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METAFUNCTIONS

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ABSTRACT. In this article a new notion is introduced. This notion (metafunction) is a generalization of the notion of multifunction. Some examples of metafunctions are shown. Various continuity properties of metafunctions are defined and investigated. It is shown that a metafunction, which represents some kind of "inverse mapping" to the Lebesque measure, has strong continuity properties.

1. Introduction

In this article a new notion will be introduced. We will define the notion of metafunction. Broadly speaking, a metafunction is a generalized multifunction. A motivation for this kind of generalization will be shown in examples. Some of the examples will be taken from real life, some from topology and related branches of mathematics.

Various continuity properties of metafunctions will be defined and investigated. The introduction of a new notion of this kind inevitably generates a great number of open questions. Several open questions are listed in the last section of the paper.

2. Sets, logical levels, metasets

Every researcher in biology knows, that investigation of properties of cells requires qualitatively different methods than investigation of properties of tissues. Using the language of Gregory Bateson [2], tissues are perceived from a higher logical level than the cells are.

From a mathematical point of view, we obtain more information about the set of cells of a concrete body, if we represent this set in a more structured form:

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$\big\{\{The\ heart\ cells\}, \{The\ blood\ cells\}, \{The\ brain\ cells\} \ldots\ etc.\big\}$

So the passage from one (logical) level of observation to a higher one can be modeled by replacing a set X by an appropriate system of its subsets.

Let's consider one more real life example: When buying lottery tickets, some people avoid buying tickets with numbers like 11111 or 33333 arguing that "such numbers never win". They prefer to buy tickets with an inconspicuous number like 23875, or 62179: "Numbers like these win most of the time."

Of course, this kind of thinking is erroneous. The probability that number 11111 wins is exactly the same as the probability that 23875 wins. Nevertheless, there is a deeply hidden intuitive knowledge of probability laws in this error. Namely the tickets from the set $A = \{11111, 22222, \ldots, 99999\}$ win less frequently than the tickets from the set $B = \{23875, 62179, \ldots, \text{similar numbers}\}$. The only mistake that an eager-to-win ticket buyer commits is skipping one logical level below: the relationship between the elements 11111 and 23875 is not inherited from the relationship of their "mother sets".

3. The notion of metafunction

In this section the notion of metafunction will be defined and some examples of metafunctions will be given.

First, we will give an informal definition of metafunction: If a multifunction is a point-to-set mapping, a metafunction will be a "point to metaset" mapping. By "metaset" or "structured set" we denote a set described through its subsets. So if $A = \{1, 2, 3, 4, 5, 6\}$ is a set, $B = \big\{\{2, 4, 6\}, \{3, 6\}\big\}$ is a metaset, describing a structure on A. The prefix "meta" is used to stress the fact that B provides more information about A than A does when presented unstructured. In our particular case, "meta-information" about A contained in B concerns the divisibility by certain numbers (e.g., $2, 3$).

In what follows, by $P(Y)$ we denote the set of all nonempty subsets of a nonempty set Y. A metaset on Y is a nonempty subset of $P(Y)$, so it is a point from $P\big(P(Y)\big)$.

Definition 1. Let X, Y be two nonempty sets. A mapping $M\colon X \to P\big(P(Y)\big)$ will be called *metafunction* from X to Y. We will denote such a metafunction by $M\colon X \uparrow Y$.

Let's remark that in some cases to consider a void set as a point from $P(Y)$ might be useful. In this paper, however, we will work only with nonempty subsets of Y and $P(Y)$.

Some metafunctions are shown in the examples below. The reader can observe that these metafunctions provide more information about certain mathematical objects than multifunctions do.

Example 1. Let λ be the Lebesgue measure defined on $\mathbb{R}$. Let $C(\mathbb{R})$ be the set of all closed nonempty subsets of $\mathbb{R}$. We define a metafunction $M\colon [0,+\infty) \uparrow \mathbb{R}$ in the following way:

$$\forall t \in [0,+\infty) \quad M(t) = \{A \in C(\mathbb{R})\colon \lambda(A) = t\}\,.$$

So that the metafunction M is an "inverse mapping" to λ defined on $C(\mathbb{R})$.

Example 2. In this example we define a metafunction, which reflects the perfect symmetry of the real line and which also evokes a periodicity phenomenon. First, let us consider the following metaset A:

$$A = \big\{\{a+k\colon k \in \mathbb{Z}\}\colon a \in \mathbb{R}\big\}\,.$$

So

$$A = \big\{\{\mathbb{Z}+a\}\colon a \in \mathbb{R}\big\}\,.$$

Imagine the set $\mathbb{Z}+a$ for certain a and try to make the picture of A. All the sets contained in A are sets for which "the periodicity gap equals 1". Let's now imagine this gap rising continuously from 1 to 2. We obtain the following metafunction $S\colon [1,2] \uparrow \mathbb{R}$:

$$\forall x \in [1,2] \quad S(x) = \big\{\{a+kx\colon k \in \mathbb{Z}\}\colon a \in \mathbb{R}\big\}\,.$$

We feel that this metafunction should have some (not yet specifically defined) nice continuity properties.

Example 3. Let X be a Banach space. Let $L\colon X \uparrow X$ be defined as follows:

$$\forall z \in X \quad L(z) = \{p\colon p \text{ is a line in } X \text{ containing } z\}\,.$$

Example 4. Let $X = \mathbb{R}^n$ be equipped with the usual euclidean metrics d. Let H denote the Hausdorff metrics derived from d. Then the mapping M: $[0,+\infty) \to P\big(C(X)\big)$ defined by

$$M(t) = \big\{A\colon A \text{ closed in } \mathbb{R}^n;\ H\big(\{0\},A\big) = t\big\}$$

is a metafunction from X to X.

Example 5. Let T be a measurable space, Y a topological space and $F\colon T \times \mathbb{R} \to Y$ a multifunction. Let $a \in \mathbb{R}$, $a > 0$. Let us define $M_a\colon T \uparrow Y$ by

$$M_a(t) = \big\{F(t,z)\colon z \in [0,a]\big\}\,.$$

So the value of t is represented by all values $F(t,\cdot)$ generated during the time interval $[0,a]$. Intuitively, the metafunction M_a for $a \in [0,1)$ might converge (in a yet not defined sense) to M_1 when a approaches 1.

4. Continuity properties of metafunctions. M-composition

Let us consider topological spaces X and Y and a metafunction $M\colon X \uparrow Y$. In the multifunction theory, there are many possibilities how to define some kind of continuity for a multifunction from X to Y (see [1]). Of course, the number of possibilities increases in the case of metafunctions. The following general way of defining a continuity seems to be quite natural:

We equip $P(Y)$ with a topology (for example Vietoris topology). Then, we can say that $M\colon X \uparrow Y$ has some continuity property if the multifunction $M\colon X \to P(Y)$, derived from M in a natural way, has some continuity property.

Below we define explicitly three different notions of a continuity for metafunctions. Then, we will present some examples and results related to these properties.

Definition 2. Let (X,S), (Y,T) be topological spaces. Let $M\colon X \uparrow Y$ be a metafunction, let $x \in X$. We will say that

1. M is *L-lower continuous* at x (*L.L.c.* in x) iff $\forall A \in M(x)$, $\forall O \in T$ such that $A \cap O \neq \emptyset$ $\exists U(x)$ (a neighbourhood of x) such that $\forall t \in U(x)$ $\exists A' \in M(t)$ such that $A' \cap O \neq \emptyset$.
2. M is *L-upper continuous* at x (*L.U.c.* in x) iff $\forall A \in M(x)$, $\forall O \in T$ such that $A \subset O$ $\exists U(x)$ (a neighbourhood of x) such that $\forall t \in U(x)$ $\exists A' \in M(t)$ such that $A' \subset O$.
3. M is *L-continuous* at x (*L.c.* in x) iff $\forall A \in M(x)$, $\forall n \in \mathbb{N}$, $\forall O$, $V_1, V_2, \dots, V_n \in T$ such that $A \subset O$, $A \cap V_i \neq \emptyset$, $i = 1,2,\dots,n$, $\exists U(x)$ (a neighbourhood of x) such that $\forall t \in U(x)$ $\exists A' \in M(t)$ such that $A' \subset O$, $A' \cap V_i \neq \emptyset$, $i = 1,2,\dots,n$.

Definition 3. Let A, B be sets, $F\colon A \to B$ be a multifunction. We will denote by F_L the metafunction defined by:

$$F_L(a) = \big\{\{t\}\colon t \in F(a)\big\} \qquad \forall a \in A.$$

We will denote by F_U the metafunction defined by:

$$F_U(a) = \big\{F(a)\big\} \qquad \forall a \in A.$$

Remark 1. It is easy to see that if F is an l.s.c. multifunction, then both F_L and F_U are L.L.c. metafunctions. Moreover, F_L is an L.U.c. metafunction. If F is a u.s.c. multifunction, F_U is an L.U.c. metafunction, in general, F_U need not be L.L.c..

It is clear that if a metafunction M is L-continuous at a point x, then it is both L.L.c. and L.U.c. at x. So for a metafunction being L.L.c. and L.U.c. at

x there are necessary conditions for being L.c. at x. As Example 6 will show, these conditions are not sufficient ones.

In the definition above we always considered only one set A from $M(x)$. That is why all the defined continuity properties were called "L"-continuities. We could define "U-continuities" (U.L.c., U.U.c., U.c.) analogously. For example:

We will say that a metafunction $M\colon X \uparrow Y$ is *U-lower continuous* at x (*U.L.c.* in x) iff

$\forall O \in T$ such that $\forall A \in M(x)$ $A \cap O \neq \emptyset$ $\exists U(x)$ (a neighbourhood of x) such that $\forall t \in U(x)$ $\forall A' \in M(t)$ $A' \cap O \neq \emptyset$.

EXAMPLE 6. Let $M\colon \mathbb{R} \uparrow \mathbb{R}$ be defined as follows:

$M(0) = \{\mathbb{R}\}$,

$M(x) = \big\{\{a\}\colon a \in \mathbb{R}\big\}$, $x \neq 0$.

M is L.L.c. and L.U.c at 0. However, M is not L-continuous at 0. Let us take, for example, $U = \mathbb{R}$, $V_1 = (0,1)$, $V_2 = (2,3)$. Then there exists $A \in M(0)$, $A \subset U$, $A \cap V_1 \neq \emptyset$, $A \cap V_2 \neq \emptyset$.

On the other hand if $t \neq 0$, $A' \in M(t)$ and $A' \cap V_1 \neq \emptyset$ then $A' \cap V_2 = \emptyset$ because A' is a singleton.

In Theorem 1 below we will prove that the metafunction M defined in Example 1 (the "inverse metafunction to the Lebesgue measure") is L-continuous. First, we show in an example that L.U. continuity is quite a strong property for a metafunction and that L-continuity is a very strong property.

EXAMPLE 7. The metafunction S from Example 2 is L.L. continuous, but it is not L.U. continuous.

a) It is easy to see that S is L.L. continuous. Namely, if V is a nonempty open set and $v \in V$, then $A_x = \{v + kx\colon k \in \mathbb{Z}\}$ is from $S(x)$ and $A_x \cap V \neq \emptyset$.

b) S is not L.U. continuous. Consider the set $A = \mathbb{Z} = \{k \cdot 1\colon k \in \mathbb{Z}\}$. Then $A \in S(1)$.

Put $U = \left(-\infty, \frac{1}{2}\right) \cup \bigcup\limits_{k=1}^{\infty} \left(k - \frac{1}{k},\ k + \frac{1}{k}\right)$. Then $A \subset U$. However, for $t \in \left(1, \frac{3}{2}\right)$ there is no $A' \in S(t)$ with the property $A' \subset U$. Therefore S is not L.U.c. So it cannot be L-continuous either.

THEOREM 1. *The metafunction* $M\colon [0,+\infty) \to C(\mathbb{R})$ *defined by*

$$M(\alpha) = \big\{A \in C(\mathbb{R})\colon \lambda(A) = \alpha \ \textit{for all}\ \alpha \in [0,+\infty)$$

(see Example 1) is L-continuous.

Proof. Let $\alpha \in [0,+\infty)$. Let $A \in M(\alpha)$. Let $U, V_1, V_2, \dots, V_n$ be open subsets of $\mathbb{R}$ such that $A \subset U$, $A \cap V_i \neq \emptyset$, $i = 1, 2, \dots, n$.

Denote $W = U - A$. W is a nonempty open set, so it contains a nonempty open interval. Therefore there exists a point $a \in W$ and an $\varepsilon > 0$ such that $[a, a+\varepsilon]$ is a subset of W.

Let us consider ε to be positive. (If $\alpha > 0$ let's take $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$.) For each t, $0 < t < \varepsilon$, let's define $A_t = A \cup [a, a+t]$.

(*) Each of the sets A_t is closed. Moreover, $A_t \in M(\alpha + t)$, $A_t \subset U$ and $A_t \cap V_i \neq \emptyset$, $i = 1, 2, \dots, n$.

If $\alpha = 0$, the L-continuity of M is proved.

Let $\alpha > 0$. For $i = 1, 2, \dots, n$ let's choose a point $p_i \in A \cap V_i$. For $\delta \geqslant 0$ let us define $B_\delta = \left(\bigcup_{i=1}^{n} [p_i - \delta,\, p_i + \delta] \right) \cap A$.

(In general, the intervals defined above are not disjoint.)

(**) The sets B_δ are closed subsets of U. Moreover, for every $\delta \geqslant 0$ and for each $i = 1, 2, \dots, n$, $B_\delta \cap V_i \neq \emptyset$.

Let's define a function $f\colon [0, +\infty) \to \mathbb{R}$ by $f(\delta) = \lambda(B_\delta)$. We see that $f(0) = 0$, $\lim_{\delta \to +\infty} f(\delta) = \lambda(A) = \alpha$.

Moreover, f is an increasing function with the Darboux property and

$$f\big([0, +\infty)\big) = [0, \alpha].$$

Because of this for every $t \in [0, \varepsilon]$ there exists $\delta_t \geqslant 0$ such that $B_{\delta_t} \in M(\alpha - t)$.

Now, (*) and (**) imply the L-continuity of M. $\square$

When we make a composition of two multifunctions, we can lose a lot of information about the two component multifunctions. We can preserve more information when "a composition" in the form of metafunction is created. We show it in the following example.

Example 8. Let G, F, H, S be multifunctions from $\mathbb{R}$ to $\mathbb{R}$ defined as follows:

$\forall x \in \mathbb{R}\ \ G(x) = x + Q$,

$\forall x \in \mathbb{R}\ \ F(x) = [x - 1, x + 1]$,

$\forall x \in \mathbb{R}\ \ H(x) = \{-x, x\}$,

$\forall x > 0\ \ S(x) = [0, +\infty)$, $\forall x < 0\ \ S(x) = (-\infty, 0]$,

$S(0) = (-\infty, +\infty)$.

It is easy to see that $F \circ G = S \circ H$. If we denote $\mathbb{Z} = F \circ G$ then

$\mathbb{Z}(x) = (-\infty, +\infty)$ for each x from $\mathbb{R}$.

Now, let's examine the following two metafunctions M and P:

$\forall x \in \mathbb{R}\ \ M(x) = \big\{F(t)\colon t \in G(x)\big\} = \big\{[x + q - 1, x + q + 1]\colon q \in Q\big\}$,

$\forall x \in \mathbb{R} \;\; P(x) = \{S(t) \colon t \in H(x)\}$ so

$\forall x \neq 0 \;\; P(x) = \{(-\infty, 0\rangle, [0, +\infty)\}$, $P(0) = \{(-\infty, +\infty)\}$.

We can see that M (P) forms "a composition" of F, G (S, H). It is a kind of a composition that preserves more information about the original components.

Example 8 serves as a motivation for the following definition.

Definition 4. Let A, B, C be sets. Let $L \colon A \uparrow B$, $M \colon B \uparrow C$ be metafunctions.

We denote the following multifunction by UL:

$$\mathrm{UL}(x) = \bigcup_{A \in L(X)} A .$$

The metafunction $K \colon A \uparrow C$ defined by

$$K(a) = \bigcup_{t \in \mathrm{UL}(a)} M(t) \quad \forall a \in A$$

will be called an *m-composition* of M and L. We will write $K = M \otimes L$.

Let $F \colon B \to C$, $G \colon A \to B$ be multifunctions. We define $F \otimes G$ by $F \otimes G = F_U \otimes G_L$ (or $F_U \otimes G_U$ which is the same). We say that $F \otimes G$ is an *m-composition* of F and G.

Remark 2. For metafunctions M, P and multifunctions F, G, S, H from Example 8

$$M = F \otimes G, \;\; P = S \otimes H \;\text{ is true}.$$

In general, if F, G are two multifunctions and if it is possible to make their m-composition $F \otimes G$, then

$$F \otimes G = \{F(t) \colon t \in G(x)\} \;\text{ for every } x \text{ that is an argument of } G .$$

We will use this fact in Example 9.

The next theorem shows that under certain assumptions continuity properties of metafunctions are preserved by m-composition.

Theorem 2. *Let (X, T), (Y, τ), $(\mathbb{Z}, \upsilon)$ be topological spaces. Let $G \colon X \uparrow Y$ be a L.L.c. metafunction, let $F \colon Y \uparrow \mathbb{Z}$ be a L.U.c. metafunction. Then the metafunction $M \colon X \uparrow \mathbb{Z}$ defined by $M = F \otimes G$ is L.U.c.*

Proof. Let $x \in X$, $U \in \upsilon$ and $A \in M(x)$ such that $A \subset U$.

(*) From the definition of M we see that there exists $B \in G(x)$ and $t \in B$ such that $A \in F(t)$.

Since F is L.U.c. there exists an open neighbourhood $O(t)$ of t such that $\forall t' \in O(t)$ there exists $A' \in F(t')$ such that $A' \subset U$.

(**) Since $t \in O(t)$ and $t \in B$, $B \cap O(t) \neq \emptyset$ holds. But $B \in G(x)$ and G is L.L. continuous. Therefore there exists a neighbourhood $U(x)$ of x such that

$$\forall x' \in U(x)\ \exists B' \in F(x')\,,\ B' \cap O(t) \neq \emptyset\,.$$

(*) and (**) imply that there exists a neighbourhood $U(x)$ of x such that for every x' from $U(x)$ there exist $A' \in F \otimes G(x')$ such that $A' \subset U$.

(Namely, A' will be of the form $A' \in F(t')$ where $t' \in O(t) \cap B'$ and $B' \in G(x')$.) M is proved to be L.U.c. □

The proofs of the two following theorems are almost identical with the proof of Theorem 2 and we omit them.

Theorem 3. *Let (X,T), (Y,τ), $(\mathbb{Z},\upsilon)$ be topological spaces. Let $G\colon X \uparrow Y$ be a L.L.c. metafunction, let $F\colon Y \uparrow \mathbb{Z}$ be a L.L.c. metafunction. Then the metafunction $M\colon X \uparrow \mathbb{Z}$ defined by $M = F \otimes G$ is L.L.c.*

Theorem 4. *Let (X,T), (Y,τ), $(\mathbb{Z},\upsilon)$ be topological spaces. Let $G\colon X \uparrow Y$ be a L.L.c. metafunction, let $F\colon Y \uparrow \mathbb{Z}$ be a L.c. metafunction. Then the metafunction $M\colon X \uparrow \mathbb{Z}$ defined by $M = F \otimes G$ is L.c.*

The following example shows that if G is a u.s.c. multifunction and F is a l.s.c. multifunction, their m-composition need not be neither L.U.c. nor L.L.c.

Let us notice that $F \otimes G = F_U \otimes G_U$ as defined in Definition 4. As mentioned in Remark 1, since $F(G)$ is l.s.c. (u.s.c.), $F_U(G_U)$ is L.L.c.(L.U.c.). So, in the following example we will show that a composition of L.U.c. and L.L.c. metafunctions need not be neither L.L.c. nor L.U.c.

Example 9. Let F, G be multifunctions from $\mathbb{R}$ to $\mathbb{R}$ defined by:

$$\forall x < 1\ \ G(x) = \{1\},\ G(1) = \{0,1\},\ \forall x > 1\ \ G(x) = \{0\},$$

$$\forall x < 1\ \ F(x) = \{x, x+1\},\ \forall x \geqslant 1\ \ F(x) = \{x\}.$$

Then $M = F \otimes G$ is defined as follows:

$$\forall x < 1\ \ M(x) = \big\{\{1\}\big\},$$

$$M(1) = \big\{\{0,1\},\{1\}\big\},$$

$$\forall x > 1\ \ M(x) = \big\{\{0,1\}\big\}.$$

M is not L.U.c. at the point 1 because $A = \{1\} \subset U = \left(\frac{1}{2},\frac{3}{2}\right)$ and for all $t > 1$ there exists no set $A' \in M(t)$ such that $A' \subset U$ would be true.

Analogously, M is not L.L.c. at the point 1 because $\{0,1\} \cap \left(-\frac{1}{2},\frac{1}{2}\right) \neq \emptyset$ but for all $A' \in M(t)$ for $t < 1$ $A' \cap \left(-\frac{1}{2},\frac{1}{2}\right) = \emptyset$.

5. Open questions

Since the notion of a metafunction is new, and at least as complex as the notion of multifunction, there are very many open questions concerning metafunctions. There are also many new related notions to be defined and to be investigated. In the list below we present only some questions concerning topological properties of spaces and mappings.

Question 1. Let (X, d) be a metric space. Define the following metafunctions:

$$M\colon \mathbb{R}^+ \uparrow X\,, \quad \forall t > 0 \;\; M(t) = \big\{\{s,u\}\colon d(s,u) = t\big\}\,,$$

$$P\colon \mathbb{R}^+ \uparrow X\,, \quad \forall t > 0 \;\; P(t) = \big\{A \subset X\colon \forall s,u \in A,\ s \neq u,\ d(s,u) = t\big\}\,.$$

For which spaces are M and P defined correctly? What is the relationship between the properties of M, P and topological properties of X (connectedness, compactness, dimension, completeness)?

Question 2. Let (X, T) be a topological space, p a collection of open covers of X. If $A \subset X$ and $c \in p$ by C-star of A we mean the set $\bigcup\limits_{\substack{U\in c\\ U\cap A\neq\emptyset}} U$. Let us define a metafunction $S\colon X \uparrow X$ as follows:

$$\forall x \in X \;\; S(x) = \big\{A \in C(X)\colon A \text{ is contained in a } c\text{-star of } x \text{ for some } c \text{ from } p\big\}\,.$$

When is S correctly defined? Investigate the relationship between continuity properties of S and topological properties of X.

Question 3. Define and investigate natural and useful continuity (monotonocity) properties for metafunctions.

Question 4. Investigate "collapsing" of metafunctions to multifunctions and the preservation of continuity properties under this operation.

Question 5. Investigate "recovering" of metafunctions from multifunctions with values in hyperspaces (points become sets). Study the preservation of continuity properties under this operation.

Question 6. Study the existence of continuous selections (multifunctions) for continuous metafunctions.

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