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THE $\langle s \rangle$-DENSITY TOPOLOGY IS NOT GENERATED

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ABSTRACT. There are presented some properties of real functions $f\colon \mathbb{R} \to \mathbb{R}$ which are continuous when $\langle s \rangle$-density topology (for a sequence $\langle s \rangle$ such that $\liminf\limits_{n\to\infty} \frac{s_n}{s_{n+1}} = 0$) is used on both the domain and the range. For example, it is shown that such functions are *Baire**1 functions. The main result concerns the property that the $\langle s \rangle$-density topology is not generated.

Let m be the standard Lebesgue measure in the real line $\mathbb{R}$, and $\mathcal{L}$ be the σ-algebra of Lebesgue measurable sets of $\mathbb{R}$. By $\mathbb{N}$ we shall denote the set of all positive integers and by S the family of all unbounded and nondecreasing sequences of positive numbers. By $\langle s \rangle$ we shall denote a sequence $\{s_n\}_{n\in\mathbb{N}} \in S$, by $\mathcal{T}_d$ the classical density topology on $\mathbb{R}$, and by $\mathcal{T}_0$ the natural topology on $\mathbb{R}$. Moreover, by $\mathcal{C}^{\mathcal{T}}_{\mathcal{T}_1}$ we shall denote the family of all continuous functions $f\colon (\mathbb{R},\mathcal{T}_1) \to (\mathbb{R},\mathcal{T})$, where $\mathcal{T}_1, \mathcal{T}$ denotes topologies on $\mathbb{R}$. The symbols $d(A,x)$, $d_{\langle s \rangle}(A,x)$ denote the ordinary density, the $\langle s \rangle$-density of the Lebesque measurable set A at point $x \in \mathbb{R}$, respectively. And by $\omega(f,x)$ we shall denote an oscillation of f at x.

In paper [FH] we can find a definition of a density point connected with a sequence belonging to S.

Definition 1. We shall say that $x \in \mathbb{R}$ is a *density point* of a set $A \in \mathcal{L}$ *with respect to a sequence* $\{s_n\}_{n\in\mathbb{N}} \in S$ (abbr. $\langle s \rangle$-density point) if

$$\lim_{n\to\infty} \frac{m\Big(A \cap \big[x - \frac{1}{s_n}, x + \frac{1}{s_n}\big]\Big)}{\frac{2}{s_n}} = 1\,.$$

Let $\langle s \rangle \in S$ and $A \in \mathcal{L}$. Putting

$$\Phi_{\langle s \rangle}(A) = \big\{x \in \mathbb{R} \colon x \text{ is } \langle s \rangle\text{-density point of } A\big\}\,,$$

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we have that $\Phi_{\langle s\rangle}$ is a lower density operator (see [FH]). Let

$$\mathcal{T}_{\langle s\rangle} = \left\{A \in \mathcal{L} \colon A \subset \Phi_{\langle s\rangle}(A)\right\}.$$

We have the following result:

THEOREM 1. (Cf. [FH].) *Let $\langle s\rangle \in S$. The family $\mathcal{T}_{\langle s\rangle}$ is a topology on the real line. We call it $\langle s\rangle$-density topology.*

Let us introduce the following notation $S_0 = \left\{\langle s\rangle \in S \colon \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\right\}$ and $S_1 = \left\{\langle s\rangle \in S \colon \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0\right\}$. We have the following theorem.

THEOREM 2. (Cf. [FH] and [W].) *Let $\langle s\rangle \in S$. Then $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_d$ if and only if $\langle s\rangle \in S_1$.*

Moreover, we have that $\mathcal{T}_0 \subset \mathcal{T}_d \subset \mathcal{T}_{\langle s\rangle}$ for every $\langle s\rangle \in S$. So $\mathbb{R}$ with the $\langle s\rangle$-density topology is a Hausdorff space for every $\langle s\rangle \in S$.

Further in this paper we show that for all $\langle s\rangle$ belonging to S the $\langle s\rangle$-density topology is not generated. Regarding Theorem 2 we should limit our consideration to sequences belonging to S_0, because just then we can obtain new results. In spite of this fact I decided to formulate further properties for every sequences belonging to S. Of course, it is well-known that they are true for the density topology (see [CL1]).

DEFINITION 2. We shall say that $\mathcal{T}_1$ topological space $(X, \mathcal{T})$ is *generated* if for every $\mathcal{T}_1$ topology τ on X such that $\mathcal{C}_{\mathcal{T}}^{\mathcal{T}} \subset \mathcal{C}_{\tau}^{\tau}$ we have that $\mathcal{T} \subset \tau$.

The generated spaces are characterized by the following theorem:

THEOREM 3. (Cf. [Wa].) *A Hausdorff topological space (X, τ) is generated if and only if the class of complements of level sets of its continuous selfmaps is a subbase for τ.*

So we will show that $\left\{\mathbb{R} \setminus f^{(-1)}(a) \colon a \in \mathbb{R},\ f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}\right\}$ is not a subbase for the topology $\mathcal{T}_{\langle s\rangle}$. However, first we need some information concerning a function belonging to the family $\mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$.

If $f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$, we shall say that f is *$\langle s\rangle$-density continuous function.*

THEOREM 4. *If a family $\mathcal{T}$ is a topology such that $\mathcal{T}_0 \subset \mathcal{T}$ then a triple $\left(\mathcal{C}_{\mathcal{T}}^{\mathcal{T}}, \max(\cdot), \min(\cdot)\right)$ is a lattice (where $\max(\cdot)$, $\min(\cdot)$ denote $\max\{f, g\}$, $\min\{f, g\}$ for $f, g \in \mathcal{C}_{\mathcal{T}}^{\mathcal{T}}$, respectively).*

P r o o f. The only difficulty of this proof is to show that

$$\forall_{f,g \in \mathcal{C}_{\mathcal{T}}^{\mathcal{T}}} \left(\max\{f, g\} \in \mathcal{C}_{\mathcal{T}}^{\mathcal{T}} \wedge \min\{f, g\} \in \mathcal{C}_{\mathcal{T}}^{\mathcal{T}}\right).$$

Let $f,g \in \mathcal{C}_{\mathcal{T}}^{\mathcal{T}}$ and put $h=\max\{f,g\}$. Let x_0 be an arbitrary real number. We have the following cases

1. $h(x_0)=f(x_0)>g(x_0)$. Let $m \in \big(g(x_0),f(x_0)\big)$ and G be a $\mathcal{T}$-neighbourhood of $h(x_0)$ ($G\in\mathcal{T}$ and $h(x_0)\in G$). Putting $G_0=G\cap(m,+\infty)$ we obtain that $G_0\in\mathcal{T}$ and $G_0\neq\emptyset$. Let $H=f^{-1}(G_0)\cap g^{-1}\big((-\infty,m)\big)$. It is easy to verify that $H\in\mathcal{T}$ and $x_0\in H$. Moreover, if $x\in H$ then $f(x)\in G_0$ and $g(x)<m$. This result, together with fact that $G_0\subset(m,+\infty)$, shows that $h(x)=f(x)\in G_0$. Consequently, h is continuous with respect to the topology $\mathcal{T}$ at the point x_0.
2. $h(x_0)=g(x_0)>f(x_0)$. In this case, an exactly similar reasoning shows that h is continuous with respect to the topology $\mathcal{T}$ at the point x_0.
3. $h(x_0)=g(x_0)=f(x_0)$. Let G_0 be $\mathcal{T}$-neighbourhood of $h(x_0)$. Putting $H=f^{-1}(G_0)\cap g^{-1}(G_0)$ we have that $H\in\mathcal{T}$ and $x_0\in H$. Moreover, $h(H)\subset G_0$, so h is continuous with respect to the topology $\mathcal{T}$ at the point x_0.

Consequently, we obtain that h is continuous with respect to the topology $\mathcal{T}$, so $\max\{f,g\}\in\mathcal{C}_{\mathcal{T}}^{\mathcal{T}}$.

Now, let $s=\min\{f,g\}$. Similarly as described above, replacing the interval $(-\infty,m)$ with an interval $(m,+\infty)$, we can prove that $s\in\mathcal{C}_{\mathcal{T}}^{\mathcal{T}}$. □

Corollary 1. *Let $\langle s\rangle\in S$. A triple $\Big(\mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}},\max(\cdot),\min(\cdot)\Big)$ is a lattice.*

We will present one further property of the $\langle s\rangle$-density topology which can be found in [FH].

Property 1.

(i) $\forall_{\langle s\rangle\in S}\ \forall_{x\in\mathbb{R}}\ (A\in\mathcal{T}_{\langle s\rangle}\Rightarrow A+x\in\mathcal{T}_{\langle s\rangle})$,

(ii) $\forall_{\langle s\rangle\in S}\ \big(A\in\mathcal{T}_{\langle s\rangle}\Rightarrow -A\in\mathcal{T}_{\langle s\rangle}\big)$.

From the above and Corollary 1 we obtain immediately

Remark 1. Let $\langle s\rangle\in S$. If $f\in\mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$ then $-f\in\mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$, $|f|\in\mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$ and $f+b\in\mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$, where $b\in\mathbb{R}$.

P r o o f . Property 1 implies immediately that $-f$, $(f+b)$ belong to $\mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$. Moreover, since $|f|=\max\big\{\max\{f,0\},\max\{-f,0\}\big\}$ it is easy to prove that $|f|\in\mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$. □

We will need some information concerning functions which are continuous when the natural topology is used on the range and the $\langle s\rangle$-density topology is used on the domain. So we recall the following results:

Definition 3. Let $f\colon \mathbb{R}\to\mathbb{R}$ and $\langle s\rangle \in S$. We shall say that f is *$\langle s\rangle$-approximately continuous* at point $x_0 \in \mathbb{R}$ if there exists a set $A_{x_0} \in \mathcal{L}$ such that

$$x_0 \in \Phi_{\langle s\rangle}(A_{x_0}) \qquad \text{and} \qquad f(x_0) = \lim_{\substack{x\to x_0,\\ x\in A_{x_0}}} f(x)\,.$$

Definition 4. Let $f\colon \mathbb{R}\to\mathbb{R}$ and $\langle s\rangle \in S$. We shall say that f is *$\langle s\rangle$-approximately continuous* if f is $\langle s\rangle$-approximately continuous at every point $x\in\mathbb{R}$.

Theorem 5. (Cf. [L].) *Let $f\colon \mathbb{R}\to\mathbb{R}$ and $\langle s\rangle \in S$. A function f belongs to the family $\mathcal{C}^{\mathcal{T}_0}_{\mathcal{T}_{\langle s\rangle}}$ if and only if f is $\langle s\rangle$-approximately continuous.*

Remark 2. Let $\langle s\rangle \in S$. If $f \in \mathcal{C}^{\mathcal{T}_{\langle s\rangle}}_{\mathcal{T}_{\langle s\rangle}}$ then f is $\langle s\rangle$-approximately continuous.

Definition 5. ([CLO]) Let $\{J_n\}_{n\in\mathbb{N}}$ be a sequence of intervals and let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of closed intervals. We shall say that the sequence $\{J_n\}_{n\in\mathbb{N}}$ *captures* the sequence $\{I_n\}_{n\in\mathbb{N}}$ if for each $n\in\mathbb{N}$ the intervals I_n and J_n have the same center and $I_n \subset J_n$.

Let a sequence $\{J_n\}_{n\in\mathbb{N}}$ captures a sequence $\{I_n\}_{n\in\mathbb{N}}$. Putting

$$N_x = \{k\in\mathbb{N}\colon x\notin J_k\} \qquad \text{and} \qquad J_x = \bigcup_{n\in N_x} I_n$$

for $x\in\mathbb{R}$ we have the following lemma:

Lemma 1. ([CLO]) *If a sequence $\{J_n\}_{n\in\mathbb{N}}$ captures a sequence $\{I_n\}_{n\in\mathbb{N}}$ and $\sum\limits_{n\in\mathbb{N}} \frac{m(I_n)}{m(J_n)} < +\infty$ then $d(J_x, x) = 0$ for every $x\in\mathbb{R}$.*

Definition 6. A function $f\colon \mathbb{R}\to\mathbb{R}$ is in the class $Baire^*1$ if and only if for each $\mathcal{T}_0$-perfect set P there exists a portion Q of P such that $f\big|_Q$ is continuous.

The idea of proving the next theorem comes from Theorem 3 in [CLO].

Theorem 6. *Let $\langle s\rangle \in S$. If $f \in \mathcal{C}^{\mathcal{T}_{\langle s\rangle}}_{\mathcal{T}_{\langle s\rangle}}$ then f is $Baire^*1$ function.*

Proof. Let $\langle s\rangle \in S$ and suppose that f is not $Baire^*1$ function. Then there exists a $\mathcal{T}_0$-perfect set P such that for each portion Q of P the function $f\big|_Q$ has a point of discontinuity. Putting

$$Z = \bigl\{x\in P\colon f\big|_P \text{ is not continuous at } x\bigr\}$$

we obtain that Z is dense in P. We will show that there exists $x\in P$ such that f is not $\langle s\rangle$-density continuous at point x. We will use an induction to find a sequence $\{x_n\}_{n\in\mathbb{N}} \subset P\cap Z$, two sequences of closed intervals $\{J_n\}_{n\in\mathbb{N}}$, $\{I_n\}_{n\in\mathbb{N}}$ and subsequence $\{s_{m_n}\}_{n\in\mathbb{N}}$ of sequence $\{s_n\}_{n\in\mathbb{N}}$ such that

(i) the sequence $\{J_n\}_{n\in\mathbb{N}}$ captures the sequence $\{I_n\}_{n\in\mathbb{N}}$,

(ii) $\forall_{n\in\mathbb{N}}\big(f(x_n)\in I_n\big)$,

(iii) $\forall_{n>1}(J_{n-1}\cap J_n=\emptyset)$,

(iv) $\forall_{n\in\mathbb{N}}\Big(0<m(J_n)<\omega\big(f\big|_P,x_n\big),\ m(I_n)=\frac{m(J_n)}{2^n}\Big)$,

(v) $\forall_{n>1}\Big[x_n-\frac{1}{s_{m_n}},x_n+\frac{1}{s_{m_n}}\Big]\subset\Big(x_{n-1}-\frac{1}{s_{m_{n-1}}},\ x_{n-1}+\frac{1}{s_{m_{n-1}}}\Big)$,

(vi) $\forall_{n\in\mathbb{N}}\bigg(m\Big(f^{-1}(I_n)\cap\big(x_n-\frac{1}{s_{m_n}},\ x_n+\frac{1}{s_{m_n}}\big)\Big)>\big(1-\frac{1}{2^n}\big)\frac{2}{s_{m_n}}\bigg)$.

Take $n=1$ and let x_1 be any point of Z. Of course, $\omega\big(f\big|_P,x_1\big)>0$. There exists a closed interval J_1 centered at $f(x_1)$ such that $0<m(J_1)<\omega\big(f\big|_P,x_1\big)$. Moreover, there exists a closed interval I_1 centered at $f(x_1)$ such that $m(I_1)=\frac{m(J_1)}{2}$. Of course, $I_1\subset J_1$. From the fact that the function f is $\langle s\rangle$-approximately continuous at x_1 it follows that there exists $m_1^0\in\mathbb{N}$ such that for each $m\geqslant m_1^0$ we have $m\Big(f^{-1}(I_1)\cap\big(x_1-\frac{1}{s_m},x_1+\frac{1}{s_m}\big)\Big)>\big(1-\frac{1}{2}\big)\frac{2}{s_m}$. Put $m_1=m_1^0$. Let $n>1$. Suppose that for each $i<n$ we have x_i,I_i,J_i,s_{m_i} which are satisfied conditions (ii)–(vi). Consider a set

$$f^{-1}(J'_{n-1})\cap\left(x_{n-1}-\frac{1}{s_{m_{n-1}}},\ x_{n-1}+\frac{1}{s_{m_{n-1}}}\right)\cap P.$$

Since $m(J_{n-1})<\omega\big(f\big|_P,x_{n-1}\big)$, we have that the above set is nonempty. Let y be any point of this set. If $y\in Z$ then $x_n=y$. If $y\notin Z$ then $f\big|_P$ is continuous at y. In consequence, the set $f^{-1}(J'_{n-1})\cap P$ is open in P. Since Z is dense in P then

$$f^{-1}(J'_{n-1})\cap\left(x_{n-1}-\frac{1}{s_{m_{n-1}}},\ x_{n-1}+\frac{1}{s_{m_{n-1}}}\right)\cap P\cap Z\neq\emptyset.$$

Let x_n be any point of this set. The facts that $x_n\in Z$ and J_{n-1} is a closed interval imply that there exists a closed interval J_n centered at $f(x_n)$ such that $0<m(J_n)<\omega\big(f\big|_P,x_n\big)$ and $J_{n-1}\cap J_n=\emptyset$. Let I_n be a closed interval centered at $f(x_n)$ such that $m(I_n)=\frac{m(J_n)}{2^n}$. Of course $I_n\subset J_n$. Moreover, we have that f is $\langle s\rangle$-approximately continuous at $x_n\in\Big(x_{n-1}-\frac{1}{s_{m_{n-1}}},\ x_{n-1}+\frac{1}{s_{m_{n-1}}}\Big)$ and $\lim\limits_{m\to\infty}\frac{1}{s_m}=0$. Hence there exists $m_n^0\in\mathbb{N}$ such that for each $m\geqslant m_n^0$ we have $m\Big(f^{-1}(I_n)\cap\big(x_n-\frac{1}{s_m},x_n+\frac{1}{s_m}\big)\Big)>\big(1-\frac{1}{2^n}\big)\frac{2}{s_m}$ and $\Big[x_n-\frac{1}{s_{m_n}},x_n+\frac{1}{s_{m_n}}\Big]\subset\Big(x_{n-1}-\frac{1}{s_{m_{n-1}}},x_{n-1}+\frac{1}{s_{m_{n-1}}}\Big)$. Put $m_n=\max\{m_n^0,m_{n-1}\}+1$.

Finally, the sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{J_n\}_{n\in\mathbb{N}}$, $\{I_n\}_{n\in\mathbb{N}}$ and subsequence $\{s_{m_n}\}_{n\in\mathbb{N}}$ satisfy conditions (i)–(vi).

Let $x \in \bigcap_{n\in\mathbb{N}} \left[x_n - \frac{1}{s_{m_n}}, x_n + \frac{1}{s_{m_n}}\right] \cap P$. We can assume that $f(x) \notin J_n$ for $n \in \mathbb{N}$. Otherwise we can consider a subsequence of the sequence $\{J_n\}_{n\in\mathbb{N}}$ which has the desirable property. The existence of this subsequence is guaranteed by condition (iii). Indeed, if such a subsequence did not exist then there would be $f(x) \in J_n$ for every n greater than some positive integer n_0. Which contradicts the condition (iii). Moreover, the condition (iv) implies that

$$\sum_{n\in\mathbb{N}} \frac{m(I_n)}{m(J_n)} = \sum_{n\in\mathbb{N}} \frac{1}{2^n} < +\infty\,.$$

From the above and Lemma 1 we have that $d\left(\bigcup_{n=1}^{\infty} I_n,\, f(x)\right) = 0$. Consequently, we obtain that $d_{\langle s\rangle}\left(\bigcup_{n=1}^{\infty} I_n,\, f(x)\right) = 0$. This fact together with the fact that f is $\langle s\rangle$-density continuous implies that $d_{\langle s\rangle}\left(f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right),\, x\right) = 0$. On the other hand, for a subsequence $\{s_{m_n}\}_{n\in\mathbb{N}}$ of the sequence $\{s_n\}_{n\in\mathbb{N}}$ we have that

$$\begin{aligned}
&\frac{m\left(f^{-1}\left(\bigcup_{k=1}^{\infty} I_k\right) \cap \left[x - \frac{1}{s_{m_n}},\, x + \frac{1}{s_{m_n}}\right]\right)}{\frac{2}{s_{m_n}}} \geqslant \frac{m\left(f^{-1}(I_n) \cap \left[x - \frac{1}{s_{m_n}},\, x + \frac{1}{s_{m_n}}\right]\right)}{\frac{2}{s_{m_n}}} \\
&= \frac{m\left(f^{-1}(I_n) \cap \left[x - \frac{1}{s_{m_n}},\, x + \frac{1}{s_{m_n}}\right]\right) - \frac{1}{s_{m_n}}}{\frac{2}{s_{m_n}}} > \frac{\left(1 - \frac{1}{2^n}\right)\frac{2}{s_{m_n}} - \frac{1}{s_{m_n}}}{\frac{2}{s_{m_n}}} \\
&\qquad = \left(1 - \frac{1}{2^n}\right) - \frac{1}{2}\,.
\end{aligned}$$

In consequence $\limsup_{n\to\infty} \frac{m\left(f^{-1}\left(\bigcup_{k=1}^{\infty} I_k\right)\cap\left[x-\frac{1}{s_{m_n}},x+\frac{1}{s_{m_n}}\right]\right)}{\frac{2}{s_{m_n}}} \geqslant \frac{1}{2}$. This contradiction finishes the proof. $\square$

The following lemma will be useful for proving the next theorem. The idea proving this lemma is adapted from [CL1].

Lemma 2. *Let $\langle s\rangle \in S$. If $f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$ and $a \in \mathbb{R}$ then $\operatorname{int}_{\mathcal{T}_0}\{x \in \mathbb{R} \colon f(x) > a\}$ is dense in $\{x \in \mathbb{R} \colon f(x) > a\}$.*

Proof. Let $G = \{x \in \mathbb{R} \colon f(x) > a\}$. It easy to observe that $G \in \mathcal{T}_{\langle s\rangle}$. Suppose that $\operatorname{int}_{\mathcal{T}_0} G$ is not dense in G. Therefore there exists an open interval

$I=(c,d)$ such that $G\cap I\neq\emptyset$ and $\operatorname{int}_{\mathcal{T}_0}G\cap I=\emptyset$. We can assume that $c,d\in G$. We will show that the sets $G\cap\operatorname{cl}_{\mathcal{T}_0}I$ and $G'\cap\operatorname{cl}_{\mathcal{T}_0}I$ are dense in $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I$. Moreover, we will show that they are G_δ sets. Putting $g=\max\{f,a\}\in\mathcal{C}^{\mathcal{T}_{\langle s\rangle}}_{\mathcal{T}_{\langle s\rangle}}$ we obtain that g is $Baire^*1$ function. So $\mathbb{R}\setminus g^{-1}(a)\in F_\sigma\cap G_\delta$ (see [CL]). Consequently, the sets G and G' are simultaneously G_δ and F_σ sets. Indeed, we have

$$\begin{aligned}F_\sigma\cap G_\delta\ni\mathbb{R}\setminus g^{-1}(a)&=\big\{x\in\mathbb{R}\colon g(x)<a\big\}\cup\big\{x\in\mathbb{R}\colon g(x)>a\big\}\\&=\big\{x\colon g(x)>a\big\}=\big\{x\in\mathbb{R}\colon f(x)>a\big\}=G\,.\end{aligned}$$

Hence $G'\in F_\sigma\cap G_\delta$. And we obtain immediately that the sets $G\cap\operatorname{cl}_{\mathcal{T}_0}I$ and $G'\cap\operatorname{cl}_{\mathcal{T}_0}I$ are G_δ sets. Moreover, the Baire Category Theorem implies that G is nowhere dense in I. Indeed, there exists a sequence of closed sets $\{F_n\}_{n\in\mathbb{N}}$ and a sequence of open sets $\{G_n\}_{n\in\mathbb{N}}$ such that $G=\bigcup\limits_{n\in\mathbb{N}}F_n=\bigcap\limits_{n\in\mathbb{N}}G_n$. Suppose that G is not nowhere dense in I. Then there exists an open interval J such that $\operatorname{cl}_{\mathcal{T}_0}J\subset I$ and for any open interval $J_1\subset\operatorname{cl}_{\mathcal{T}_0}J$ we have $J_1\cap G\neq\emptyset$. Consequently, G_n is dense in $\operatorname{cl}_{\mathcal{T}_0}J$ for every $n\in\mathbb{N}$. It implies that $\operatorname{cl}_{\mathcal{T}_0}J\setminus G_n$ is nowhere dense in $\operatorname{cl}_{\mathcal{T}_0}J$ for every $n\in\mathbb{N}$. Hence

$$\bigcup_{n\in\mathbb{N}}\big(\operatorname{cl}_{\mathcal{T}_0}J\setminus G_n\big)=\operatorname{cl}_{\mathcal{T}_0}J\cap G'$$

is a meager set in $\operatorname{cl}_{\mathcal{T}_0}J$. Moreover, we have that $\operatorname{int}_{\mathcal{T}_0}F_n\cap\operatorname{cl}_{\mathcal{T}_0}J\subset\operatorname{int}_{\mathcal{T}_0}G\cap I=\emptyset$ for every $n\in\mathbb{N}$. Consequently, F_n is nowhere dense in $\operatorname{cl}_{\mathcal{T}_0}J$ and G is a meager set in $\operatorname{cl}_{\mathcal{T}_0}J$. Finally, we have that $\operatorname{cl}_{\mathcal{T}_0}J=\operatorname{cl}_{\mathcal{T}_0}J\cap G'\cup\operatorname{cl}_{\mathcal{T}_0}J\cap G$ is a meager set in $\operatorname{cl}_{\mathcal{T}_0}J$. But it is impossible, so G is nowhere dense in I.

It implies that $\operatorname{cl}_{\mathcal{T}_0}G$ is nowhere dense in I. Consequently, G' is dense in $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I$. Indeed, let J be an open interval such that $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I\cap J\neq\emptyset$. From the above and in fact that $c,d\in G$ it follows that $\operatorname{cl}_{\mathcal{T}_0}G\cap I\cap J\neq\emptyset$ so there exists an open interval $J_1\subset J\cap I$ such that $\operatorname{cl}_{\mathcal{T}_0}G\cap J_1=\emptyset$. Consequently, $J_1\cap G'\neq\emptyset$. It implies that G' is dense in $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I$.

Moreover, it is easy to verify that G is dense in $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I$. Indeed, let J be an open interval such that $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I\cap J\neq\emptyset$. Let $x\in\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I\cap J$. We consider two cases

1. if $x\in G$ then $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I\cap J\cap G\neq\emptyset$,
2. if $x\in(G^d\setminus G)\cap I$ then $\emptyset\neq J\cap I\cap G\subset\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I\cap J\cap G$.

It means that G is dense in $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I$. Finally, we have that $G\cap\operatorname{cl}_{\mathcal{T}_0}I$ and $G'\cap\operatorname{cl}_{\mathcal{T}_0}I$ are dense in $\operatorname{cl}_{\mathcal{T}_0}G\cap\operatorname{cl}_{\mathcal{T}_0}I$ and they are G_δ sets. But it contradicts the Baire Category Theorem. $\square$

THEOREM 7. *For all $\langle s\rangle \in S$ the $\langle s\rangle$-density topology on $\mathbb{R}$ is not generated.*

Proof. Let $\tau = \left\{\mathbb{R}\setminus f^{-1}(a)\colon f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}, a \in \mathbb{R}\right\}$. First, we will show that $\tau = \left\{\mathbb{R}\setminus f^{-1}(0)\colon f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}\right\}$. Let $A \in \tau$. There exists a function $f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$ and $a \in \mathbb{R}$ such that $A = \mathbb{R}\setminus f^{-1}(a)$. Put $h = f - a$. From Remark 1 we have that $h \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$. Moreover, $h^{-1}(0) = f^{-1}(a)$, so $A = \mathbb{R}\setminus h^{-1}(0) \in \left\{\mathbb{R}\setminus f^{-1}(0)\colon f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}\right\}$. The opposite inclusion is obvious. Moreover, Corollary 1 and Remark 1 imply that τ is closed under finite intersection. Indeed, let $f, h \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$, then $\mathbb{R}\setminus h^{-1}(0) \cap \mathbb{R}\setminus f^{-1}(0) = \mathbb{R}\setminus g^{-1}(0)$, where $g = \min\{|f|, |h|\}$. Moreover, $\tau = \left\{\{x \in \mathbb{R}\colon f(x) > 0\}\colon f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}\right\}$. Indeed, if there exists a function $f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$ such that $A = \mathbb{R}\setminus f^{-1}(0)$ then $A = \{x \in \mathbb{R}\colon g(x) > 0\}$, where $g = \min\{|f|, |h|\}$. If $A = \{x \in \mathbb{R}\colon f(x) > 0\}$ for some $f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$ then we have that $A = \mathbb{R}\setminus h^{-1}(0)$, where $h = \max\{f, 0\}$. Finally, from the above and Theorem 3 we obtain that it suffices to show that $\tau = \left\{\{x \in \mathbb{R}\colon f(x) > 0\}\colon f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}\right\}$ is not a basis of the topology $\mathcal{T}_{\langle s\rangle}$. Let $E = \mathbb{R}\setminus\mathbb{Q}$ and suppose that E can be written as a union of elements from τ. Then there exists $f \in \mathcal{C}_{\mathcal{T}_{\langle s\rangle}}^{\mathcal{T}_{\langle s\rangle}}$ such that $\emptyset \neq \{x \in \mathbb{R}\colon f(x) > 0\} \subset E$. Consequently, $\operatorname{int}_{\mathcal{T}_0}\{x \in \mathbb{R}\colon f(x) > a\} = \emptyset$, which violates Lemma 2. $\square$

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