1\}\big)$. Because $\overline{U} = \overline{\operatorname{Int} U}$ we have the inclusion $G(g_1) \cup G(g_2) \subset \overline{G(f)}$. The functions g_1, g_2 are quasi-continuous so f is graph quasi-continuous. As the functions g_1, g_2 are different, we get the contradiction to (c), which finishes the proof. $\square$

Let $(X, \mathcal{T})$ be a topological space. We denote:

$$\mathcal{T}_q = \{U \setminus H : U \in \mathcal{T},\ H \text{ is of first category}\}\ ([9]),$$
$$\mathcal{T}_\alpha = \{U \setminus H : U \in \mathcal{T},\ H \text{ is nowhere dense}\}\ ([16]).$$

Lemma 4.

(a) ([7]) *Let $(X, \mathcal{T})$ be a Baire space and Y a regular one. A function $f : (X, \mathcal{T}) \to Y$ is quasi-continuous if and only if $f : (X, \mathcal{T}_q) \to Y$ is quasi-continuous.*

(b) ([16]) *Let $(X, \mathcal{T})$ be a topological space and Y a uniform one. A function $f : (X, \mathcal{T}) \to Y$ is quasi-continuous if and only if $f : (X, \mathcal{T}_\alpha) \to Y$ is quasi-continuous.*

Theorem 6. *Let $(X, \mathcal{T})$ be a Baire space and $(Y, \mathcal{P})$ a uniform one.*

(a) *If a function $f : (X, \mathcal{T}_q) \to Y$ is graph quasi-continuous then $f : (X, \mathcal{T}) \to Y$ is graph quasi-continuous.*

(b) *If a function $f : (X, \mathcal{T}) \to Y$ is graph quasi-continuous and cliquish then $f : (X, \mathcal{T}_q) \to Y$ is graph quasi-continuous.*

P r o o f. (a) Assume that $f : (X, \mathcal{T}_q) \to Y$ is graph quasi-continuous. Then there exists a quasi-continuous function $g : (X, \mathcal{T}_q) \to Y$ such that $G(g) \subset \overline{G(f)}^{\,q}$. By Lemma 4 (a) the function $g : (X, \mathcal{T}) \to Y$ is quasi-continuous, too. Since $\overline{G(f)}^{\,q} \subset \overline{G(f)}$, it follows that $G(g) \subset \overline{G(f)}$. It means that $f : (X, \mathcal{T}) \to Y$ is graph quasi-continuous.

(b) Assume that $f : (X, \mathcal{T}) \to Y$ is graph quasi-continuous and cliquish. Then there exists a quasi-continuous function g such that $G(g) \subset \overline{G(f)}$. Let $\varrho \in \mathcal{P}$ and $\varepsilon > 0$. According to Theorem 2 the set $A(f, g, \varrho, \varepsilon)$ is dense in X and semi-open. Let us take a non-empty set $U \setminus H \in \mathcal{T}_q$, so $U \in \mathcal{T}$, H is of the first category in X. Then the set $U \cap A(f, g, \varrho, \varepsilon)$ is non-empty and semi-open. Because X is a Baire space then $U \cap A(f, g, \varrho, \varepsilon)$ is of the second category. So $\big(U \cap A(f, g, \varrho, \varepsilon)\big) \setminus H \neq \emptyset$. Hence $(U \setminus H) \cap A(f, g, \varrho, \varepsilon) \neq \emptyset$. It means that the set $A(f, g, \varrho, \varepsilon)$ is dense in $(X, \mathcal{T}_q)$. With respect to Lemma 4 (a) the function $g : (X, \mathcal{T}_q) \to Y$ is quasi-continuous, too. Then $f : (X, \mathcal{T}_q) \to Y$ is graph quasi-continuous. $\square$

Theorem 7. *Let $(X,\mathcal{T})$ be a topological space and $(Y,\mathcal{P})$ be a uniform one. A function $f\colon (X,\mathcal{T}) \to Y$ is graph quasi-continuous if and only if a function $f\colon (X,\mathcal{T}_\alpha) \to Y$ is graph quasi-continuous.*

P r o o f. Assume that $f\colon (X,\mathcal{T}) \to Y$ is graph quasi-continuous. Let us take a non-empty set $U \setminus H \in \mathcal{T}_\alpha$, so $U \in \mathcal{T}$, H is nowhere dense in X. Let us remark that $U \cap (X \setminus \overline{H})$ is non-empty and open in $(X,\mathcal{T})$. Assuming on f there exists a quasi-continuous function $g\colon (X,\mathcal{T}) \to Y$ such that $G(g) \subset \overline{G(f)}$. Let $\varrho \in \mathcal{P}$ and $\varepsilon > 0$. By Theorem 1 the set $A(f,g,\varrho,\varepsilon)$ is dense in $(X,\mathcal{T})$. So we have $\big(U \cap (X \setminus H)\big) \cap A(f,g,\varrho,\varepsilon) \neq \emptyset$ since the set $\big(U \cap (X \setminus \overline{H})\big) \cap A(f,g,\varrho,\varepsilon)$ is non-empty. It means that the set $A(f,g,\varrho,\varepsilon)$ is dense in $(X,\mathcal{T}_\alpha)$. According to Lemma 4 (b) the function $g\colon (X,\mathcal{T}_\alpha) \to Y$ is quasi-continuous, too, so by Theorem 1, the function $f\colon (X,\mathcal{T}_\alpha) \to Y$ is graph quasi-continuous. The proof of the second part is the same as in Theorem 6 (a), we only use Lemma 4 (b). □

In [13] it is shown that the uniform convergence preserves neither graph continuity nor graph quasi-continuity. However, if we have additional assumptions, the quasi-uniform convergence preserves the graph quasi-continuity.

Definition 2. Let X be a topological space, $(Y,\mathcal{P})$ a uniform one, and let Σ denote a directed set. A net $\{f_\sigma\}_{\sigma\in\Sigma}$, $f_\sigma\colon X \to Y$, is called *quasi-uniformly convergent* to some $f\colon X \to Y$ at x_0 if for each $\varrho \in \mathcal{P}$, $\varepsilon > 0$ there exists $\sigma_0 \in \Sigma$ such that for each $\sigma \geqslant \sigma_0$ there exists a neighbourhood U_σ such that for each $x \in U_\sigma$ we have $\varrho\big(f_\sigma(x), f(x)\big) < \varepsilon$.

Theorem 8. *Let X be an extremally disconnected space and $(Y,\mathcal{P})$ a complete uniform one. If the net $\{f_\sigma\}_{\sigma\in\Sigma}$ of cliquish functions $f_\sigma\colon X \to Y$, $\sigma \in \Sigma$, is quasi-uniformly convergent to $f\colon X \to Y$, functions $g_\sigma\colon X \to Y$ are quasi-continuous and $G(g_\sigma) \subset \overline{G(f_\sigma)}$ for $\sigma \in \Sigma$, then:*

(a) *the net $\{g_\sigma\}_{\sigma\in\Sigma}$ is pointwise convergent to $g\colon X \to Y$;*

(b) *$G(g) \subset \overline{G(f)}$;*

(c) *the function g is quasi-continuous.*

P r o o f. (a) Let $\varrho \in \mathcal{P}$, $\varepsilon > 0$, and let $x_0 \in X$. First, we show that $\{g_\sigma\}_{\sigma\in\Sigma}$ is the Cauchy net. As the net $\{f_\sigma\}_{\sigma\in\Sigma}$ is quasi-uniformly convergent to f at x_0, we have the existence of $\sigma_0 \in \Sigma$ such that for $\sigma \geqslant \sigma_0$ there exists a neighbourhood U_σ of x_0 with the property $\varrho\big(f_\sigma(x), f(x)\big) < \frac{\varepsilon}{7}$ for each $x \in U_\sigma$. Let us take $\sigma_1, \sigma_2 \geqslant \sigma_0$, then a set $U_\sigma = U_{\sigma_1} \cap U_{\sigma_2}$ is a neighbourhood of x_0. Due to the quasi-continuity of g_{σ_1} at x_0 we can choose a semi-neighbourhood $U_1 \subset U_\sigma$ of x_0 such that $\varrho\big(g_{\sigma_1}(x_0), g_{\sigma_1}(x)\big) < \frac{\varepsilon}{7}$ for each $x \in U_1$. Moreover, g_{σ_2} is quasi-continuous at x_0 so we can choose a semi-neighbourhood $U_2 \subset U_\sigma$ of x_0 such that $\varrho\big(g_{\sigma_2}(x_0), g_{\sigma_2}(x)\big) < \frac{\varepsilon}{7}$ for each $x \in U_2$. Assuming X, the set

$U_1 \cap U_2$ is semi-open. We put $A = \mathrm{Int}(U_1 \cap U_2)$. Then A is non-empty and open. The function f_{σ_1} is cliquish which yields to the existence of a non-empty open set $W \subset A$ such that for $x_1, x_2 \in W$ we have $\varrho\big(f_{\sigma_1}(x_1), f_{\sigma_1}(x_2)\big) < \frac{\varepsilon}{7}$. The sets $A(f_{\sigma_1}, g_{\sigma_1}, \varrho, \frac{\varepsilon}{7})$ and $A(f_{\sigma_2}, g_{\sigma_2}, \varrho, \frac{\varepsilon}{7})$ are dense in X by Theorem 1, there exist $x_1, x_2 \in W$ and we have: $\varrho\big(f_{\sigma_1}(x_1), g_{\sigma_1}(x_1)\big) < \frac{\varepsilon}{7}$ and $\varrho\big(f_{\sigma_2}(x_2), g_{\sigma_2}(x_2)\big) < \frac{\varepsilon}{7}$.

From the above it follows that

$$\begin{aligned}\varrho\big(g_{\sigma_1}(x_0), g_{\sigma_2}(x_0)\big) \leqslant\ & \varrho\big(g_{\sigma_1}(x_0), g_{\sigma_1}(x_1)\big) + \varrho\big(g_{\sigma_1}(x_1), f_{\sigma_1}(x_1)\big)\\ &+ \varrho\big(f_{\sigma_1}(x_1), f_{\sigma_1}(x_2)\big) + \varrho\big(f_{\sigma_1}(x_2), f(x_2)\big)\\ &+ \varrho\big(f(x_2), f_{\sigma_2}(x_2)\big) + \varrho\big(f_{\sigma_2}(x_2), g_{\sigma_2}(x_2)\big)\\ &+ \varrho\big(g_{\sigma_2}(x_2), g_{\sigma_2}(x_0)\big) < \varepsilon\,.\end{aligned}$$

Hence $\big\{g_\sigma(x_0)\big\}_{\sigma\in\Sigma}$ is the Cauchy net. If Y is complete, then $\big\{g_\sigma(x_0)\big\}_{\sigma\in\Sigma}$ is convergent to $g(x_0)$. Finally we can say the net $\{g_\sigma\}_{\sigma\in\Sigma}$ is pointwise convergent to some function g.

(b) We next claim that $G(g) \subset \overline{G(f)}$. Let us fix $\big(x_0, g(x_0)\big) \in G(g)$ and a neighbourhood $U \times K\big(g(x_0), \varrho, \varepsilon\big)$ of this point. Since $\{f_\sigma\}_{\sigma\in\Sigma}$ is quasi-uniformly convergent at x_0 there exists $\sigma_0 \in \Sigma$ such that for $\sigma \geqslant \sigma_0$ we can take a neighbourhood U_σ of x_0 with the property $\varrho\big(f_\sigma(x), f(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U_\sigma$. The net $\big\{g_\sigma(x_0)\big\}_{\sigma\in\Sigma}$ is convergent to $g(x_0)$, so there exists $\sigma_1 \in \Sigma$ such that for $\sigma \geqslant \sigma_1$ we have $\varrho\big(g_\sigma(x_0), g(x_0)\big) < \frac{\varepsilon}{4}$.

By the property of the set Σ we can find $\sigma_2 \in \Sigma$ such that $\sigma_2 \geqslant \sigma_0$ and $\sigma_2 \geqslant \sigma_1$. Let $\sigma \geqslant \sigma_2$. By quasi-continuity of g_σ at $x_0 \in U_\sigma$ there exists a semi-neighbourhood $U_1 \subset U_\sigma$ of x_0 such that $\varrho\big(g_\sigma(x_0), g_\sigma(x)\big) < \frac{\varepsilon}{4}$ for each $x \in U_1$. It may be concluded that there exists a non-empty open set $U_2 = \mathrm{Int}\, U_1 \cap U_\sigma \cap U$. Because $G(g_\sigma) \subset \overline{G(f_\sigma)}$ then a set $U_2 \cap A(f_\sigma, g_\sigma, \varrho, \frac{\varepsilon}{4})$ is non-empty. Then there exists $x_1 \in U_2$ for which $\varrho\big(g_\sigma(x_1), f_\sigma(x_1)\big) < \frac{\varepsilon}{4}$. According to the above we have $\varrho\big(f(x_1), g(x_0)\big) \leqslant \varrho\big(f(x_1), f_\sigma(x_1)\big) + \varrho\big(f_\sigma(x_1), g_\sigma(x_1)\big) + \varrho\big(g_\sigma(x_1), g_\sigma(x_0)\big) + \varrho\big(g_\sigma(x_0), g(x_0)\big) < \varepsilon$. We have shown that $f(x_1) \in K\big(g(x_0), \varrho, \varepsilon\big)$. In consequence the set $\big(U \times K\big(g(x_0), \varrho, \varepsilon\big)\big) \cap G(f)$ is non-empty, so finally $G(g) \subset \overline{G(f)}$.

(c) Let $\varrho \in \mathcal{P}, \varepsilon > 0$ and $x_0 \in X$ be established and let $U \subset X$ be a neighbourhood of x_0. Since the net $\big\{g_\sigma(x_0)\big\}_{\sigma\in\Sigma}$ is convergent to $g(x_0)$ by (a), there exists $\sigma_0 \in \Sigma$ such that for $\sigma \geqslant \sigma_0$ we have $\varrho\big(g_\sigma(x_0), g(x_0)\big) < \frac{\varepsilon}{5}$. As the net $\{f_\sigma\}_{\sigma\in\Sigma}$ is quasi-uniformly convergent to f at x_0 we have the existence of $\sigma_1 \in \Sigma$ such that for $\sigma \geqslant \sigma_1$ we can find a neighbourhood U_σ of x_0 such that $\varrho\big(f_\sigma(x), f(x)\big) < \frac{\varepsilon}{5}$ for each $x \in U_\sigma$. By the property of the set Σ we have $\sigma_2 \in \Sigma$ such that $\sigma_2 \geqslant \sigma_0$ and $\sigma_2 \geqslant \sigma_1$. Let $\sigma \geqslant \sigma_2$. We define $W = U \cap U_\sigma$, a neighbourhood of x_0. The quasi-continuity of g_σ implies the existence of a non-empty semi-open set $U_1 \subset W$ satisfying the following condition $\varrho\big(g_\sigma(x), g_\sigma(x_0)\big) < \frac{\varepsilon}{5}$ for each $x \in U_1$. The set $B(f_\sigma, g_\sigma, \varrho, \frac{\varepsilon}{5})$ is nowhere dense, so there exists a non-empty open set $U_2 \subset \operatorname{Int} U_1$ such that $U_2 \subset A(f_\sigma, g_\sigma, \varrho, \frac{\varepsilon}{5})$. Let $x \in U_2$. Since $G(g) \subset \overline{G(f)}$ there exists $x_1 \in U_2$ such that $\varrho\big(f(x_1), g(x)\big) < \frac{\varepsilon}{5}$.

We obtain $\varrho\big(g(x_0), g(x)\big) \leqslant \varrho\big(g(x_0), g_\sigma(x_0)\big) + \varrho\big(g_\sigma(x_0), g_\sigma(x_1)\big) + \varrho\big(g_\sigma(x_1), f_\sigma(x_1)\big) + \varrho\big(f_\sigma(x_1), f(x_1)\big) + \varrho\big(f(x_1), g(x)\big) < \varepsilon$. It means that g is quasi-continuous and the proof is completed. □

Lemma 5. *Let X be a topological space and $(Y, \mathcal{P})$ a uniform one. If the net $\{f_\sigma\}_{\sigma\in\Sigma}$ of cliquish functions $f_\sigma \colon X \to Y$ is quasi-uniformly convergent to some $f \colon X \to Y$, then f is cliquish.*

Proof. Let $\varrho \in \mathcal{P}$, $\varepsilon > 0$ and $x_0 \in X$ be established and let $U \subset X$ be a neighbourhood of x_0. Since the net $\{f_\sigma\}_{\sigma\in\Sigma}$ is quasi-uniformly convergent to f there exists $\sigma_0 \in \Sigma$ such that for $\sigma \geqslant \sigma_0$ there exists a neighbourhood U_σ of x_0 such that $\varrho\big(f_\sigma(x), f(x)\big) < \frac{\varepsilon}{3}$ for each $x \in U_\sigma$. We take $\sigma \geqslant \sigma_0$. The function f_σ is cliquish, so there exists a non-empty open set $U_1 \subset U \cap U_\sigma$ such that $\varrho\big(f_\sigma(x_1), f_\sigma(x_2)\big) < \frac{\varepsilon}{3}$ for each $x_1, x_2 \in U_1$. Let $x_1, x_2 \in U_1$. Then we have $\varrho\big(f(x_1), f(x_2)\big) \leqslant \varrho\big(f(x_1), f_\sigma(x_1)\big) + \varrho\big(f_\sigma(x_1), f_\sigma(x_2)\big) + \varrho\big(f_\sigma(x_2), f(x_2)\big) < \varepsilon$. It means that f is cliquish and the proof is completed. □

From Theorem 8 and Lemma 5 we can write

Corollary 1. *Let X be an extremally disconnected space and let $(Y, \mathcal{P})$ be a complete uniform one. If the net $\{f_\sigma\}_{\sigma\in\Sigma}$ of graph quasi-continuous and cliquish functions $f_\sigma \colon X \to Y$ is quasi-uniformly convergent to some $f \colon X \to Y$, then f is graph quasi-continuous and cliquish.*

We will now use the symbol $F(X, Y)$ to denote the family of all functions from X to Y, GQ—the family of all graph quasi-continuous from X to Y and GQ_C-the family of all graph quasi-continuous and cliquish functions from X to Y. The convergence in the uniform space $\big(F(X, Y), \mathcal{W}\big)$ coincides with the quasi-uniform convergence ([17]).

From Corollary 1 we have

Theorem 9. *Let X be an extremally disconnected space and $(Y, \mathcal{P})$ a complete uniform one. Then the set of all graph quasi-continuous and cliquish functions $f \colon X \to Y$ is closed in the space $\big(F(X, Y), \mathcal{W}\big)$ equipped with the topology of the quasi-uniform convergence.*

Now, let $(Y, \mathcal{T})$ be a locally convex topological vector space.

Lemma 6. ([1]) *In each locally convex topological space $(Y, \mathcal{T})$ there exists a family of seminorms $\mathcal{S}$ with the property:*

(a) *for each $p_1, p_2 \in \mathcal{S}$ there exists $p_3 \in \mathcal{S}$ such that* $\max\{p_1, p_2\} \leqslant p_3$,

(b) *for each* $p_1 \in \mathcal{S}$ *there exists* $p_2 \in \mathcal{S}$ *with* $p_1(x) \leqslant \frac{1}{2}p_2(x)$ *such that the sets* $\{x\colon p(x) \leqslant 1\}$ *for* $p \in \mathcal{S}$ *give a base for neighbourhoods of null.*

The topology induced by the family $\mathcal{S}$ *coincides with the topology* $\mathcal{T}$.

For $p \in \mathcal{S}$ we consider the functions $\varsigma_p\colon Y \times Y \to [0,\infty)$ such that $\varsigma_p(x,y) = p(x-y)$ for $x, y \in Y$. The function ς_p is a pseudometric for each $p \in \mathcal{S}$. We will use the symbol $\mathcal{P}_{\mathcal{S}}$ to denote the family of pseudometrics ς_p. Now, we denote $V(\varsigma_p, \varepsilon) = \{(x,y)\colon p(x-y) < \varepsilon\}$ for $p \in \mathcal{S}$, $\varepsilon > 0$. Then the family $\{V(\varsigma_p, \varepsilon)\colon p \in \mathcal{S},\ \varepsilon > 0\}$ give a base for a uniform structure on Y. We denote this uniform space as $(Y, \mathcal{P}_{\mathcal{S}})$. The topology induced by the family $\mathcal{P}_{\mathcal{S}}$ coincides with the given topology on X. So, we consider the locally convex topological vector space as the uniform space $(Y, \mathcal{P}_{\mathcal{S}})$.

LEMMA 7. *Let* X *be a* T_1 *first countable space without isolated points.*

(a) *Then there exists disjoint dense* $A, B \subset X$ *such that* $X = A \cup B$.

(b) *If* $A \subset X$ *is dense then the subspace* A *is a* T_1 *first countable space without isolated points.*

Proof. (a) in [10];

(b) the simple proof is omitted. □

THEOREM 10. *Let* X *be a* T_1 *first countable space without isolated points and let* $(Y, \mathcal{P}_{\mathcal{S}})$ *be a locally convex topological vector space. Then the set of all graph quasi-continuous and cliquish functions* $f\colon X \to Y$ *is co-dense in the space of all graph quasi-continuous functions from* X *to* Y *equipped with the topology of quasi-uniform convergence.*

Proof. We will show that $\mathrm{GQ} \subset \overline{\mathrm{GQ} \setminus \mathrm{GQ}_C}^{\mathrm{GQ}}$, where $\overline{A}^{\mathrm{GQ}}$ denotes the closure of A in the space GQ equipped with the topology of quasi-uniform convergence. If $f \in \mathrm{GQ} \setminus \mathrm{GQ}_C$ then $f \in \overline{\mathrm{GQ} \setminus \mathrm{GQ}_C}^{\mathrm{GQ}}$. Now let $f \in \mathrm{GQ}_C$. Then there exists a quasi-continuous function $g\colon X \to Y$ such that $G(g) \subset \overline{G(f)}$. For a seminorm $p \in \mathcal{S}$ and for $n \in \mathbb{N}$, by Theorem 1, the set $A(f, g, p, \frac{2}{n})$ is dense in X. According to Lemma 7 we select disjoint and dense sets $B_{p,n}, C_{p,n}$ such that $A(f, g, p, \frac{2}{n}) = B_{p,n} \cup C_{p,n}$ for each $p \in \mathcal{S}$ and for $n \in \mathbb{N}$. Let us put

$$(p_1, n_1) \geqslant (p_2, n_2) \quad \text{if and only if} \quad p_1 \geqslant p_2 \ \text{ and } \ n_1 \geqslant n_2\,,$$

so the set $\Sigma = \{\sigma = (p,n)\colon p \in \mathcal{S},\ n \in \mathbb{N}\}$ is directed by the relation $\geqslant$. Let $\{y_\sigma\}_{\sigma \in \Sigma}$ be a net of members of Y satisfying condition $\frac{1}{n} < p(y_\sigma) < \frac{2}{n}$ where $\sigma = (p,n)$.

We define functions $f_\sigma\colon X \to Y$ by $f_\sigma(x) = f(x)$ for $x \in B_{p,n}$ and $f_\sigma(x) = f(x) + y_\sigma$ for $x \in X \setminus B_{p,n}$.

We will show that the net $\{f_\sigma\}_{\sigma \in \Sigma}$ is uniformly convergent to f. Let $p \in \mathcal{S}$,

$\varepsilon > 0$ be given and let us $n \in \mathbb{N}$ such that $\frac{2}{n} < \varepsilon$. We take $\sigma, \upsilon \in \Sigma$ such that $\sigma = (p, n)$, $\upsilon = (q, m)$ and $\upsilon \geqslant \sigma$. If $x \in B_{p,m}$, we have $p\big(f_\upsilon(x) - f(x)\big) = p\big(f(x) - f(x)\big) = 0$, and if $x \in X \setminus B_{p,m}$, we have $p\big(f_\upsilon(x) - f(x)\big) = p\big(f(x) + y_\upsilon - f(x)\big) \leqslant p(y_\upsilon) < \frac{2}{m} \leqslant \frac{2}{n} < \varepsilon$. It means that the net $\{f_\sigma\}_{\sigma\in\Sigma}$ is uniformly convergent to f.

Now, we define a net $\{g_\sigma\}_{\sigma\in\Sigma}$ by $g_\sigma(x) = g(x)$ for $x \in X$, $\sigma \in \Sigma$. It is evident that the function g_σ for $\sigma \in \Sigma$ is quasi-continuous.

We will show that f_σ for $\sigma \in \Sigma$ is graph quasi-continuous. Let us take $\sigma, \upsilon \in \Sigma$ such that $\sigma = (p, n)$ and $\upsilon = (q, m)$. Since f is cliquish, g is quasi-continuous and $G(g) \subset \overline{G(f)}$, according to Theorem 2 it follows that the set $A(f, g, p, \frac{1}{n})$ is dense in X and semi-open in X. By Lemma (1b) the set $B_{q,m} \cap A\big(f, g, p, \frac{1}{n}\big)$ is dense in X, too. Let us take $x \in B_{q,m} \cap A\big(f, g, p, \frac{1}{n}\big)$. Then $p\big(f_\upsilon(x) - g_\upsilon(x)\big) = p\big(f(x) - g(x)\big) < \frac{1}{n}$. It means that $x \in A\big(f_\upsilon, g_\upsilon, p, \frac{1}{n}\big)$. From the inclusion $A\big(f_\upsilon, g_\upsilon, p, \frac{1}{n}\big) \supset B_{q,m} \cap A\big(f, g, p, \frac{1}{n}\big)$ it follows that the set $A\big(f_\upsilon, g_\upsilon, p, \frac{1}{n}\big)$ is dense in X and by quasi-continuity of g_υ we have the inclusion

$$G(g_\upsilon) \subset \overline{G(f_\upsilon)}\,. \tag{4}$$

It means that f_υ is graph quasi-continuous for each $\upsilon \in \Sigma$.

Now, we show that $f_\sigma \notin \mathrm{GQ}_C$. Let us take a non-empty open set $U \subset X$, $\sigma \in \Sigma$ such that $\sigma = (p, n) \in \Sigma$ and $\varepsilon = \frac{1}{n}$. The function f is cliquish, so there exists a non-empty open set $V \subset U$ satisfying the inclusion $p\big(f(x_1) - f(x_2)\big) < \frac{1}{2n}$ for each $x_1, x_2 \in V$. Suppose that f_σ is cliquish. Then there exists a non-empty open set $W \subset V$ such that

$$p\big(f_\sigma(x_1) - f_\sigma(x_2)\big) < \frac{1}{2n} \tag{5}$$

for each $x_1, x_2 \in W$. The sets $B_{p,n}$, $X \setminus B_{p,n}$ are dense, so we can choose $x_1 \in W \cap B_{p,n}$ and $x_2 \in W \cap (X \setminus B_{p,n})$. Then we have

$$\begin{aligned} p\big(f_\sigma(x_1) - f_\sigma(x_2)\big) &= p\big(f(x_1) - f(x_2) - y_\sigma\big) \\ &\geqslant \big|p\big(f(x_1) - f(x_2)\big) - p(y_\sigma)\big| > \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}\,. \end{aligned}$$

It is a contradiction to (5), so f_σ is not cliquish for $\sigma \in \Sigma$. By (4) we have $f_\sigma \in \mathrm{GQ} \setminus \mathrm{GQ}_C$ for each $\sigma \in \Sigma$. Finally, we have shown that there exists the net $\{f_\sigma\}_{\sigma\in\Sigma}$ of graph quasi-continuous functions, not cliquish, which is quasi-uniformly convergent to f, so $f \in \overline{\mathrm{GQ} \setminus \mathrm{GQ}_C}^{\,\mathrm{GQ}}$. □

In this part we shall study the right-sided (left-sided) graph quasi-continuous function.

Definition 3. ([3]) A function $g\colon \mathbb{R} \to \mathbb{R}$ is said to be *right-sided* (*left-sided*) *quasi-continuous* if for each $\varepsilon > 0$ and for each $\delta > 0$ there exists a non-empty open set $U \subset (x_0, x_0 + \delta)$ $(U \subset (x_0 - \delta, x_0))$ such that $|g(x_0) - g(x)| < \varepsilon$ for each $x \in U$.

Definition 4. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *right-sided* (*left-sided*) *graph quasi-continuous* if there exists a right-sided quasi-continuous (left-sided quasi-continuous) function $g\colon \mathbb{R} \to \mathbb{R}$ such that $G(g) \subset \overline{G(f)}$.

From Theorem 1 and Theorem 2 we have the corollary for real functions.

Theorem 11. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$. If g is right-sided (left-sided) quasi-continuous, then the following conditions are equivalent:*

(a) $G(g) \subset \overline{G(f)}$;

(b) *the set $A(f, g, \varepsilon)$ is dense in $\mathbb{R}$ for each $\varepsilon > 0$.*

Theorem 12. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$. If g is right-sided (left) quasi-continuous, then the following conditions are equivalent:*

(a) $G(g) \subset \overline{G(f)}$ *and f is cliquish;*

(b) *the set $B(f, g, \varepsilon)$ is nowhere dense in $\mathbb{R}$ for each $\varepsilon > 0$.*

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