Tatra Mt. Math. Publ. **34** (2006), 37–47

Tatra Mountains
Mathematical Publications

A GENERALIZATION OF THE DENSITY TOPOLOGY

Małgorzata Filipczak — Tomasz Filipczak

ABSTRACT. In the paper we introduce a notion of f-density point of the measurable set on the real line. This notion is a generalization of Lebesgue density and also of ψ-density which was introduced in [S. J. Taylor: *On strengthening the Lebesgue density theorem*, Fund. Math. **46** (1959) 395–315]. We examine basic properties of an f-density and of a topology generated by it. Moreover, we investigate relationships between f-density and Lebesgue density and between f-densities for different functions f.

Throughout the paper we use standard notation: $\mathbb{R}$ denotes the set of real numbers, S the family of Lebesgue measurable subsets of $\mathbb{R}$ and $|E|$ the Lebesgue measure of a measurable set E. We also write $A \sim B$ if $|A \triangle B| = 0$.

Let $\mathcal{A}$ be a family of all functions $f\colon (0,\delta) \to (0,\infty)$ (δ depends on f) such that

(A1) $\lim\limits_{x\to 0^+} f(x) = 0$,

(A2) f is nondecreasing,

(A3) f is continuous,

(A4) $\limsup\limits_{x\to 0^+} \dfrac{x}{f(x)} > 0$.

Let $f \in \mathcal{A}$, $E \in S$ and $x \in E$. We say that x is an *f-density point* (*right-hand f-density point*, *left-hand f-density point*) of E if

$$\lim_{\substack{h\to 0,\, k\to 0\\ h\geqslant 0,\, k\geqslant 0,\, h+k>0}} \frac{\big|(x-h, x+k)\setminus E\big|}{f(h+k)} = 0$$

$$\Big(\lim_{h\to 0^+} \frac{\big|(x, x+h)\setminus E\big|}{f(h)} = 0\,, \quad \lim_{h\to 0^+} \frac{\big|(x-h, x)\setminus E\big|}{f(h)} = 0 \Big).$$

2000 Mathematics Subject Classification: 54A10, 26A15, 28A05.
Keywords: density points, density topology, comparison of topologies.

If x is an f-density point of a complement of E, then we say that x is an f-dispersion point of E. By $\varphi_f(E)$ we denote the set of all f-density points of E. In the same way *right-* and *left-hand f-dispersion points* of E, as well as the sets $\varphi_f^+(E)$ ($\varphi_f^-(E)$) of all right-hand (left-hand) f-density points of E may be defined.

Remark 1. The condition (A4) can be expressed in the following way:
There is a sequence $\{h_n\}$ of positive numbers convergent to 0 and a positive number M such that $f(h_n) < Mh_n$ for every n.

Remark 2. In the definition of family $\mathcal{A}$ one can assume that functions are defined on an interval $(0,\delta]$ or on the half-line $(0,\infty)$. Furthermore, if we put $f(0)=0$ then we may formulate condition (A4) as follows: $\underline{D}\, f^+(0) < \infty$.

Unless otherwise stated, we assume that $f \in \mathcal{A}$. We will also write $\lim\limits_{I\to x} \frac{|I\setminus E|}{f(|I|)}$ instead of $\lim\limits_{\substack{h\to 0,k\to 0\\ h\geqslant 0,k\geqslant 0,h+k>0}} \frac{|(x-h,x+k)\setminus E|}{f(h+k)}$.

Obviously, if we put $f(x)=x$ in the definition of f-density, we get Lebesgue density. On the other hand, putting $f(x) = x\psi(x)$, where ψ is a function which fulfils the conditions (A1)–(A3), we obtain ψ-density defined by S. J. Taylor in [T]. The notion of ψ-density was also considered (in "symmetric" version) in [TW-B], [W-B] and [W].

Proposition 1. *$\varphi_f(E) = \varphi_f^+(E) \cap \varphi_f^-(E)$ for every measurable set E.*

Proof. For any positive h and k we have

$$\begin{aligned} \frac{\big|[x-h,x+k]\setminus E\big|}{f(h+k)} &= \frac{\big|[x-h,x]\setminus E\big|}{f(h+k)} + \frac{\big|[x,x+k]\setminus E\big|}{f(h+k)} \\ &\leqslant \frac{\big|[x-h,x]\setminus E\big|}{f(h)} + \frac{\big|[x,x+k]\setminus E\big|}{f(k)}. \end{aligned}$$

Thus $\varphi_f^+(E) \cap \varphi_f^-(E) \subset \varphi_f(E)$. This finishes the proof as the inverse implication is evident. □

Theorem 1. (Compare [TW-B, Th. 1.2].) *The sets $\varphi_f(E)$, $\varphi_f^+(E)$ and $\varphi_f^-(E)$ are measurable for every measurable set E.*

Proof. It is sufficient to show that $\varphi_f^+(E) \in S$. For $h>0$ put

$$F_h(x) = \frac{\big|[x,x+h]\setminus E\big|}{f(h)} \qquad \text{for } x \in \mathbb{R}.$$

The function F_h is continuous. Moreover,

$$\varphi_f^+(E) = \bigcap_{n\in\mathbb{N}} \ \bigcup_{\delta\in\mathbb{Q}_+} \ \bigcap_{h\in(0,\delta)} F_h^{-1}\left(\left[0,\frac{1}{n}\right]\right)$$

and, consequently, $\varphi_f^+(E)$ is a set of type $F_{\sigma\delta}$. □

In the following three theorems we investigate relationships between f-density and Lebesgue density. The set of all Lebesgue density points of a measurable set E will be denoted by $\varphi(E)$.

THEOREM 2. *If $x \in \varphi_f^+(E)$ then $\liminf_{h\to 0+} \frac{|[x,x+h]\setminus E|}{h} = 0$.*

Proof. Suppose that $\liminf_{h\to 0+} \frac{|[x,x+h]\setminus E|}{h} > 0$. We will prove that $x \notin \varphi_f^+(E)$. From (A4) it follows that there is a positive number M and a sequence $\{h_n\}$ of positive numbers convergent to 0 such that $f(h_n) < M h_n$ for every n. Hence

$$\begin{aligned}\limsup_{h\to 0+} \frac{\big|[x,x+h]\setminus E\big|}{f(h)} &\geqslant \limsup_{n\to\infty} \frac{\big|[x,x+h_n]\setminus E\big|}{f(h_n)} \\ &\geqslant \limsup_{n\to\infty} \frac{\big|[x,x+h_n]\setminus E\big|}{M h_n} \\ &\geqslant \frac{1}{M} \liminf_{h\to 0+} \frac{\big|[x,x+h]\setminus E\big|}{h} > 0\,.\end{aligned}$$

Consequently, x is not a right-hand f-density point of E. □

COROLLARY 1. *For every measurable set E we have:*

(a) *if $x \in \varphi_f(E)$ then $\liminf_{h\to 0+} \frac{|[x,x+h]\setminus E|}{h} = 0$ and $\liminf_{h\to 0+} \frac{|[x-h,x]\setminus E|}{h} = 0$;*

(b) *$\varphi_f(E) \subset \mathbb{R}\setminus\varphi(\mathbb{R}\setminus E)$, i.e., no f-density point of E is its Lebesgue dispersion point;*

(c) $\big|\varphi_f(E)\setminus E\big| = 0$.

Proof. The conditions (a) and (b) are evident. From Lebesgue Density Theorem it follows that $\big|\mathbb{R}\setminus E\setminus\varphi(\mathbb{R}\setminus E)\big| = 0$. Thus the inclusion $\varphi_f(E)\setminus E \subset \mathbb{R}\setminus E\setminus\varphi(\mathbb{R}\setminus E)$ implies (c). □

THEOREM 3. *Let E be a measurable set for which $\liminf_{h\to 0+} \frac{|[x,x+h]\setminus E|}{h} = 0$. Then there exists a function $f \in \mathcal{A}$ such that $x \in \varphi_f^+(E)$.*

Proof. Let $\{h_n\}$ be an decreasing sequence tending to 0 such that $\lim_{n\to\infty} \frac{|[x,x+h_n]\setminus E|}{h_n} = 0$ and

$$f(x) = \begin{cases} \frac{h_{n+1}+h_n}{2} & \text{for } x \in \left[h_{n+1}, \frac{h_{n+1}+h_n}{2}\right],\ n\in\mathbb{N}, \\ \text{linear} & \text{on } \left[\frac{h_{n+1}+h_n}{2}, h_n\right],\ n\in\mathbb{N}. \end{cases}$$

Obviously, f satisfies (A1)–(A3). Since for $a_n = \frac{h_{n+1}+h_n}{2}$ we have $f(a_n) = a_n$, f satisfies also (A4) and, consequently, $f \in \mathcal{A}$. Notice that for $h \in [h_{n+1}, h_n]$

$$\frac{\big|[x, x+h] \setminus E\big|}{f(h)} \leqslant \frac{\big|[x, x+h_n] \setminus E\big|}{\frac{h_{n+1}+h_n}{2}} \leqslant 2\frac{\big|[x, x+h_n] \setminus E\big|}{h_n} \xrightarrow[n\to\infty]{} 0\,.$$

Thus $\lim\limits_{h\to 0+} \frac{|(x,x+h)\setminus E|}{f(h)} = 0$, i.e., $x \in \varphi_f^+(E)$. □

A bilateral version of the above theorem is not true.

Theorem 4. *There is a measurable set E such that*

$$\liminf_{h\to 0+} \frac{\big|[0,h] \setminus E\big|}{h} = 0\,, \quad \liminf_{h\to 0+} \frac{\big|[-h,0] \setminus E\big|}{h} = 0\,, \tag{$*$}$$

and for any $f \in \mathcal{A}$, 0 is not an f-density point of E.

P r o o f. Let

$$A = \bigcup_{n=1}^{\infty} \left(\frac{1}{(6n+5)!}, \frac{1}{(6n+4)!}\right),$$

$$B = \bigcup_{n=1}^{\infty} \left(\frac{1}{(6n+2)!}, \frac{1}{(6n+1)!}\right),$$

$$E = A \cup (-B)\,.$$

Obviously, E satisfies ($*$). Suppose, by contradiction, that there is $f \in \mathcal{A}$, such that $0 \in \varphi_f(E)$. Thus there exists a positive number M and a sequence $\{h_n\}$ tending to 0 such that $f(h_n) < Mh_n$ for every n. Put

$$P_1 = \bigcup_{n=1}^{\infty} \left[\frac{1}{(6n+6)!}, \frac{1}{(6n+3)!}\right], \quad P_2 = \bigcup_{n=1}^{\infty} \left[\frac{1}{(6n+3)!}, \frac{1}{(6n)!}\right].$$

From $P_1 \cup P_2 = \left(0, \frac{1}{6!}\right]$ it follows that $h_n \in P_1$ for infinitely many numbers n, or $h_n \in P_2$ for infinitely many numbers n. Without loss of generality we may assume that $h_n \in P_2$ for every n. Then for each $n \in \mathbb{N}$ one can find $k_n \in \mathbb{N}$ with $h_n \in \left[\frac{1}{(6k_n+3)!}, \frac{1}{(6k_n)!}\right]$. Hence

$$\begin{aligned}\frac{\big|[0, h_n] \setminus E\big|}{f(h_n)} &\geqslant \frac{h_n - \frac{1}{(6k_n+4)!}}{Mh_n} = \frac{1}{M}\left(1 - \frac{1}{h_n(6k_n+4)!}\right) \\ &\geqslant \frac{1}{M}\left(1 - \frac{1}{6k_n+4}\right) \xrightarrow[n\to\infty]{} \frac{1}{M} > 0\,.\end{aligned}$$

Consequently, $0 \notin \varphi_f(E)$. This contradicts our assumptions. □

Now, we will define an f-density topology, and we examine some properties of it. This definition is similar to that of density topology (compare [W]).

Theorem 5. *For each $A, B \in S$ we have:*

(a) *if $A \subset B$ then $\varphi_f(A) \subset \varphi_f(B)$;*

(b) *if $A \sim B$ then $\varphi_f(A) = \varphi_f(B)$;*

(c) *$\varphi_f(\emptyset) = \emptyset$ and $\varphi_f(\mathbb{R}) = \mathbb{R}$;*

(d) *$\varphi_f(A \cap B) = \varphi_f(A) \cap \varphi_f(B)$.*

The operation φ_f need not be so-called lower density because it need not fulfil the condition $A \sim \varphi_f(A)$. S. J. Taylor has proved the following theorem

Theorem 6. ([T, Th. 4]) *Let $f(x) = x\psi(x)$, where ψ satisfies the conditions* (A1)–(A3). *There exists a perfect set E such that $|E| > 0$ and $\varphi_f(E) = \emptyset$.*

Put

$$\mathcal{T}_f = \{E \in S \colon E \subset \varphi_f(E)\}.$$

Similarly as in [W, Th. 2.2] we can prove

Theorem 7. *$\mathcal{T}_f$ is a topology stronger than the natural topology on the real line.*

The topology $\mathcal{T}_f$ will be called *f-density topology.* Many properties of f-density topology are similar to those of the density topology $\mathcal{T}$. Thus the following theorem may be proved in the same way as theorems for the density topology (see [W, Th. 2.8–2.11]).

Theorem 8. *Let $E \subset \mathbb{R}$.*

(a) *If $|E| = 0$, then E is a $\mathcal{T}_f$-nowhere dense set.*

(b) *$|E| = 0$ if and only if E is a $\mathcal{T}_f$-closed and a $\mathcal{T}_f$-discrete set.*

(c) *$E \in S$ if and only if E is a $\mathcal{T}_f$-Borel set.*

(d) *If E is a $\mathcal{T}_f$-compact set, then E is finite.*

(e) *The topology $\mathcal{T}_f$ is neither separable nor Lindelöf nor first countable.*

Notice that if $f(x) = x$ then the implication (a) may be reversed, and $\mathcal{T}_f$ is a Baire space (see [W, Th. 2.8, 2.12]). On the other hand if there is a set E such that $|E| > 0$ and $\varphi_f(E) = \emptyset$ then E is a $\mathcal{T}_f$-nowhere dense set and $\mathcal{T}_f$ is of the first category. The first fact is obvious, and the second one has been proved in [W-B, Th. 5]. Thus we have

Proposition 2. *If $f(x) = x\psi(x)$, where ψ satisfies* (A1)–(A3), *then*

(a) *there exists a $\mathcal{T}_f$-nowhere dense set E such that $|E| > 0$;*

(b) *$\mathbb{R}$ is of $\mathcal{T}_f$-first category.*

We shall examine the invariantness of the f-density topology with respect to linear transformations. We immediately obtain

PROPOSITION 3. *$\mathcal{T}_f$ is invariant with respect to translations and point symmetries.*

PROPOSITION 4. *If $\alpha \geqslant 1$ then $\mathcal{T}_f$ is invariant with respect to multiplication by α.*

Proof. Notice first that $0 \in \varphi_f(E)$ implies $0 \in \varphi_f(\alpha E)$. Indeed, if $\lim_{I\to 0} \frac{|I\setminus E|}{f(|I|)} = 0$ then

$$\frac{\left|I \setminus \alpha E\right|}{f\left(|I|\right)} = \frac{\alpha\left|\frac{1}{\alpha}I \setminus E\right|}{f\left(\alpha\left|\frac{1}{\alpha}I\right|\right)} \leqslant \alpha \frac{\left|\frac{1}{\alpha}I \setminus E\right|}{f\left(\left|\frac{1}{\alpha}I\right|\right)} \xrightarrow[I\to 0]{} 0 .$$

Let $E \in \mathcal{T}_f$ and $y \in \alpha E$. Then $E \subset \varphi_f(E)$ and $y = \alpha x$ for some $x \in E$. Hence $0 \in \alpha(E-x)$. Consequently, from $E-x \in \mathcal{T}_f$ it follows that $0 \in \varphi_f\big(\alpha(E-x)\big) = \varphi_f(\alpha E - y)$. Thus $y \in \varphi_f(\alpha E)$ and therefore $\alpha E \in \mathcal{T}_f$. $\square$

Evidently, for some $f \in \mathcal{A}$, $\mathcal{T}_f$ is invariant with respect to multiplication by each $\alpha \neq 0$ (for example if $f(x) = x$). Below we find a function $f \in \mathcal{A}$, for which this is not true.

PROPOSITION 5. *For each $\alpha \in (0,1)$ there exists $f \in \mathcal{A}$ such that $\mathcal{T}_f$ is not invariant with respect to multiplication by α.*

Proof. Let $\alpha \in (0,1)$. Put $\beta = \frac{1}{\alpha}$, n_0—a positive integer greater than 2β,

$$E = (-\infty, 0] \;\cup \bigcup_{n=n_0}^{\infty} \left(\frac{\beta}{n!}, \frac{n}{n!}\right),$$

$$f(x) = \begin{cases} \frac{n}{n!} & \text{for } \; x \in \left[\frac{\beta}{n!}, \frac{n}{n!}\right], \; n \geqslant n_0 \, , \\ \frac{1}{n!} + \frac{n-1}{\beta-1}\left(x - \frac{1}{n!}\right) & \text{for } \; x \in \left[\frac{1}{n!}, \frac{\beta}{n!}\right], \; n \geqslant n_0 \, . \end{cases}$$

Obviously, $f \in \mathcal{A}$. To finish the proof it is sufficient to show that

$$0 \in \varphi_f^+(E) \tag{1}$$

and

$$0 \notin \varphi_f^+(\alpha E). \tag{2}$$

For $h \in \left[\frac{\beta}{n!}, \frac{n}{n!}\right]$ we have

$$\frac{\left|[0,h] \setminus E\right|}{f(h)} \leqslant \frac{\left|\left[0, \frac{\beta}{n!}\right]\right|}{\frac{n}{n!}} = \frac{\beta}{n} \, . \tag{3}$$

Suppose now that $h \in \left[\frac{1}{n!}, \frac{\beta}{n!}\right]$. Then

$$\frac{\left|[0,h] \setminus E\right|}{f(h)} \leqslant \frac{\left|\left[0, \frac{\beta}{(n+1)!}\right] \cup \left[\frac{1}{n!}, h\right]\right|}{\frac{1}{n!} + \frac{n-1}{\beta-1}\left(h - \frac{1}{n!}\right)} = \frac{hn! - 1 + \frac{\beta}{n+1}}{1 + \frac{n-1}{\beta-1}(hn! - 1)}.$$

Put $y = hn! - 1$. Evidently, $y \in [0, \beta - 1]$ if and only if $h \in \left[\frac{1}{n!}, \frac{\beta}{n!}\right]$. It is easy to see that function $g(y) = \dfrac{y + \frac{\beta}{n+1}}{\frac{n-1}{\beta-1}y + 1}$ is decreasing in the interval $[0, \beta - 1]$. In fact,

$$g'(y) = \frac{1 - \frac{(n-1)\beta}{(\beta-1)(n+1)}}{\left(\frac{n-1}{\beta-1}y + 1\right)^2} = \frac{2\beta - n - 1}{(\beta - 1)(n+1)\left(\frac{n-1}{\beta-1}y + 1\right)^2} < 0$$

for $n \geqslant n_0$. Hence

$$\frac{\left|[0,h] \setminus E\right|}{f(h)} \leqslant g(h) < g(0) = \frac{\beta}{n+1}. \tag{4}$$

The conditions (3) and (4) imply (1). On the other hand,

$$\alpha E = (-\infty, 0] \ \cup \bigcup_{n=n_0}^{\infty} \left(\frac{1}{n!}, \frac{\alpha n}{n!}\right).$$

Thus for $h_n = \frac{1}{n!}$ we have

$$\frac{\left|[0,h_n] \setminus \alpha E\right|}{f(h_n)} \geqslant \frac{\left|\left[\frac{\alpha}{n!}, \frac{1}{n!}\right]\right|}{\frac{1}{n!}} = 1 - \alpha > 0$$

which implies (2). □

From Theorem 7 and Theorem 8 it follows that $\mathcal{T}_f \supset \mathcal{T}^*$, where $\mathcal{T}^* = \{G \setminus P \colon G \text{ is open and } |P| = 0\}$ is so-called *Hashimoto topology* (see [H]). It is possible to prove a stronger result.

THEOREM 9. $\mathcal{T}_f \supsetneqq \mathcal{T}^*$.

P r o o f. It is sufficient to find a set E such that $E \in \mathcal{T}_f \setminus \mathcal{T}^*$.

Put $b_n = \frac{1}{2^{n-1}}$, $t_n = \min\left\{\frac{1}{2^{n+1}}, \frac{f(b_{n+1})}{n2^n}\right\}$, $a_n = b_{n+1} + t_n$, and

$$E = (-\infty, 0] \ \cup \bigcup_{n=1}^{\infty} (a_n, b_n).$$

Thus

$$b_{n+1} < a_n = \frac{1}{2^n} + t_n \leqslant \frac{1}{2^n} + \frac{1}{2^{n+1}} < b_n,$$

and for $h \in [b_{n+1}, b_n]$

$$\frac{\big|[0,h] \setminus E\big|}{f(h)} \leqslant \frac{\sum\limits_{k=n}^{\infty} t_k}{f(b_{n+1})} \leqslant \frac{\sum\limits_{k=n}^{\infty} \frac{f(b_{k+1})}{k2^k}}{f(b_{n+1})} \leqslant \sum_{k=n}^{\infty} \frac{1}{k2^k} < \frac{1}{n} .$$

Hence $0 \in \varphi_f^+(E)$ and, consequently, $E \in \mathcal{T}_f$. The condition $E \notin \mathcal{T}^*$ is evident. $\square$

In [TW-B], M. Terepeta and E. Wagner-Bojakowska considered the notion of ψ-density point of a measurable set. They said that x is a ψ-density point of $E \in S$ if $\lim\limits_{h \to 0+} \frac{|[x-h,x+h] \setminus E|}{2h\psi(2h)} = 0$ (ψ satisfies (A1)–(A3)). This is a symmetric version of the definition of f-density point for $f(x) = x\psi(x)$ and, obviously, $\varphi_f(E) \subset \varphi_\psi(E)$ and $\mathcal{T}_f \subset \mathcal{T}_\psi$ (where $\varphi_\psi(E)$ is a set of all ψ-density points of E and $\mathcal{T}_\psi = \big\{E \in S\colon E \subset \varphi_\psi(E)\big\}$). Thus [TW-B, Th. 2.11] implies

Proposition 6. $\bigcap\limits_{f \in \mathcal{A}} \mathcal{T}_f = \mathcal{T}^*$.

To compare f-density topologies we need definitions which were introduced in [TW-B].

Let $f_1, f_2 \in \mathcal{A}$. We say that f_1 precedes f_2 if $\limsup\limits_{h \to 0+} \frac{f_1(h)}{f_2(h)} < \infty$. We denote it by $f_1 \prec f_2$. If $f_1 \prec f_2$ and $f_2 \prec f_1$ then we say that f_1 and f_2 are equivalent. Evidently, functions f_1 and f_2 are equivalent if and only if there are positive numbers m, M, δ such that

$$m < \frac{f_1(x)}{f_2(x)} < M \qquad \text{for all } x \in (0,\delta) .$$

Theorem 10. (Compare [TW-B, Th. 2.2].) *If $f_1 \prec f_2$ then $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$.*

P r o o f. Let $E \in \mathcal{T}_{f_1}$. Then for an arbitrary $x \in E$

$$\lim_{I \to x} \frac{|I \setminus E|}{f_1(|I|)} = 0 \quad \text{and} \quad \limsup_{h \to 0+} \frac{f_1(h)}{f_2(h)} < \infty .$$

Consequently,

$$\limsup_{I \to x} \frac{|I \setminus E|}{f_2(|I|)} \leqslant \limsup_{I \to x} \frac{|I \setminus E|}{f_1(|I|)} \limsup_{I \to x} \frac{f_1(|I|)}{f_2(|I|)} = 0$$

and $E \in \mathcal{T}_{f_2}$. $\square$

Corollary 2. *If f_1 and f_2 are equivalent then $\mathcal{T}_{f_1} = \mathcal{T}_{f_2}$.*

Corollary 3. *If there are positive numbers m, M, δ such that $m < \frac{f(x)}{x} < M$ for all $x \in (0,\delta)$ then $\mathcal{T}_f = \mathcal{T}$ (where $\mathcal{T}$ is the density topology).*

From [TW-B, Th. 1.4] it follows

Proposition 7. *Let $f(x) = x\psi(x)$ where ψ satisfies (A1)–(A3). Then $\mathcal{T}_f \subsetneqq \mathcal{T}$.*

Proposition 8. *There is a function $f \in \mathcal{A}$ such that $\mathcal{T} \subsetneqq \mathcal{T}_f$.*

Proof. Let

$$f(x) = \begin{cases} \frac{n}{(2n)!} & \text{for } x \in \left[\frac{1}{(2n+1)!}, \frac{n}{(2n)!}\right], \ n \in \mathbb{N}, \\ \text{linear} & \text{on } \left[\frac{n}{(2n)!}, \frac{1}{(2n-1)!}\right], \ n \in \mathbb{N}. \end{cases}$$

Obviously, $f \in \mathcal{A}$. For $x \in \left[\frac{1}{(2n+1)!}, \frac{1}{(2n-1)!}\right]$ we have

$$\frac{x}{f(x)} \leqslant \frac{\frac{1}{(2n-1)!}}{\frac{n}{(2n)!}} = 2 .$$

Thus $\mathrm{id}_{\mathbb{R}} \prec f$ and $\mathcal{T} \subset \mathcal{T}_f$.

Put

$$A = \bigcup_{n=1}^{\infty} \left(\frac{1}{(2n)!}, \frac{1}{(2n-1)!}\right)$$

and

$$E = A \cup (-A) \cup \{0\} .$$

It is clear that 0 is not a density point of E, so $E \notin \mathcal{T}$. On the other hand, if $h \in \left[\frac{1}{(2n+1)!}, \frac{1}{(2n-1)!}\right]$, then

$$\frac{\left|[0,h] \setminus A\right|}{f(h)} \leqslant \frac{\frac{1}{(2n)!}}{\frac{n}{(2n)!}} = \frac{1}{n} .$$

Consequently, $0 \in \varphi_f^+(A)$ and $E \in \mathcal{T}_f$. □

Proposition 9. *There exists a function $f \in \mathcal{A}$ such that $\mathcal{T} \not\subseteq \mathcal{T}_f$ and $\mathcal{T}_f \not\subseteq T$.*

Proof. Let $a_n = \frac{1}{(2n+1)!}$, $c_n = \frac{1}{(2n-1)!} - \frac{1}{(2n)!} = \left(1 - \frac{1}{2n}\right) a_{n-1}$ for $n \geqslant 2$ and

$$f(x) = \begin{cases} \frac{1}{(2n)!} & \text{for } x \in [a_n, c_n] , \\ \text{linear} & \text{on } [c_n, a_{n-1}] . \end{cases}$$

Evidently, $f \in \mathcal{A}$. Put

$$A = (-\infty, 0] \cup \bigcup_{n=2}^{\infty} \left(\sqrt{2n+1}\,a_n\,, a_{n-1}\right).$$

Since

$$\frac{\left|A \cap \left(0, \sqrt{2n+1}\,a_n\right)\right|}{\sqrt{2n+1}\,a_n} \leqslant \frac{a_n}{\sqrt{2n+1}\,a_n} = \frac{1}{\sqrt{2n+1}} \xrightarrow[n\to\infty]{} 0\,.$$

0 is not a density point of A and $A \notin \mathcal{T}$. On the other hand, for $h \in [a_n, a_{n-1}]$ we have

$$\frac{\left|[0,h] \setminus A\right|}{f(h)} \leqslant \frac{\left|\left(0, \sqrt{2n+1}\,a_n\right)\right|}{\frac{1}{(2n)!}} \leqslant \frac{1}{\sqrt{2n+1}}\,.$$

Hence $A \in \mathcal{T}_f$, so $\mathcal{T}_f \nsubseteq T$.

Let $J_n = \left[\frac{1}{n!} - \frac{2}{(n+1)!}\,, \frac{1}{n!} - \frac{1}{(n+1)!}\right]$ for $n \geqslant 2$ and

$$B = (-\infty, 0] \;\cup \bigcup_{n=2}^{\infty} \left(\left(\frac{1}{(n+1)!}\,, \frac{1}{n!}\right) \setminus J_n\right).$$

One can easily check that 0 is a density point of B and therefore $B \in \mathcal{T}$. However,

$$\frac{\left|[0, c_n] \setminus B\right|}{f(c_n)} \geqslant \frac{\left|J_{2n-1}\right|}{f(c_n)} = 1\,,$$

so $0 \notin \varphi_f(B)$. Consequently, $B \notin \mathcal{T}_f$ and $\mathcal{T} \nsubseteq \mathcal{T}_f$. □

Finally we observe that each f-density topology is generated by some piecewise linear function.

Theorem 11. (Compare [TW-B, Th. 2.10].) *For arbitrary function $f \in \mathcal{A}$ there exists a piecewise linear function $l \in \mathcal{A}$ such that $\mathcal{T}_f = \mathcal{T}_l$.*

P r o o f. Let $a = f(1)$. Since f is continuous, there is a sequence $\{x_n\}$ of positive numbers such that $f(x_n) = \frac{a}{2^n}$. Put

$$l(x) = \begin{cases} \frac{a}{2^n} & \text{for } x = x_n\,,\ n \in \mathbb{N}\,,\\ \text{linear} & \text{on } [x_n, x_{n+1}]\,,\ n \in \mathbb{N}\,. \end{cases}$$

Obviously, $f \in \mathcal{A}$. For $x \in [x_{n+1}, x_n]$ we have

$$\frac{1}{2} = \frac{f(x_{n+1})}{l(x_n)} \leqslant \frac{f(x)}{l(x)} \leqslant \frac{f(x_n)}{l(x_{n+1})} = 2\,.$$

Thus the functions f and l are equivalent and $\mathcal{T}_f = \mathcal{T}_l$. □

REFERENCES

[GNN] GOFFMAN, C.—NEUGEBAUER, C. J.—NISHURA, T.: *The density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–505.

[H] HASHIMOTO, H.: *On the *topology and its application*, Fund. Math. **91** (1976), 5–10.

[T] TAYLOR, S. J.: *On strengthening the Lebesgue density theorem*, Fund. Math. **46** (1959), 305–315.

[TW-B] TEREPETA, M.—WAGNER-BOJAKOWSKA, E.: *ψ-density topology*, Rend. Circ. Mat. Palermo, (2) **48** (1999), 451–476.

[W-B] WAGNER-BOJAKOWSKA, E.: *Remarks on ψ-density topology*, Atti Sem. Mat. Fis. Univ. Modena **49** (2001), 79–87.

[W] WILCZYŃSKI, W.: *Density topologies*, in: Handbook of Measure Theory (E. Pap, ed.), Elsevier Science B. V., North Holland, Amsterdam, 2002, pp. 675–702.

Received July 31, 2004

Małgorzata Filipczak
Faculty of Mathematics
Łódź University
ul. Stefana Banacha 22
PL–90-238 Łódź
POLAND
E-mail: malfil@math.uni.lodz.pl

Tomasz Filipczak
Institute of Mathematics
Łódź Technical University
Al. Politechniki 11
PL–90-924 Łódź
POLAND
E-mail: tfil@math.uni.lodz.pl