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C-UPPER SEMICONTINUOUS AND C⋆-UPPER SEMICONTINUOUS MULTIFUNCTIONS

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ABSTRACT. We express c-upper semicontinuous and c⋆-upper semicontinuous multifunctions in terms of active boundary (Frac) of multifunctions. Characterizations of locally compact and locally countably compact space in terms of c-upper semicontinuous and c⋆-upper semicontinuous multifunctions having closed graphs are given. We give some results when c-upper semicontinuity (c⋆-upper semicontinuity) of multifunctions implies c-upper semicontinuity (c⋆-upper semicontinuity) of corresponding graph multifunctions.

1. Introduction

In what follows let X, Y be Hausdorff topological spaces. Let Z be a topological space. The symbol B^c, $\overline{B}$, will stand for the complement and the closure of $B \subset Z$, respectively. We denote by 2^Z the space of all closed subsets of Z, by $\mathrm{CL}(Z)$ the space of all nonempty closed sets in Z and by $\mathrm{K}(Z)$ the space of all nonempty compact sets in Z.

Let (Z,d) be a metric space. The open d-ball with center $z_0 \in Z$ and radius $\varepsilon > 0$ will be denoted by $S_\varepsilon(z_0)$ and the ε-parallel body $\bigcup_{a\in A} S_\varepsilon(a)$ for a subset A of Z will be denoted by $S_\varepsilon(A)$.

The distance between a point z of Z and a nonempty set $A \subset Z$ will be denoted by $d(z,A)$, where

$$d(z,A) = \inf\{d(z,a) \colon a \in A\}.$$

A multifunction $F \colon Y \mapsto X$ is said to be *upper semicontinuous* (USC) at y_0, if for every open set V containing $F(y_0)$ there exists a neighbourhood U of y_0 such that $F(U) \subset V$ $(F(U) = \bigcup\{F(y) \colon y \in U\})$.

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A multifunction $F\colon Y \mapsto X$ is said to be *c-upper semicontinuous* (c-USC) at y_0 ([Ne]) if for every open set V containing $F(y_0)$ and having a compact complement, there is an open neighbourhood U of y_0 such that $F(U) \subset V$. We say that $F\colon Y \mapsto X$ is *c-upper semicontinuous* if it is c-upper semicontinuous at every $y \in Y$.

It is clear that $F\colon Y \mapsto X$ is c-USC iff $F^{-}(K)$ is a closed set in Y for every compact $K \subset X$.

A multifunction F with closed values with the property $F^{-}(K)$ being a closed set in Y for every compact $K \subset X$ is called *upper Fell semicontinuous* in [Mc]. A multifunction $F\colon Y \mapsto X$ is said to be *c*-upper semicontinuous* (c*-USC) at y_0 if for every open set V containing $F(y_0)$ and having a countably compact complement, there is an open neighbourhood U of y_0 such that $F(U) \subset V$.

Clearly every c*-USC multifunction is also c-USC. If X is a metrizable space then also every c-USC multifunction is c*-USC. We say that $F\colon Y \mapsto X$ has a closed graph, if the set $\{(y,x)\colon x \in F(y)\}$ is a closed set in $Y \times X$. We denote the graph of a mapping F by $\mathrm{Gr}(F)$. Of course, if $F\colon Y \mapsto X$ has a closed graph then F is c-USC.

Proposition 1.1. ([Ne]) *Let X be a locally compact space and Y be a topological space. Let $F\colon Y \mapsto X$ be c-USC with closed values. Then F has a closed graph.*

Proposition 1.2. ([H]) *Let X,Y be first countable topological spaces. Let $F\colon Y \mapsto X$ be c-USC with closed values. Then F has a closed graph.*

By the active boundary of $F\colon Y \mapsto X$ at y_0 ($\operatorname{Frac} F(y_0)$) we mean

$$\operatorname{Frac} F(y_0) = \bigcap\bigl\{\mathrm{cl}\bigl(F(W) \setminus F(y_0)\bigr)\colon W \in \mathcal{B}(y_0)\bigr\}\,,$$

where $\mathcal{B}(y_0)$ stands for a neighbourhood base at y_0 ([Do]).

Remark. The definition does not depend on the choice of a base of the neighbourhoods of y_0.

2.

Proposition 2.1. *Let $F\colon Y \mapsto X$ be a multifunction. If $\operatorname{Frac} F(y_0) \subset F(y_0)$, then F is c-USC at y_0.*

Proof. Suppose F is not c-USC at y_0. Then there exists an open set $V \subset X$ with the compact complement, such that $F(y_0) \subset V$, that for every neighbourhood U of y_0, $F(U) \cap V^c \neq \emptyset$. Then there is a net $\{y_W\colon W \in \mathcal{B}(y_0)\}$

in Y converging to y_0 and a net $\{x_W : W \in \mathcal{B}(y_0)\}$, where $x_W \in F(y_W)$, converging to a point $x_0 \in V^c$. We have $x_0 \in \operatorname{Frac} F(y_0)$ and $x_0 \notin F(y_0)$, a contradiction. $\square$

Theorem 2.2. *Let X be a locally compact space and Y be a topological space, and let $F\colon Y \mapsto X$ be a multifunction. Suppose that $F(y_0)$ is closed. Then F is c-USC at y_0 if and only if $\operatorname{Frac} F(y_0) \subset F(y_0)$.*

Proof. $\Leftarrow$ It follows from Proposition 2.1.
$\Rightarrow$ Suppose that $\operatorname{Frac} F(y_0)$ is not a subset of $F(y_0)$. Let $x_0 \in \operatorname{Frac} F(y_0)$ such that $x_0 \notin F(y_0)$. Since $F(y_0)$ is closed and X is locally compact, there exists a compact neighbourhood K of x_0 such that $K \cap F(y_0) = \emptyset$. Since $x_0 \in \operatorname{Frac} F(y_0)$ we have $F(U) \cap K \neq \emptyset$ for all $U \in \mathcal{B}(y_0)$ where $\mathcal{B}(y_0)$ is a family of all neighbourhoods of y_0, a contradiction with c-USC of F at y_0. $\square$

Theorem 2.3. *Let X be first countable and Y be first countable at y_0. Let $F\colon Y \mapsto X$ be a multifunction. Suppose that $F(y_0)$ is closed. Then F is c-USC at y_0 if and only if $\operatorname{Frac} F(y_0) \subset F(y_0)$.*

Proof. $\Leftarrow$ It follows from Proposition 2.1.
$\Rightarrow$ Suppose that $\operatorname{Frac} F(y_0)$ is not a subset of $F(y_0)$. Let $x_0 \in \operatorname{Frac} F(y_0)$ such that $x_0 \notin F(y_0)$. Since X, Y are first countable at y_0, there are sequences $\{y_n : n \in \mathbb{Z}^+\}$ in Y and $\{x_n : n \in \mathbb{Z}^+\}$ in X such that $\{(y_n, x_n) : n \in \mathbb{Z}^+\}$ converges to (y_0, x_0) and such that $x_n \in F(y_n)$ for all $n \in \mathbb{Z}^+$. The openness of $X \setminus F(y_0)$ and the convergence of $\{x_n : n \in \mathbb{Z}^+\}$ to x_0 imply that there exists $n_0 \in \mathbb{Z}^+$ such that $x_n \notin F(y_0)$ for every $n \geqslant n_0$. Now $K = \{x_n : n \geqslant n_0\} \cup \{x_0\}$ is a compact set in X with $F(y_0) \cap K = \emptyset$. The c-USC of F at y_0 implies that there is an open neighbourhood U of y_0 with $F(U) \cap K = \emptyset$, a contradiction. $\square$

Proposition 2.4. *Let X be a regular locally countably compact space and Y be a topological space. Let $F\colon Y \mapsto X$ be $c^\star$-USC multifunction with closed values. Then F has a closed graph.*

The proof of the next Proposition is similar to the proof of Proposition 2.1.

Proposition 2.5. *Let Y be first countable at y_0. If $\operatorname{Frac} F(y_0) \subset F(y_0)$ then F is $c^\star$-USC at y_0.*

Theorem 2.6. *Let X be a locally countably compact and Y be first countable and, let $F\colon Y \mapsto X$ be a multifunction. Suppose that $F(y_0)$ is closed. Then F is $c^\star$-USC at y_0 if and only if $\operatorname{Frac} F(y_0) \subset F(y_0)$.*

Proof. $\Leftarrow$ It follows from Proposition 2.5.
$\Rightarrow$ The proof is similar to the proof of the necessary condition of c-USC in Theorem 2.2. $\square$

The proof of the next Theorem follows from Theorem 1.1 and Theorem 1 in [HM].

THEOREM 2.7. *Let Y be a topological space. The following are equivalent:*

(a) *Y is locally compact;*

(b) *For every space X, every c-USC multifunction F with closed values from Y to X has a closed graph.*

The proof of the next Theorem follows from Proposition 2.4 and Theorem 1' in [HM].

THEOREM 2.8. *Let Y be a regular topological space. The following are equivalent:*

(a) *Y is locally countably compact;*

(b) *For every space X, every $c^\star$-USC multifunction with closed values from Y to X has a closed graph.*

3.

For any function $f\colon Y \mapsto X$ the corresponding graph function $\widehat{f}\colon Y \mapsto Y \times X$ is defined by $\widehat{f}(y) = \big(y, f(y)\big)$ for each $y \in Y$. Some results about relation between c-continuity f and $\widehat{f}$ are given in [HM].

For any multifunction $F\colon Y \mapsto X$ we define the corresponding graph multifunction $\widehat{F}$ as follows:

$$\widehat{F}(y) = \big\{(y,x)\colon x \in F(y)\big\}.$$

We give some results when c-upper semicontinuity ($c^\star$-upper semicontinuity) of multifunctions implies c-upper semicontinuity ($c^\star$-upper semicontinuity) of corresponding graph multifunctions.

THEOREM 3.1. *Let X, Y be topological spaces. If $F\colon Y \mapsto X$ is USC at y_0 and $F(y_0)$ is compact then $\widehat{F}\colon Y \mapsto Y \times X$ is USC at y_0.*

P r o o f. Suppose that $\widehat{F}\colon Y \mapsto Y \times X$ is not USC at y_0. Let $\mathcal{B}$ be an open neighbourhood base at y_0. Then there is an open neighbourhood W of $\widehat{F}(y_0)$ such that $\widehat{F}(V) \cap W^c \neq \emptyset$ for every $V \in \mathcal{B}$. The set

$$\{y_0\} \times F(y_0) = \big\{(y_0, x)\colon x \in F(y_0)\big\}$$

is compact in $Y \times X$. There are open neighbourhoods U of y_0 and O of $F(y_0)$ such that $\widehat{F}(y_0) \subset U \times O \subset W$. There is an open neighbourhood base $\mathcal{H}$ of y_0

such that $\mathcal{H} \subset \mathcal{B}$ and such that $V \subset U$ for all $V \in \mathcal{H}$. For all $V \in \mathcal{H}$ we have $F(V) \cap O^c \neq \emptyset$, a contradiction. □

The following example shows that the condition of compactness of $F(y)$ in Theorem 3.1 is essential.

EXAMPLE 3.2. Let $X = \mathbb{Z}^+$ and $Y = \{0\} \cup \left\{\frac{1}{n} \colon n \in \mathbb{Z}^+\right\}$. Consider the inherited euclidean topologies on Y and X. It is easy to verify that the multifunction $F \colon Y \mapsto X$ defined by

$$F(y) = \begin{cases} X, & y = 0, \\ \{n+1\}, & y = \frac{1}{n} \end{cases}$$

is USC at 0. The multifunction $\widehat{F} \colon Y \mapsto Y \times X$ is not USC at 0. (The set $A = \bigcup\limits_n \left\{\{\frac{1}{n}\} \times \{1, \dots, n\}\right\} \cup \left\{\{0\} \times \mathbb{Z}^+\right\}$ is an open set such that $\widehat{F}(0) \subset A$ but $\widehat{F}(V) \cap A^c \neq \emptyset$ for all $V \subset \mathcal{B}$, where $\mathcal{B}$ is an open neighbourhood base $\mathcal{B}$ at 0.)

THEOREM 3.3. *Let X,Y be topological spaces. If $F \colon Y \mapsto X$ is c-USC at y_0 and $F(y_0)$ is compact then $\widehat{F} \colon Y \mapsto Y \times X$ is c-USC at y_0.*

Proof. Suppose that $\widehat{F} \colon Y \mapsto Y \times X$ is not c-USC at y_0. Let $\mathcal{B}$ be an open neighbourhood base of y_0. Then there is an open neighbourhood W of $\widehat{F}(y_0)$ having compact complement such that $\widehat{F}(V) \cap W^c \neq \emptyset$ for every $V \in \mathcal{B}$. For every $V \in \mathcal{B}$ choose $y_V \in V$ and $x_V \in F(V)$ such that $(y_V, x_V) \in W^c$. The set

$$\{y_0\} \times F(y_0) = \left\{(y_0, x) \colon x \in F(y_0)\right\}$$

is compact in $Y \times X$. Then there are open neighbourhoods U of y_0 and O of $F(y_0)$ such that $\widehat{F}(y_0) \subset U \times O \subset W$. It is easy to see that there exists an open neighbourhood base $\mathcal{H}$ at y_0 where $\mathcal{H} \subset \mathcal{B}$ such that $x_V \in O^c$ for all $V \in \mathcal{H}$. Define

$$A = \mathrm{cl}\left\{(y_V, x_V) \colon V \in \mathcal{H}\right\}.$$

Since $A \subset W^c$ and A is compact, then $\Pi_x A$ is compact where $\Pi_x A \subset O^c$ (Π_x is the projection map of $Y \times X$ onto X). It now follows that $X \setminus \Pi_x A$ is an open neighbourhood of $F(y_0)$ with compact complement such that $F(V) \cap \Pi_x A \neq \emptyset$ for all $V \in \mathcal{H}$. Therefore F is not c-USC at y_0, a contradiction. □

THEOREM 3.4. *Let X be a regular topological space and Y be a topological space. If $F \colon Y \mapsto X$ is c-USC at y_0 and $F(y_0)$ is a closed set, then $\widehat{F} \colon Y \mapsto Y \times X$ is c-USC at y_0.*

Proof. Suppose that $\widehat{F}$ is not c-USC at y_0. Let $\mathcal{B}$ be an neighbourhood base of y_0. Then there is an open neighbourhood W of $\widehat{F}(y_0)$ in $Y \times X$ having compact complement such that $\widehat{F}(V) \cap W^c \neq \emptyset$ for every $V \in \mathcal{B}$. Then for every

$V \in \mathcal{B}$ there exists a point $(y_V, x_V) \in Y \times X$ where $y_V \in V$ and $x_V \in F(V)$ such that $(y_V, x_V) \in W^c$. W^c is a compact set, then the net $\{(y_V, x_V) \colon V \in \mathcal{B}\}$ has a cluster point, which is a point of line $y_0 \times X$. We denote this point by (y_0, x_0). We have that $x_0 \in (F(y_0))^c$, X is a regular space, then there exists a closed neighbourhood U of x_0 such that $U \cap F(y_0) = \emptyset$. Define

$$A = \mathrm{cl}\{(y_V, x_V) \colon V \in \mathcal{B}\}.$$

Since $A \subset W^c$, A is compact and then $\Pi_x A$ is a compact set in X, where Π_x is the projection map of $Y \times X$ onto X. Then $U \cap \Pi_x A$ is a compact set such that $(U \cap \Pi_x A) \cap F(y_0) = \emptyset$ and $(U \cap \Pi_x A) \cap F(V) \neq \emptyset$ for all $V \in \mathcal{H}$ where $\mathcal{H} \subset \mathcal{B}$ is a neighbourhood base of y_0, a contradiction. $\square$

The following example shows that the condition of closedness of $F(y)$ in Theorem 3.4 is essential.

EXAMPLE 3.5. Consider $X = Y = \{\frac{1}{n} \colon n \in \mathbb{Z}^+\} \cup \{0\}$ with the inherited euclidean topology. Let

$$F(y) = \begin{cases} \{\frac{1}{n}\} & y = \frac{1}{n}, \\ \{\frac{1}{n} \colon n \in \mathbb{Z}^+\} & y = 0. \end{cases}$$

The multifunction $F \colon Y \mapsto X$ is c-USC at 0 but the multifunction $\widehat{F} \colon Y \mapsto Y \times X$ is not c-USC at 0 since $\left(\{(0,0)\} \cup \{(\frac{1}{n}, \frac{1}{n}) \colon n \in \mathbb{Z}^+\}\right)^c$ is the open neighbourhood of $\widehat{F}(0)$ having compact complement such that $\widehat{F}(V) \cap \left(\{(0,0)\} \cup \{(\frac{1}{n}, \frac{1}{n}) \colon n \in \mathbb{Z}^+\}\right) \neq \emptyset$ for all neighbourhood V of 0.

Since the projection of a countable compact set is countably compact, a countably compact set in first countable spaces is a closed set, the proof of the next theorem is similar to the proof of Theorem 3.3.

THEOREM 3.6. *Let X, Y be first countable topological spaces. If $F \colon Y \mapsto X$ is $c^\star$-USC at y_0 and $F(y_0)$ is compact, then $\widehat{F} \colon Y \mapsto Y \times X$ is $c^\star$-USC at y_0.*

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