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ON COMBINATORIAL PROPERTIES OF BOREL GENERATED σ-IDEALS RELATED TO THE PROPERTY (P)

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ABSTRACT. We consider a strengthening of (P) property and other notions of σ-ideals. In particular, we prove that every transitive weakly G_δ generated σ-ideal has the strongest possible property in the hierarchy of properties related to (P) property.

1. Notations and definitions

Let Bor and Perf denote the collection of all Borel and all perfect subsets of 2^ω, respectively.

We say that a σ-ideal $\mathcal{I} \subseteq P(2^\omega)$ *has a Borel base* iff $\forall_{X\in\mathcal{I}}\ \exists_{B\in\mathrm{Bor}}\ B \in \mathcal{I} \wedge X \subseteq B$.

We say that a σ-ideal $\mathcal{I}$ is *transitive invariant* iff $\forall_{A\in\mathcal{I}}\ \forall_{t\in 2^\omega}\ A + t \in \mathcal{I}$.

Let us denote:

$\mathcal{N}$, $\mathcal{M}$ = the σ-ideal of Lebesgue measure zero subsets of 2^ω and of meager subsets, respectively.

Let $[\omega]^\omega$ and $[\omega]^{<\omega}$ be the set of all infinite and finite subsets of ω, respectively and let $\omega^{\uparrow\omega}$ denote the family of all strictly increasing functions from ω^ω.

Suppose that $K \subseteq \omega$ and $A \subseteq 2^\omega$ then we denote $A \restriction K = \{x \restriction K : x \in A\}$.

We often conflate the power set $P(K)$ with the subset of 2^ω of characteristic functions: $P(K) = \{x \in 2^\omega : \forall_{i\in\omega\setminus K}\ x(i) = 0\}$.

Let us denote by $\mathbf{d}$ the minimal size of a dominating family in ω^ω, i.e., the minimal size of a family $\mathcal{F} \subseteq \omega^\omega$ such that $\forall_{f\in\omega^\omega}\ \exists_{g\in\mathcal{F}}\ f \prec g$, where $f \prec g \iff \forall_{k\in\omega}\ f(k) \leqslant g(k)$.

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If $x, y \in 2^\omega$ then we write $x \sim y$ iff $\exists_{n\in\omega} \forall_{k>n}\ x(k) = y(k)$, i.e., x and y are almost equal.

We frequently consider the field Δ^0_1 of clopen subsets of 2^ω.

Let $\mathcal{I}$ be a σ-ideal in $P(2^\omega)$ and let $\mathcal{F}$ be a field of subsets of 2^ω.

For any two partitions $\mathcal{B}$ and $\mathcal{C}$ of set X let us denote by $\mathcal{B} \wedge \mathcal{C}$ the partition

$$\{B \cap C \colon B \in \mathcal{B},\ C \in \mathcal{C},\ B \cap C \neq \emptyset\}$$

and we say that $\mathcal{B}$ is finer than $\mathcal{C}$ (abbreviation: $\mathcal{B} \prec \mathcal{C}$) iff $\forall_{B\in\mathcal{B}} \exists_{C\in\mathcal{C}}\ B \subseteq C$.

Let us recall that a set X is in the class (s_0) of Marczewski null sets iff every perfect set P contains a perfect set Q which misses X.

The following properties were defined and considered in [N]:

Definition 1.1. We say that the σ-ideal $\mathcal{I}$ has the

(UP) *property* iff for every continuous function $F \colon 2^\omega \to 2^\omega$ there exists a set $A \in [\omega]^\omega$ such that $F\big[P(A)\big] \in \mathcal{I}$.

(BUP) *property* iff for every continuous function $C \colon [\omega]^\omega \to 2^\omega$ there exists a set $A \in [\omega]^\omega$ such that $C\big[[A]^\omega\big] \in \mathcal{I}$.

$P^{\mathrm{Tree}}(\mathcal{F})$ *property* iff there exists a sequence $\mathcal{B}_0, \mathcal{B}_1, \dots$ of finite partitions of 2^ω into sets from the field $\mathcal{F}$ (we call this sequence *a tree system*) such that we have:

1. For each $i \in \omega$, $\mathcal{B}_{i+1}$ is finer than $\mathcal{B}_i$.
2. If $(B_t)_{t\in 2^{<\omega}}$ is any system of sets such that
   (a) $\forall_{t\in 2^{<\omega}}\ B_t \in \mathcal{B}_{|t|}$,
   (b) $\forall_{t\in 2^{<\omega}} \forall_{i\in 2}\ B_{t\frown\langle i\rangle} \subseteq B_t$;
   then $\mathcal{A}(\langle B_t\rangle) = \bigcup\limits_{x\in 2^\omega} \bigcap\limits_{k\in\omega} B_{x|k} \in \mathcal{I}$
   (we call such a system the *Cantor system* for the tree system $\langle \mathcal{B}_n \colon n \in \omega\rangle$, note that it will cause no confusion if we write simply *the Cantor system*).

We frequently consider the case $\mathcal{F} = \Delta^0_1$. Let us recall another couple of definitions. First we define two notions of Ramsey type sets.

If $|s| < \omega$, $A \in [\omega]^\omega$ and $\max s < \min A$ then we define $[s, A] = \{x \in [\omega]^\omega \colon s \subseteq x \subseteq s \cup A\}$. These sets are called the *Ellentuck neighbourhoods*. Recall that $X \subseteq [\omega]^\omega$ is *completely Ramsey null* (or for short X is CR_0) iff for every Ellentuck neighbourhood $[s, A]$ there exists $B \in [A]^\omega$ such that $[s, B] \cap X = \emptyset$.

We say that a set $X \subseteq 2^\omega$ is *uniformly completely Ramsey null* ($X \in \mathrm{UCR}_0$) iff for each continuous function $F \colon 2^\omega \to 2^\omega$ there exists $A \in [\omega]^\omega$ such that $\big|F\big[P(A)\big] \cap X\big| \leqslant \omega$.

Analogously, we say that a set $X \subseteq 2^\omega$ is BUCR_0 iff for each Borel function $F\colon 2^\omega \to 2^\omega$ there exists $A \in [\omega]^\omega$ such that $\big|F\big[P(A)\big] \cap X\big| \leqslant \omega$, which is equivalent to the statement that for each continuous function $C\colon [\omega]^\omega \to 2^\omega$ there exists $A \in [\omega]^\omega$ such that $\big|C\big[[A]^\omega\big] \cap X\big| \leqslant \omega$ ([N, Definition 4.1 and Theorem 4.2]).

The following definitions were considered by M. Balcerzak in [B1] and [B2].

Definition 1.2. Suppose that $\mathcal{I}$ is a σ-ideal. We say that $\mathcal{I}$

- is *weakly G_δ generated* iff for every countable $D \subseteq 2^\omega$ we have that there exists a G_δ set $G \in \mathcal{I}$ such that $D \subseteq G$;
- has the (P) *property* iff $\forall_{P \in \mathrm{Perf}}\ \exists_{Q \in \mathrm{Perf}}\ Q \in \mathcal{I} \wedge Q \subseteq P$.

By virtue of Theorem 4.5 from [N] we know that every σ-ideal with the $P^{\mathrm{Tree}}(\Delta^0_1)$ has the (BUP) property. We have also the following easy fact:

Fact 1.3. *Every weakly G_δ generated σ-ideal has the property* (UP).

Proof. Suppose that $\mathcal{I}$ is a weakly G_δ generated σ-ideal. Let $F\colon 2^\omega \to 2^\omega$ be a continuous function. Since $\mathcal{I}$ is weakly G_δ generated, there exists a G_δ set $G \in \mathcal{I}$ such that $F\big[[\omega]^{<\omega}\big] \subseteq G$. Then $F^{-1}[G]$ is a G_δ set and $[\omega]^{<\omega} \subseteq F^{-1}[G]$. Therefore, there exists an $A \in [\omega]^\omega$ such that $P(A) \subseteq F^{-1}[G]$, hence $F\big[P(A)\big] \subseteq G$, thus $F\big[P(A)\big] \in \mathcal{I}$. This proves that $\mathcal{I}$ has the property (UP). $\square$

Therefore we have the following diagram of inclusions for σ-ideals containing all singletons:

$$\begin{array}{ccccccc} P^{\mathrm{Tree}}(\Delta^0_1) & \longrightarrow & (\mathrm{BUP}) & \longrightarrow & (\mathrm{UP}) & \longrightarrow & (\mathrm{P}) \\ & & & & \uparrow & & \\ & & & & \text{weakly } G_\delta \text{ generated} & & \end{array}$$

Figure 1. Basic inclusions

For every σ-ideal $\mathcal{I}$ we consider the following σ-ideal.

$$\mathrm{Lusin}(\mathcal{I}) = \big\{X \subseteq 2^\omega \colon \forall_{A \in \mathcal{I}}\ |A \cap X| \leqslant \aleph_0\big\}.$$

Let us notice that $\mathrm{Lusin}(\mathcal{M})$ are Lusin sets (together with countable sets) while $\mathrm{Lusin}(\mathcal{N})$ are Sierpiński sets (together with countable sets).

2. Main result

Our first example shows that in diagram (1) there is no inclusion from "weakly G_δ generated" to "$P^{\mathrm{Tree}}(\Delta^0_1)$", namely we have:

Theorem 2.1. *Assume CH. Then there exists a σ-ideal $\mathcal{I}$ such that*

1. *$\mathcal{I}$ is weakly G_δ generated.*
2. *$\mathcal{I}$ does not have the $P^{\mathrm{Tree}}(\Delta^0_1)$ property.*

Before we start the construction let us note the following fact that will be used in the construction.

Fakt 2.2. *Every σ-ideal which has the $P^{\mathrm{Tree}}(\Delta^0_1)$ property has the following property:*

There exists a perfect set $P \in \mathrm{Perf}$ such that

$$\forall_{z\in 2^\omega}\ P + z \in \mathcal{I}.$$

P r o o f. There is an increasing function $x\colon \omega \to \omega$ such that $\mathcal{B}_n = \{[t]\colon t \in 2^{x(n)}\}$ is a tree system such that for every Cantor system $\langle B_r\colon r \in 2^{<\omega}\rangle$ for $\langle \mathcal{B}_n\colon n \in \omega\rangle$ we have $\mathcal{A}(\langle B_r\rangle) \in \mathcal{I}$. Let us choose a Cantor system $\langle t_r\colon r \in 2^{<\omega}\rangle$ such that $\mathcal{A}(\langle t_r\rangle) \in \mathrm{Perf}$. Suppose that $z \in 2^\omega$. Let us consider the following Cantor system: $\langle s_r\colon r \in 2^{<\omega}\rangle$, where $s_r = t_r + z \restriction [0, x(|r|))$. Then $\mathcal{A}(\langle s_r\rangle) = \mathcal{A}(\langle t_r\rangle) + z$. Therefore $\forall_{z\in 2^\omega}\ \mathcal{A}(\langle t_r\rangle) + z \in \mathcal{I}$. $\square$

P r o o f o f T h e o r e m 2.1. Assume CH. We will construct a σ-ideal $\mathcal{I}$ such that

- $\mathcal{I}$ is weakly G_δ generated,
- $\forall_{P\in\mathrm{Perf}}\ \exists_{z\in 2^\omega}\ P + z \notin \mathcal{I}$,
- $\mathcal{I}$ has a Borel basis.

Let $\{D_\xi\colon \xi < 2^\omega\}$ be an enumeration of all countable subsets of the space 2^ω and let $\{P_\xi\colon \xi < 2^\omega\}$ be an enumeration of all perfect subsets of 2^ω.

By induction on $\theta < \omega_1$ we will construct the following sequences:

1. $\langle G_\theta\colon \theta \in \omega_1\rangle$, $G_\theta \in G_\delta \cap \mathcal{N}$;
2. $\langle z_\theta\colon \theta \in \omega_1\rangle$, $z_\theta \in 2^\omega$;
3. $\langle h_\theta\colon \theta \in \omega_1\rangle$, where $h_\theta\colon 2^\omega \to P_\theta + z_\theta$ is a homeomorphism.

At stage θ we assume that all

1. $\langle G_\xi\colon \xi < \theta\rangle$,
2. $\langle z_\xi\colon \xi < \theta\rangle$,
3. $\langle h_\xi\colon \xi < \theta\rangle$,

are chosen, and moreover, we assume that $\forall_{\mu,\gamma<\theta}\, h_\mu^{-1}(G_\gamma) \in \mathcal{N}$.

Let us formulate a useful lemma for the next part of our proof.

Lemma 2.3. *Suppose that a map $h\colon 2^\omega \to N$ is a homeomorphism onto a perfect set N and suppose that $N \in \mathcal{N}$. Then there exists $z \in 2^\omega$ such that $h^{-1}(N+z) \in \mathcal{N}$.*

P r o o f. The following interesting proof of this lemma was discovered by I. R e c ł a w:

Let us consider the following set:

$$G = \{\langle x, y\rangle\colon h(x) + y \in N\}.$$

It is obvious that G is a Borel set, and moreover, $(G)_{x_0} = \{y\colon y \in N + h(x_0)\} \in \mathcal{N}$ for every $x_0 \in 2^\omega$, where $(X)_x$ and $(X)^y$ denote as usually $\{y\colon \langle x,y\rangle \in X\}$ and $\{x\colon \langle x,y\rangle \in X\}$, respectively. Hence, by the Fubini theorem there exists $z \in 2^\omega$ such that $(G)^z \in \mathcal{N}$, i.e., $\{x \in 2^\omega\colon h(x) \in N + z\} \in \mathcal{N}$. □

First choose a G_δ set $G_\theta \in \mathcal{N}$ such that $D_\theta \subseteq G_\theta$ and

$$\forall_{\mu<\theta}\, h_\mu^{-1}(G_\theta) \in \mathcal{N}.$$

Next, choose a G_δ set $G_\theta^* \in \mathcal{N}$ such that $\bigcup_{\mu\leqslant\theta} G_\mu \subseteq G_\theta^*$. Let $h'_\theta\colon 2^\omega \to P_\theta$ be any homeomorphism.

By Lemma 2.3 choose $z_\theta \in 2^\omega$ such that the set $h_\theta'^{-1}(G_\theta^* + z_\theta)$ is of measure zero. Now define a homeomorphism $h_\theta\colon 2^\omega \to P_\theta + z_\theta$ as follows: $h_\theta(x) = h'_\theta(x) + z_\theta$. Then we have

$$h_\theta^{-1}(G_\theta^*) = \{x\colon h'_\theta(x) + z_\theta \in G_\theta^*\} = \{x\colon h'_\theta(x) \in G_\theta^* + z_\theta\} = h_\theta'^{-1}(G_\theta^* + z_\theta),$$

therefore $h_\theta^{-1}(G_\theta^*) \in \mathcal{N}$. This finishes the inductive construction.

Define $\mathcal{I}$ to be the σ-ideal generated by sets from the sequence $\langle G_\theta : \theta \in \omega_1\rangle$.

It is obvious that $\mathcal{I} \subseteq \mathcal{N}$ since $\forall_{\theta\in\omega_1} G_\theta \in \mathcal{N}$. Moreover, $\mathcal{I}$ is weakly G_δ generated. Suppose that $P \in \mathrm{Perf}$. Then there exists $\theta \in \omega_1$ such that $P = P_\theta$. We have $h_\theta^{-1}(G_\mu) \in \mathcal{N}$ for each $\mu \in \omega_1$. From this we conclude that $h_\theta(2^\omega) = P_\theta + z_\theta \notin \mathcal{I}$.

It follows that

$$\forall_{P\in\mathrm{Perf}}\, \exists_{z\in 2^\omega}\, P + z \notin \mathcal{I}.$$

Now, by Fact 2.2, $\mathcal{I}$ does not have the $P^{\mathrm{Tree}}(\Delta_1^0)$ property. □

The ideal $\mathcal{M}$ of meager sets has the $P^{\mathrm{Tree}}(\Delta_1^0)$ property and is not weakly G_δ generated. Therefore Theorem 2.1 shows that these two properties are independent. The next theorem shows that in case of transitive invariant σ-ideals there is an implication between them.

Theorem 2.4. *Suppose that $\mathcal{I}$ is a transitive invariant σ-ideal, such that $\mathcal{I}$ is weakly G_δ generated. Then $\mathcal{I}$ has the $P^{\mathrm{Tree}}(\Delta_1^0)$ property.*

P r o o f. At first we will show the following lemma:

Lemma 2.5. *Let $M \subseteq 2^\omega$ be a meager set. Then the σ-ideal $\{X \subseteq 2^\omega\colon X + M \in \mathcal{M}\}$ has the $P^{\mathrm{Tree}}(\Delta_1^0)$ property.*

P r o o f. For a meager set $M \subseteq 2^\omega$ there is a real $z \in 2^\omega$ and there are finite intervals I_n, $n \geqslant 1$, in ω such that $\max I_n = \min I_{n+1} - 1$, $\bigcup_{n\geqslant 1} I_n = \omega$, and $M \subseteq \{x \in 2^\omega\colon \forall_n^\infty x \restriction I_n \neq z \restriction I_n\}$. Without loss of generality let us assume that $M = \{x \in 2^\omega\colon \forall_n^\infty x \restriction I_n \neq z \restriction I_n\}$.

Define $J_i = I_{2^i} \cup \cdots \cup I_{2^{i+1}-1}$. Let us define also $\mathcal{B}_n = \{[t]\colon t \in 2^{J_0\cup\cdots\cup J_n}\}$. It is easy to see that $\mathcal{B}_{n+1} \prec \mathcal{B}_n$.

Suppose that $(t_r)_{r\in 2^{<\omega}}$ is a system of finite sequences from $2^{<\omega}$ such that

1. $\forall_{r\in 2^{<\omega}} \forall_{i\in 2}\ t_r \subseteq t_{r^\frown\langle i\rangle}$,
2. $\forall_{r\in 2^{<\omega}}\ t_r \in 2^{J_0\cup\cdots\cup J_{|r|}}$.

It is easy to see that $\mathcal{A}(\langle t_r\rangle) = \bigcup\limits_{x\in 2^\omega} \bigcap\limits_{k\in\omega} [t_{x\restriction k}] = \left\{\bigcup\limits_{k\in\omega} t_{x\restriction k}\colon x \in 2^\omega\right\}$.

For each $i \in \omega$ let $r_0^{(i)}, \ldots, r_{2^i-1}^{(i)}$ be any enumeration of all sequences from 2^i. Let us choose $w \in 2^\omega$ such that

$$\forall_{i\in\omega} \forall_{0\leqslant j\leqslant 2^i-1}\ w \restriction I_{2^i+j} = z \restriction I_{2^i+j} + t_{r_j^{(i)}} \restriction I_{2^i+j}\,.$$

Define:

$$M_1 = \{x\colon \forall_n^\infty x \restriction J_n \neq w \restriction J_n\}\,.$$

Let us choose any $y \in 2^\omega$ and $i \in \omega$. There exists $j \in [0, 2^i)$ such that $y \restriction i = r_j^{(i)}$. Let us define $a = \bigcup\limits_{i\in\omega} t_{y\restriction i}$. It is easy to see that

$$a \restriction J_0 \cup \cdots \cup J_i = t_{y\restriction i} = t_{r_j^{(i)}}\,.$$

Let us choose an arbitrary $x \in M$. We have $\forall_n^\infty x \restriction I_n \neq z \restriction I_n$.

For $i \in \omega$ there exists $j \in [0, 2^i)$ such that $y \restriction i = r_j^{(i)}$. Then we have

$$w \restriction I_{2^i+j} = z \restriction I_{2^i+j} + t_{r_j^{(i)}} \restriction I_{2^i+j}\,.$$

But

$$t_{r_j^{(i)}} = t_{y\restriction i} = a \restriction J_0 \cup \cdots \cup J_i\,.$$

Thus

$$w \restriction I_{2^i+j} = z \restriction I_{2^i+j} + a \restriction I_{2^i+j}\,.$$

Let $i_0 \in \omega$ be such that $x \restriction I_n \neq z \restriction I_n$ for $n \geqslant 2^{i_0}$. Now, if $i > i_0$ then we have

$$x \restriction I_{2^i+j} \neq z \restriction I_{2^i+j} .$$

Therefore

$$(x+a) \restriction I_{2^i+j} = x \restriction I_{2^i+j} + a \restriction I_{2^i+j} \neq z \restriction I_{2^i+j} + a \restriction I_{2^i+j} = w \restriction I_{2^i+j} .$$

Finally, we have proved that

$$\forall_{i>i_0} \exists_{0 \leqslant j < 2^i} \ (x+a) \restriction I_{2^i+j} \neq w \restriction I_{2^i+j} .$$

Thus

$$\forall_k^\infty \ (x+a) \restriction J_k \neq w \restriction J_k .$$

Finally, we obtain

$$\mathcal{A}(\langle t_r \rangle) + M \subseteq M_1 .$$

□

Since $\mathcal{I}$ is weakly G_δ generated we can find $G \subseteq 2^\omega$, a G_δ set such that $[\omega]^{<\omega} \subseteq G$ and $G \in \mathcal{I}$. Let us consider the following σ-ideal $\mathcal{J} = \{X \subseteq 2^\omega \colon X + (2^\omega \setminus G) \in \mathcal{M}\}$. By Lemma 2.5 we conclude that $\mathcal{J}$ has the $P^{\mathrm{Tree}}(\Delta_1^0)$ property. So let $\mathcal{B}_0, \mathcal{B}_1, \dots$ be a tree system such that for each Cantor system $\langle B_t \rangle$ for this tree system we have $2^\omega A(\langle B_t \rangle) \in \mathcal{J}$. Since we have $\mathcal{A}(\langle B_t \rangle) + (2^\omega \setminus G) \in \mathcal{M}$, there exists $x_0 \in 2^\omega$ such that $x_0 \notin \mathcal{A}(\langle B_t \rangle) + (2^\omega \setminus G)$. Therefore $x_0 + \mathcal{A}(\langle B_t \rangle) \subseteq G$, and since the σ-ideal $\mathcal{I}$ is transitive invariant we conclude, that $\mathcal{A}(\langle B_t \rangle) \in \mathcal{I}$. This proves that $\mathcal{I}$ has the $P^{\mathrm{Tree}}(\Delta_1^0)$ property. □

Observe that the problem whether the implication (BUP) $\to$ (UP) is reversible is still unsolved, even in the realm of non transitive σ-ideals, in contrast of the following easy result:

THEOREM 2.6. *There exists a transitive, Borel generated σ-ideal which has the* (P) *property and does not have the* (UP) *property.*

P r o o f. Let us define

$$\mathcal{I}_r = \{X \subseteq 2^\omega \colon \exists_{B \in \mathrm{Bor}} X \subseteq B \wedge \forall_{t \in 2^\omega} B + t \in \mathrm{CR}_0\} .$$

It is easy to see that I_r is indeed a transitive invariant, Borel generated σ-ideal. If we take as F the id function ($\mathrm{id}(x) = x$), then we have $\forall_{A \in [\omega]^\omega} F[P(A)] \notin \mathcal{I}_r$, which proves that I_r does not have the (UP) property. On the other hand, suppose that $P \in \mathrm{Perf}$. Then there exists a perfect set $Q \subseteq P$ such that $\forall_{x,y \in P} \ x \neq y \Rightarrow \neg(x \sim y)$. We will prove that $Q \in \mathcal{I}_r$. So let $t \in 2^\omega$. Suppose that $x, y \in (Q+t) \cap [\omega]^{<\omega}$. Then $x+t, y+t \in Q$ and $x+t \sim y+t$, which gives us that $x+t = y+t$, hence $x = y$. Therefore $|(Q+t) \cap [\omega]^{<\omega}| \leqslant 1$ and by virtue of the fact that Q is closed we conclude that $Q + t \in \mathrm{CR}_0$. This ends the proof that $\mathcal{I}_r$ has the (P) property. $\square$

As result we obtain finally the following diagram:

$$\begin{array}{ccccccc} P^{\mathrm{Tree}}(\Delta^0_1) & \longrightarrow & (\mathrm{BUP}) & \overset{?}{\Longleftrightarrow} & (UP) & \longrightarrow & (\mathrm{P}) \\ \uparrow & & & & \uparrow & & \\ \text{for transitive invariant} & & & & \text{weakly } G_\delta \text{ generated} & & \end{array}$$

FIGURE 2. Extended diagram.

3. Balcerzak–Plewik's σ-ideals

M. Balcerzak and S. Plewik considered the following (transitive invariant) σ-ideal (see [BP]):

DEFINITION 3.1. (M. Balcerzak and S. Plewik) Suppose that $\mathcal{K} \subseteq [\omega]^\omega$. Assume also that $\mathcal{K} \neq \emptyset$ and $\forall_{A \in \mathcal{K}}\ \exists_{A_0, A_1 \in \mathcal{K}}\ A_0 \cap A_1 = \emptyset \wedge A_0 \cup A_1 \subseteq A$ (in this case we say that $\mathcal{K}$ is a *normal system*). Then we put:

$$\mathcal{I}_{\mathcal{K}} = \left\{ A \subseteq 2^\omega \colon \forall_{K \in \mathcal{K}}\ A \restriction K \neq 2^K \right\}.$$

The following theorem is a generalization of Theorem 1 from [BP] (which states that for a normal system $\mathcal{K} \subseteq [\omega]^\omega$ of the size less than $\mathbf{d}$, the σ-ideal $\mathcal{I}_{\mathcal{K}}$ has the (P) property):

THEOREM 3.2. *Suppose that $\mathcal{K} \subseteq [\omega]^\omega$ and $|\mathcal{K}| < \mathbf{d}$. Assume also that $\mathcal{K}$ is a normal system. Then the σ-ideal $\mathcal{I}_{\mathcal{K}}$ is weakly G_δ generated.*

Proof. First let us recall the well-known lemma:

LEMMA 3.3. *Suppose that $\kappa < \mathbf{d}$ and $\{H_\alpha\}_{\alpha<\kappa}$ is a sequence of G_δ sets such that $\forall_{\alpha<\kappa}\ [\omega]^{<\omega} \subseteq H_\alpha$. Then there exists a G_δ set $H \subseteq 2^\omega$ such that $[\omega]^{<\omega} \subseteq H$ and $\forall_{\alpha<\kappa}\ H \subseteq H_\alpha$.*

Suppose that $D \subseteq 2^\omega$ is a countable set. We are looking for a G_δ set $G \subseteq 2^\omega$ such that $D \subseteq G$ and $G \in \mathcal{I}_{\mathcal{K}}$. Without loss of generality we may assume that D is dense in 2^ω. Enumerate $\mathcal{K}$ as $\{K_\alpha\}_{\alpha<\kappa}$. For each $\alpha < \kappa$ let us choose a G_δ set $G_\alpha \subseteq 2^\omega$ such that $D \subseteq G_\alpha$ and $2^{K_\alpha} \neq G_\alpha \restriction K_\alpha$. Since D is a countable dense set, there exists a homeomorphism $h \colon 2^\omega \to 2^\omega$ such that $h[D] = [\omega]^{<\omega}$. Since $\kappa < \mathbf{d}$ there exists a G_δ set $G^* \subseteq 2^\omega$ such that $[\omega]^{<\omega} \subseteq G^*$

and $\forall_{\alpha<\kappa} G^* \subseteq h[G_\alpha]$. Then it is easy to see that the set $G = h^{-1}[G^*]$ has the required properties. $\square$

In virtue of Theorem 2.4 we obtain the following strengthening of Balcerzak–Plewik's Theorem 1 of [BP].

Corollary 3.4. *Suppose that $\mathcal{K} \subseteq [\omega]^\omega$ and $|\mathcal{K}| < \mathbf{d}$. Assume also that $\mathcal{K}$ is a normal system. Then the σ-ideal $\mathcal{I}_{\mathcal{K}}$ has the $P^{\mathrm{Tree}}(\Delta^0_1)$ property.*

4. "Lusin" sets

Let us start with a (folklore) fact:

Fact 4.1. *Assume CH. Suppose that $\mathcal{I}$ is a σ-ideal with a Borel basis. The following conditions are equivalent*

1. $\mathrm{Lusin}(\mathcal{I}) \subseteq (s_0)$,
2. *$\mathcal{I}$ has the* (P) *property.*

Let us formulate an analogue of this fact in case of (UP) property.

Theorem 4.2. *Assume CH. Suppose that $\mathcal{I}$ is a σ-ideal with a Borel basis. The following conditions are equivalent*

1. $\mathrm{Lusin}(\mathcal{I}) \subseteq \mathrm{UCR}_0$,
2. *$\mathcal{I}$ has the* (UP) *property.*

P r o o f. (2) $\Rightarrow$ (1). Obvious: If $X \in \mathrm{Lusin}(\mathcal{I})$ and $F\colon 2^\omega \to 2^\omega$ is a continuous function, then there exists $A \in [\omega]^\omega$ such that $F[P(A)] \in \mathcal{I}$ and then $\big|F\big[P(A)\big] \cap X\big| \leqslant \omega$, hence $X \in \mathrm{UCR}_0$.

$\neg$(2) $\Rightarrow$ $\neg$(1). Assume that $\forall_{A\in[\omega]^\omega} F\big[P(A)\big] \notin \mathcal{I}$ for some continuous function $F\colon 2^\omega \to 2^\omega$. Let $(B_\alpha\colon \alpha < 2^\omega)$ be an enumeration of all Borel elements of $\mathcal{I}$. Next, let $\{A_\alpha : \alpha < 2^\omega\}$ be an enumeration of $[\omega]^\omega$ with each set appearing 2^ω many times. For every $\alpha < 2^\omega$ choose

$$x_\alpha \in F\big[P(A_\alpha)\big] \setminus \left[\bigcup_{\mu<\alpha} B_\mu \cup \{x_\mu : \mu < \alpha\}\right].$$

Define:

$$X = \{x_\alpha : \alpha \in 2^\omega\}.$$

It is obvious that the set X belongs to $\mathrm{Lusin}(\mathcal{I})$. For each $\alpha \in 2^\omega$, we have $\Big|F\big[P(A_\alpha)\big] \cap X\Big| = 2^\omega$, hence $X \notin \mathrm{UCR}_0$. $\square$

Analogously we can prove the following theorem.

Theorem 4.3. *Assume CH. Suppose that $\mathcal{I}$ is a σ-ideal with a Borel basis. The following conditions are equivalent*

1. $\mathrm{Lusin}(\mathcal{I}) \subseteq \mathrm{BUCR}_0$,
2. *$\mathcal{I}$ has the* (BUP) *property.*

P r o o f. $(2) \Rightarrow (1)$. Obvious: If $X \in \mathrm{Lusin}(\mathcal{I})$ and $C\colon [\omega]^\omega \to 2^\omega$ is a continuous function, then there exists $A \in [\omega]^\omega$ such that $C\big[[A]^\omega\big] \in \mathcal{I}$ and then $|C\big[[A]^\omega\big] \cap X| \leqslant \omega$, hence $X \in \mathrm{BUCR}_0$.

$\neg(2) \Rightarrow \neg(1)$. Assume that $\forall_{A \in [\omega]^\omega}\ C\big[[A]^\omega\big] \notin \mathcal{I}$ for some continuous function $C\colon [\omega]^\omega \to 2^\omega$. Let $(B_\alpha \colon \alpha < 2^\omega)$ be an enumeration of all Borel elements of $\mathcal{I}$. Next, let $\{A_\alpha \colon \alpha < 2^\omega\}$ be an enumeration of $[\omega]^\omega$ with each set appearing 2^ω many times. For every $\alpha < 2^\omega$ choose

$$x_\alpha \in C\big[[A_\alpha]^\omega\big] \setminus \left[\bigcup_{\mu<\alpha} B_\mu \cup \{x_\mu \colon \mu < \alpha\}\right].$$

Define:

$$X = \{x_\alpha \colon \alpha \in 2^\omega\}.$$

It is obvious that the set X belongs to $\mathrm{Lusin}(\mathcal{I})$. For each $\alpha \in 2^\omega$, we have $\Big|C\big[[A_\alpha]^\omega\big] \cap X\Big| = 2^\omega$, hence $X \notin \mathrm{BUCR}_0$. $\square$

It is natural to expect that one can find a notion of smallness $\mathcal{I}$ such that an analogy to previous theorems will hold in case that $\mathcal{I}$ has the $P^{\mathrm{Tree}}(\Delta^0_1)$ property. The next theorem shows that the answer is affirmative. Before to prove this theorem we introduce the following notion of smallness.

Definition 4.4. We say that $A \subseteq 2^\omega$ is in the $P^{\mathrm{Tree}}(\Delta^0_1)$ ideal iff there exists a tree system $\langle \mathcal{B}_n \colon n \in \omega\rangle$ such that for every Cantor system $\langle B_r \colon r \in 2^{<\omega}\rangle$ we have

$$\big|\mathcal{A}(\langle B_r\rangle) \cap X\big| \leqslant \omega.$$

Fact 4.5. *Without loss of generality we can assume that the tree system is of the form $\mathcal{B}_n = \big\{[t] \colon t \in 2^{x(n)}\big\}$, where $x \in \omega^{\uparrow\omega}$.*

Fact 4.6. *If for each $n \in \omega$, $X_n \in P^{\mathrm{Tree}}(\Delta^0_1)$, then $\bigcup_{n\in\omega} X_n \in P^{\mathrm{Tree}}(\Delta^0_1)$.*

P r o o f. For each $n \in \omega$ let us choose a *tree system* $\big\langle \mathcal{B}^{(n)}_k \colon k \in \omega\big\rangle$ such that for every Cantor system $\big\langle B^{(n)}_r \colon r \in 2^{<\omega}\big\rangle$ we have

$$\big|\mathcal{A}(\langle B^{(n)}_r\rangle) \cap X\big| \leqslant \omega.$$

Let us consider the following system:

$$\begin{aligned}
\mathcal{D}_0 &= \mathcal{B}_0^{(0)}\\
\mathcal{D}_1 &= \mathcal{B}_1^{(0)} \wedge \mathcal{B}_1^{(1)}\\
\mathcal{D}_2 &= \mathcal{B}_2^{(0)} \wedge \mathcal{B}_2^{(1)} \wedge \mathcal{B}_2^{(2)}\\
&\vdots\\
\mathcal{D}_n &= \mathcal{B}_n^{(0)} \wedge \mathcal{B}_n^{(1)} \wedge \dots \wedge \mathcal{B}_n^{(n)}\\
&\vdots
\end{aligned}$$

It is routine to verify that this system is a tree system. Suppose that $\langle B_r\colon r \in 2^{<\omega}\rangle$ is a Cantor system. Since for each $n \in \omega$ the set $\mathcal{A}(\langle B_r\rangle)$ is included in $\mathcal{A}(\langle B_r^{(n)\prime}\rangle)$, where $\langle B_r^{(n)\prime}\colon r \in 2^{<\omega}\rangle$ is some Cantor system for $\langle \mathcal{B}_k^{(n)}\colon k \in \omega\rangle$, we conclude that

$$\left|\mathcal{A}(\langle B_r\rangle) \cap X_n\right| \leqslant \omega\,,$$

hence

$$\left|\mathcal{A}(\langle B_r\rangle) \cap \bigcup_{n\in\omega} X_n\right| \leqslant \omega\,.$$

□

Let us formulate the following theorem:

THEOREM 4.7. *Assume CH. Suppose that $\mathcal{I}$ is a σ-ideal with a Borel basis. The following conditions are equivalent*

1. $\mathrm{Lusin}(\mathcal{I}) \subseteq P^{\mathrm{Tree}}(\Delta_1^0)$,
2. *$\mathcal{I}$ has the $P^{\mathrm{Tree}}(\Delta_1^0)$ property.*

Proof. (2) $\Rightarrow$ (1). Obvious: Suppose that $X \in \mathrm{Lusin}(\mathcal{I})$ and let $\mathcal{B}_n = \{[t]\colon t \in 2^{x(n)}\}$ be a tree system such that $\mathcal{A}(\langle B_r\rangle) \in \mathcal{I}$ for every Cantor system $\langle B_r\colon r \in 2^{<\omega}\rangle$. Since $X \in \mathrm{Lusin}(\mathcal{I})$, we have $|X \cap \mathcal{A}(\langle B_r\rangle)| \leqslant \omega$, hence $X \in P^{\mathrm{Tree}}(\Delta_1^0)$.

$\neg(2) \Rightarrow \neg(1)$. Suppose that $\mathcal{I}$ has not the property $P^{\mathrm{Tree}}(\Delta_1^0)$.

Let $(B_\theta)_{\theta\in 2^\omega}$ be an enumeration of some Borel basis of $\mathcal{I}$. We will construct an $X \in \mathrm{Lusin}(\mathcal{I})$ such that $X \notin P^{\mathrm{Tree}}(\Delta_1^0)$. Let

$$\mathcal{B}_n^{(\theta)} = \{[t]\colon t \in 2^{x_\theta(n)}\}$$

be an enumeration of all tree systems with 2^ω many repetitions. For each tree system $(\mathcal{B}_n^{(\theta)})$ let $A_\theta = \mathcal{A}(\langle B_r^{(\theta)}\colon r \in 2^{<\omega}\rangle)$ be a limit of a Cantor system such that $A_\theta \notin \mathcal{I}$. For each θ choose an

$$x_\theta \in A_\theta \setminus \left[\bigcup_{\mu<\theta} B_\mu \cup \{x_\mu\colon \mu < \theta\}\right].$$

Let us define $X = \{x_\theta\colon \theta \in \omega_1\}$. It is easy to conclude that the set X has all the desired properties. $\square$

Corollary 4.8. *Assume CH. Then there exists a set $X \subseteq 2^\omega$ such that X has the UCR_0 property and $X \notin P^{\mathrm{Tree}}(\Delta^0_1)$.*

Proof. By Example 2.1 we know that there exists a σ-ideal $\mathcal{I}$ such that $\mathcal{I}$ has the (UP) property and does not have the $P^{\mathrm{Tree}}(\Delta^0_1)$ property. Since $\mathcal{I}$ does not have the $P^{\mathrm{Tree}}(\Delta^0_1)$ property, we conclude that there exists $X \in \mathrm{Lusin}(\mathcal{I})$ such that $X \notin P^{\mathrm{Tree}}(\Delta^0_1)$. Since X is a $\mathcal{I}$-Lusin set, i.e., $X \in \mathrm{Lusin}(\mathcal{I})$, by Theorem 4.2 we conclude that $X \in \mathrm{UCR}_0$. $\square$

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