Tatra Mt. Math. Publ. **34** (2006), 151–157

Tatra Mountains
Mathematical Publications

ON I-DENSITY TOPOLOGIES WITH RESPECT TO A FIXED SEQUENCE—FURTHER PROPERTIES

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ABSTRACT. Some properties of topologies introduced similarly as the I-density topology are investigated. We also consider approximately continuous functions for these topologies.

We use a standard notation here. Let $\mathbb{N}$ be the set of all positive integers, $\mathcal{B}$—the family of subsets of the real line having a Baire property, I—the σ-ideal of meager sets. For every set A and $x, t \in \mathbb{R}$ the set $A + x = \{a + x : a \in A\}$ and $t \cdot A = \{t \cdot a : a \in A\}$, χ_A denotes a characteristic function of A and A' is a complement of the set A.

Let S be a family of all nondecreasing and unbounded sequences of positive real numbers. Every sequence $\{s_n\}_{n\in\mathbb{N}} \in S$ is denoted by $\langle s \rangle$.

Definition 1. ([HH]) Let $\langle s \rangle \in S$. The point 0 is an $\langle s \rangle$-*I-density point* of a set $A \in \mathcal{B}$ if for every subsequence $\{s_{n_m}\}_{m\in\mathbb{N}} \subset \langle s \rangle$ there exists a subsequence $\{s_{n_{m_p}}\}_{p\in\mathbb{N}}$ such that $\chi_{(s_{n_{m_p}} \cdot A)\cap[-1,1]} \underset{p\to\infty}{\longrightarrow} 1$ I-a.e. on $[-1,1]$.

A point $x \in \mathbb{R}$ is an $\langle s \rangle$-*I-density point* of A if 0 is an $\langle s \rangle$-I-density point of the set $A - x$.

A point $x \in \mathbb{R}$ is an $\langle s \rangle$-*I-dispersion point* of A if x is an $\langle s \rangle$-I-density point of A'.

We can define one-sided $\langle s \rangle$-I-density points in the natural way.

Putting

$$\Phi_{\langle s \rangle I}(A) = \{x \in \mathbb{R} : x \text{ is an } \langle s \rangle\text{-}I\text{-density point of } A\}$$

for any $\langle s \rangle \in S$ and $A \in \mathcal{B}$ we get that $\Phi_{\langle s \rangle I} : \mathcal{B} \to \mathcal{B}$ is a lower density operator (see [HH], [LMZ]).

2000 Mathematics Subject Classification: 54C30, 26A15.
Keywords: I-density point, approximately continuous function.

Applying this operator, we define a topology $\mathcal{T}_{\langle s\rangle I} = \{A \in \mathcal{B}\colon A \subset \Phi_{\langle s\rangle I}(A)\}$ for every fixed sequence $\langle s\rangle$, which fulfils the inclusion: $\mathcal{T}_I \subset \mathcal{T}_{\langle s\rangle I}$, where $\mathcal{T}_I$ denotes the I-density topology ([HH]).

For $\langle s\rangle = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ we obtain the I-density topology (see [PWW1]). Some properties of such topologies has been already established in [H], [HH].

Theorem 1. ([HH]) *Let* $\langle s\rangle \in \mathcal{S}$. *Then* $\mathcal{T}_I = \mathcal{T}_{\langle s\rangle I}$ *if and only if* $\liminf\limits_{n} \frac{s_n}{s_{n+1}} > 0$.

Properties of a topology $\mathcal{T}_{\langle s\rangle I}$ for any sequence $\langle s\rangle \in \mathcal{S}$:

1) The topology $\mathcal{T}_{\langle s\rangle I}$ is obviously stronger than the natural topology on the real line, so it is Hausdorff topology.

2) Every countable subset of $\mathbb{R}$ is closed in $\mathcal{T}_{\langle s\rangle I}$, so $(\mathbb{R}, \mathcal{T}_{\langle s\rangle I})$ is not a separable space and every compact subspace of $(\mathbb{R}, \mathcal{T}_{\langle s\rangle I})$ is finite.

3) $(\mathbb{R}, \mathcal{T}_{\langle s\rangle I})$ is not a regular (T_3) space, since the set $\mathbb{Q} \setminus \{0\}$ and point 0 cannot be separated by $\mathcal{T}_{\langle s\rangle I}$-open sets. On the contrary, suppose that we can find open sets G_0 and $G \in \mathcal{T}_{\langle s\rangle I}$ such that $\mathbb{Q} \subset G$, $0 \in G_0$ and $G_0 \cap G = \emptyset$. Then G and G_0 are not meager sets (they are $\mathcal{T}_{\langle s\rangle I}$-open and nonempty). It is easy to verify that for any open set W (in the natural topology) the intersection $W \cap G$ is not a meager set (if only W is nonempty). Since $G \in \mathcal{B}$ there exists a set A which is G_δ and a set $M \in I$ such that $G = A \cup M$, so for any nonempty open W the intersection $A \cap W$ is not a meager set, hence A is a residual set, so is G. Therefore the intersection $G \cap G_0$ should not be empty, a contradiction.

4) The family of $\mathcal{T}_{\langle s\rangle I}$-Borel sets coincides with the family $\mathcal{B}$.

5) Every $\mathcal{T}_{\langle s\rangle I}$-Borel sets is the sum of $\mathcal{T}_{\langle s\rangle I}$-open set and $\mathcal{T}_{\langle s\rangle I}$-closed one.

6) A set A is $\mathcal{T}_{\langle s\rangle I}$-nowhere dense if and only if it is a meager set.

The three last properties can be proved by the same methods as in the case of $\mathcal{T}_I$ topology (see [PWW2]).

Now, we shall study some basic properties of continuous functions from $(\mathbb{R}, \mathcal{T}_{\langle s\rangle I})$ into $\mathbb{R}$ equipped with the natural topology.

Let $\langle s\rangle \in \mathcal{S}$.

Definition 2. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *$\langle s\rangle$-I-approximately continuous at x_0* if and only if for every $\varepsilon > 0$ the point x_0 is an $\langle s\rangle$-I-density point of the set $f^{-1}\big((f(x_0) - \varepsilon, f(x) + \varepsilon)\big)$.

Definition 3. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *$\langle s\rangle$-I-approximately continuous* if and only if for every interval (y_1, y_2) the set $f^{-1}\big((y_1, y_2)\big)$ is open in $\mathcal{T}_{\langle s\rangle I}$.

Remark 1. Since $(\mathbb{R}, \mathcal{T}_{\langle s\rangle I})$ is not a regular space, the topology $\mathcal{T}_{\langle s\rangle I}$ is not the coarsest topology for which every $\langle s\rangle$-I-approximately continuous function is continuous.

Remark 2. From the above definitions it is easy to see that a function $f\colon \mathbb{R} \to \mathbb{R}$ is I-approximately continuous if and only if it is I-approximately continuous at every point.

In real analysis there is also another way of defining an approximate continuity, so called "restrictional" one. (The approximate continuity presented above would be called a "topological" one.)

Definition 4. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *approximately continuous at x_0 in the "restrictional" sense* if and only if in $\mathcal{T}_{\langle s\rangle I}$ topology there exists a neighbourhood E of x_0 such that $f|E$ is continuous at x_0 (in the natural topology relativized to E).

Remark 3. The "restrictional" continuity implies the "topological" one.

Remark 4. (see [J, Th. 5] or [T, pp. 81–83, condition (J_3)]) The "restrictional" approximate continuity is equivalent to the "topological" one for topology $\mathcal{T}$ invariant under translations if and only if the following condition holds:

(W^*) For every descending sequence $\{E_n\}_{n\in\mathbb{N}}$ right-hand (left-hand) $\mathcal{T}$-neighbourhoods of 0 there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ such that $h_n \searrow 0$ and the set $\{0\} \cup \bigcup_{n=1}^{\infty} (E_n \cap [0, h_n))$ $\left(\{0\} \cup \bigcup_{n=1}^{\infty} (E_n \cap (-h_n, 0])\right)$ induces a right-hand (left-hand) $\mathcal{T}$-neighbourhood of 0.

The next theorem states that for any $\mathcal{T}_{\langle s\rangle I}$ topology the condition (W^*) is not fulfilled.

Theorem 2. *Let $\langle t\rangle \in S$. There exists an increasing sequence $\{A_k\}_{k\in\mathbb{N}}$ of sets having the Baire property such that for every natural $k \in \mathbb{N}$, 0 is a right-hand $\langle t\rangle$-I-dispersion point of A_k and for any sequence $\{h_k\}_{k\in\mathbb{N}}$ of real numbers tending decreasingly to zero a point 0 is not a right-hand $\langle t\rangle$-I-dispersion point of the set $A = \bigcup_{k=1}^{\infty} (A_k \cap [0, h_k))$.*

Proof. Let $\langle t\rangle \in S_0 = \left\{\langle s\rangle \in S\colon \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\right\}$. We choose a subsequence $\{t_{n_i}\}_{i\in\mathbb{N}} \subset \{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{i\to\infty} \frac{t_{n_i}}{t_{n_i+1}} = 0$ and $\frac{t_{n_i}}{t_{n_i+1}} < \frac{1}{2}$ for every $i \in \mathbb{N}$.

Let $t_{n_i} = s_i$ and $t_{n_i+1} = w_i$ for every $i \in \mathbb{N}$. Then $\frac{s_i}{w_i} < \frac{1}{2}$ for $i \in \mathbb{N}$.

Assume that $\{\varepsilon_i\}_{i\in\mathbb{N}}$ is a decreasing sequence of real numbers tending to 0 and $\varepsilon_1 < \frac{1}{2}$. Define

$$A_{k,i} = \bigcup_{z=2^{k-1}+1}^{2^k} \left(\left(\frac{z}{2^k} - \varepsilon_i \right) \cdot \frac{1}{s_i}, \frac{z}{2^k} \cdot \frac{1}{s_i} \right]$$

for $k, i \in \mathbb{N}$ and let $A_k = \bigcup_{i=1}^{\infty} A_{k,i}$ for $k \in \mathbb{N}$. It is obvious that $\{A_k\}_{k\in\mathbb{N}}$ is an increasing sequence of sets having the Baire property.

Fix $k \in \mathbb{N}$. First let us prove that 0 is a right-hand $\langle t\rangle$-I-dispersion point of the set A_k.

Consider an arbitrary subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ of the sequence $\langle t\rangle$. For any term of this subsequence we can find a term $s_{i(m)}$ of the sequence $\{s_i\}_{i\in\mathbb{N}}$ such that $s_{i(m)-1} < t_{n_m} \leqslant s_{i(m)}$. We claim that $\frac{1}{t_{n_m}} < \frac{1}{2} \cdot \frac{1}{s_{i(m)-1}}$ for every $m \in \mathbb{N}$. Indeed, if $\frac{1}{t_{n_m}} < \frac{1}{s_{i(m)-1}}$ then $\frac{1}{t_{n_m}} \leqslant \frac{1}{w_{i(m)-1}} < \frac{1}{2} \cdot \frac{1}{s_{i(m)-1}}$.

We choose a subsequence $\{m_p\}_{p\in\mathbb{N}}$ such that the limit of the sequence $\dfrac{t_{n_{m_p}}}{s_{i(m_p)}}$ exists. We denote it by g. Of course $g \in [0, 1]$.

To deal with $\limsup_p (t_{n_{m_p}} \cdot A_k) \cap [0, 1]$ for a fixed $p \in \mathbb{N}$ we consider a set $(t_{n_{m_p}} \cdot A_k) \cap [0, 1]$ which is equal to the following sum:

$$\bigcup_{z=2^{k-1}+1}^{2^k} \left(\left(\frac{z}{2^k} - \varepsilon_{i(m_p)} \right) \cdot \frac{t_{n_{m_p}}}{s_{i(m_p)}}, \frac{z}{2^k} \cdot \frac{t_{n_{m_p}}}{s_{i(m_p)}} \right] \cup \bigcup_{j=i(m_p)+1}^{\infty} \bigcup_{z=2^{k-1}+1}^{2^k} \left(\left(\frac{z}{2^k} - \varepsilon_j \right) \cdot \frac{t_{n_{m_p}}}{s_j}, \frac{z}{2^k} \cdot \frac{t_{n_{m_p}}}{s_j} \right],$$

since for $j < i(m_p)$ we have $\frac{t_{n_{m_p}}}{s_j} > 1$. This set is contained in

$$\bigcup_{z=2^{k-1}+1}^{2^k} \left(\left(\frac{z}{2^k} - \varepsilon_{i(m_p)} \right) \cdot \frac{t_{n_{m_p}}}{s_{i(m_p)}}, \frac{z}{2^k} \cdot \frac{t_{n_{m_p}}}{s_{i(m_p)}} \right] \cup \left(0, \frac{t_{n_{m_p}}}{w_{i(m_p)}} \right].$$

Since $0 \leqslant \frac{t_{n_{m_p}}}{w_{i(m_p)}} \leqslant \frac{s_{i(m_p)}}{w_{i(m_p)}}$ for $p \in \mathbb{N}$, and since $\dfrac{s_{i(m_p)}}{w_{i(m_p)}}$ tends to 0, then

$$\limsup_p \big(t_{n_{m_p}} \cdot A_k\big) \cap [0, 1] \subset \bigcup_{z=2^{k-1}+1}^{2^k} \left\{ \frac{z}{2^k} \cdot g \right\},$$

hence it is a meager set, which is our claim.

Now the task is to show that for any sequence $\{h_k\}_{k\in\mathbb{N}}$ of real numbers tending decreasingly to 0 there exists an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers such that for any subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ the sequence of the characteristic functions of the sets $(t_{n_{m_p}}\cdot A)\cap[0,1]$, where $A=\bigcup\limits_{k=1}^{\infty}(A_k\cap[0,h_k)]$, does not converge to zero I almost everywhere.

Let $\{n_m\}_{m\in\mathbb{N}}$ be a subsequence which has been already chosen at the beginning of the proof, it means $\{t_{n_m}\}_{m\in\mathbb{N}}=\{s_i\}_{i\in\mathbb{N}}$. Let $\{s_{i_p}\}_{p\in\mathbb{N}}$ be an arbitrary subsequence of $\{s_i\}_{i\in\mathbb{N}}$. Then, for every $p\in\mathbb{N}$,

$$(t_{n_{m_p}}\cdot A)\cap[0,1]=(s_{i_p}\cdot A)\cap[0,1]=s_{i_p}\cdot\left(A\cap\left[0,\frac{1}{s_{i_p}}\right]\right)\supset s_{i_p}\cdot\left(A_{k(p)}\cap\left[0,\frac{1}{s_{i_p}}\right]\right),$$

where $k(p)$ denotes such a natural number k that $\frac{1}{s_{i_p}}\in\left(h_{k(p)+1},h_{k(p)}\right]$. Then $\lim\limits_{p\to\infty}k(p)=+\infty$ since $\left\{h_{k(p)}\right\}_{p\in\mathbb{N}}$ tends to 0. Therefore

$$\limsup_p\left(t_{n_{m_p}}\cdot A\right)\cap[0,1]\supset\bigcap_{r=1}^{\infty}\bigcup_{p=r}^{\infty}s_{i_p}\cdot\left(A_{k(p)}\cap\left[0,\frac{1}{s_{i_p}}\right]\right)$$

and since

$$s_{i_p}\cdot\left(A_{k(p)}\cap\left[0,\frac{1}{s_{i_p}}\right]\right)\supset\bigcup_{z=2^{k-1}+1}^{2^k}\left(\frac{z}{2^k}-\varepsilon_{i_p},\frac{z}{2^k}\right]\cap[0,1],$$

we conclude that

$$\bigcup_{p=r}^{\infty}s_{i_p}\left(A_{k(p)}\cap\left[0,\frac{1}{s_{i_p}}\right]\right)$$

contains a set which is open and dense in $\left[\frac{1}{2},1\right]$, so which is of the second category. Thus $\limsup\limits_p(t_{n_{m_p}}\cdot A)\cap[0,1]$ is not a meager set, and the proof is complete for $\langle t\rangle\in S_0$.

When $\langle t\rangle\in S\setminus S_0$, we have $\mathcal{T}_{\langle t\rangle I}=\mathcal{T}_I$. An analogous theorem for I-density topology was proved in [PWW2]. □

A construction of a sequence $\{A_k\}_{k\in\mathbb{N}}$ which fulfils the last theorem is different from that used in the case of I-density, since taking sequences from the family S_0 we get topologies which are not invariant under multiplication.

Property 7. ([HH, Theorem 7 (iv)])

$$\underset{\langle s\rangle\in S_0}{\forall}\ \underset{|m|<1}{\forall}\ \underset{A\in\mathcal{B}}{\exists}\left(A\in\mathcal{T}_{\langle s\rangle}\wedge m\cdot A\notin\mathcal{T}_{\langle s\rangle I}\right).$$

This property can be proved in a stronger version:

Property 7'.

$$\underset{\langle s\rangle \in S_0}{\forall}\ \underset{A\in\mathcal{B}}{\exists}\ \underset{|m|<1}{\forall}\ \big(A \in \mathcal{T}_{\langle s\rangle I} \wedge m\cdot A \notin \mathcal{T}_{\langle s\rangle I}\big)\,.$$

P r o o f. Let $\langle s\rangle \in S_0$. Then there exists a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ of $\langle s\rangle$ such that $\lim\limits_{k\to\infty} \frac{s_{n_k}}{s_{n_k+1}} = 0$.

Put $X = \bigcup\limits_{j=1}^{\infty} \Big[\frac{1}{s_{n_j+1}}, \frac{1}{\sqrt{s_{n_j}\cdot s_{n_j+1}}}\Big]$. In the proof of Theorem 4 from [HH] it is shown that 0 is $\langle s\rangle$-I-dispersion point of a set X. Defining $Y = -X \cup X$ we have $A = \{0\} \cup (\mathbb{R} \setminus Y) \in \mathcal{T}_{\langle s\rangle I}$.

For $m = 0$ it is obvious that $m\cdot A \notin \mathcal{T}_{\langle s\rangle I}$.

Now, we want to show that 0 is not a right $\langle s\rangle$-I-dispersion point of a set $m\cdot X$ for $m \in (-1,1)\setminus\{0\}$. Without loss of generality we assume that $m \in (0,1)$. We can find $k_0 \in \mathbb{N}$ such that for any $k > k_0$ we have $\sqrt{\frac{s_{n_k}}{s_{n_k+1}}} < m$. Then 0 is not a right $\langle s\rangle$-I-dispersion point of the set $m\cdot \bigcup\limits_{j=k_0}^{\infty} \Big[\frac{1}{s_{n_j+1}}, \frac{1}{\sqrt{s_{n_j}\cdot s_{n_j+1}}}\Big]$ (see [HH]), so it is neither of the set $m\cdot X$. Hence $m\cdot A = \{0\} \cup (\mathbb{R} \setminus m\cdot Y) \notin \mathcal{T}_{\langle s\rangle I}$. □

Since for all $\langle s\rangle \in S$ the topology $\mathcal{T}_{\langle s\rangle I}$ is defined on the base of a lower density operator (see [HH]), then applying a Denjoy–Stepanoff type theorem from [LMZ] we obtain

THEOREM 3. *For an arbitrary $\langle s\rangle \in S$ the following conditions are equivalent.*

(i) *a function $f\colon \mathbb{R} \to \mathbb{R}$ has the Baire property,*

(ii) *it is "topologically" $\langle s\rangle$-I-approximately continuous I-a.e.*

COROLLARY. *A function $f\colon \mathbb{R} \to \mathbb{R}$ has the Baire property if and only if there exists a sequence $\langle s\rangle \in S$ such that f is "topologically" $\langle s\rangle$-I-approximately continuous I-a.e.*

The analysis similar to that in the case of I-density topology ([PWW1], [PWW2]) shows

THEOREM 4. *A function $f\colon \mathbb{R} \to \mathbb{R}$ has the Baire property if and only if there exists a sequence $\langle s\rangle \in S$ such that f is "restrictionally" $\langle s\rangle$-I-approximately continuous I-a.e.*

COROLLARY. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is I-approximately continuous ("topologically" or "restrictionally", respectively). I-a.e. if and only if there exists a sequence $\langle s\rangle \in S$ such that f is $\langle s\rangle$-I-approximately continuous I-a.e.*

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Received December 14, 2004

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