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SOME REMARKS ABOUT BIRKHOFF AND RIEMANN-LEBESGUE INTEGRABILITY OF VECTOR VALUED FUNCTIONS

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ABSTRACT. We investigate a notion of the Birkhoff integral for Banach space valued functions defined on a measure space. We discuss connections with a similar notion introduced recently by Kadets and others. We also consider a simple version of the Birkhoff integral where countable partitions are replaced by finite ones.

1. Introduction

Integration of vector valued functions is a nontrivial and important theory (see [4], [6], [11]). In 1935 Garrett Birkhoff proposed to widen a Lebesgue-type integral for Banach space valued functions. He modified Fréchet's approach to the Lebesgue integral and gave his own definition. His notion is located between Bochner and Pettis integrability. If the range space is separable, Birkhoff and Pettis integrability coincide [12]. The Birkhoff integral seems to have been forgotten for several years. In the last period it is again of some interest. An equivalent definition was used in Fremlin's preprint [5]; for some applications, see [9]. Lately, an interesting notion of Birkhoff integrability for multifunctions has been invented in [2].

The purpose of the paper is to investigate some aspects of Birkhoff integrability. In Section 1, we prove that Birkhoff integrability and Riemann-Lebesgue integrability introduced in [8] are the same and we give some other equivalent conditions. We also study several Cauchy-type conditions that are necessary and sufficient for the existence of integrals in the respective sense. In Section 2, we consider a simple version of the Birkhoff integral approximated by finite sums when the measure in the domain is finite. Again, a corresponding version of the

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Riemann-Lebesgue integral, defined as in [8] is the same. We obtain a characterization of such an integral. The usual Riemann integral is a particular case of this notion.

Let us recall some basic notation, definitions and facts from [1]. Throughout our paper, (Ω, Σ, μ) stands for a measure space, where μ is σ-finite, and X denotes a real Banach space.

For a bounded set $A \subset X$, we write $\|A\| = \sup_{x \in A} \|x\|$. For $A \subset X$ we denote by $\overline{A}$ and $Co(A)$ (respectively) the closure and the convex hull of A.

For sets $A_1, A_2, \ldots, A_k \subset X$ and real numbers $b_1, b_2, \ldots, b_k$ we denote

$$\sum_{i=1}^{k} b_i A_i = \left\{ \sum_{i=1}^{k} b_i a_i \colon a_i \in A_i, i \in \{1, \ldots, k\} \right\}.$$

In particular, $A - A = \{x - y \colon x, y \in A\}$ for $A \subset X$; note that $\|A - A\|$ is the diameter $\rho(A)$ of A. If $B = \{x\}$, we abbreviate $A \pm B$ as $A \pm x$. Let us list some simple properties.

Fact 1. *For any bounded sets $A, B \subset X$ we have:*

(1) $\|\overline{A}\| = \|A\|$,

(2) $\|Co(A)\| = \|A\|$ *and* $\rho(Co(A)) = \rho(A)$; [1, Th. 4],

(3) $\|\overline{A} - B\| = \|A - B\|$,

(4) $\|Co(A) - B\| = \|A - B\|$.

(5) $Co(A + B) = Co(A) + Co(B)$; [1, Th. 3].

Now, let $A_n \subset X$ for $n \in \mathbb{N}$ and $A \subset X$. We say that $\sum_{i=1}^{\infty} A_i = A$ if each series $\sum_{i=1}^{\infty} a_i$ with any $a_i \in A_i$, $i \in \mathbb{N}$, is unconditionally convergent, and A consists exactly of all sums of such series.

Fact 2. [1, Th. 9]. *Let $b_{i,j} \geq 0$ for $i, j \in \mathbb{N}$ and $\sum_j b_{ij} = b_i$ for every i. Let $A_{ij} \subset A_i \subset X$ where every A_i is bounded and $\sum_i b_i A_i = B$. Then $\sum_{ij} b_{ij} A_{ij} \subset \overline{Co(A)}$. Consequently, $\overline{Co\left(\sum_{ij} b_{ij} A_{ij}\right)} \subset \overline{Co(A)}$.*

A partition of Ω is a countable family Π of pairwise disjoint measurable sets E_i of finite measure such that $\bigcup_{i \in \mathbb{N}} E_i = \Omega$. A function $f \colon \Omega \to X$ is called *summable* under a partition $\Pi = \{E_i\}_{i \in \mathbb{N}}$ of Ω, if each set $f(E_i)$ is bounded whenever $\mu(E_i) > 0$ and each series of the form $\sum_{i=1}^{\infty} f(t_i)\mu(E_i)$,

$t_i \in E_i$, is unconditionally convergent. Thus we denote $\sum_{i=1}^{\infty} f(E_i)\mu(E_i) = \left\{ \sum_{i=1}^{\infty} f(t_i)\mu(E_i) \colon t_i \in E_i, i \in \mathbb{N} \right\}$. If $f \colon \Omega \to X$ and each series of the form $\sum_{i=1}^{\infty} f(t_i)\mu(E_i)$, $t_i \in E_i$, is unconditionally convergent, the set

$$J_\Pi(f) = \overline{Co\left(\sum_{i=1}^{\infty} f(E_i)\mu(E_i)\right)}$$

is called *the integral range of function f under the partition* Π.

For partitions $\Gamma = \{D_i\}_{i\in\mathbb{N}}$, $\Pi = \{E_k\}_{k\in\mathbb{N}}$, we say that Γ is finer than Π if each set from Γ is contained in a set from Π.

Immediately from Fact 2 we can derive

COROLLARY 3. *If a function $f \colon \Omega \to X$ is summable under a partition Π, then it is summable under every finer partition Γ, and $J_\Gamma(f) \subset J_\Pi(f)$.*

DEFINITION 4. Birkhoff [1] A function $f \colon \Omega \to X$ is called B-integrable in the Birkhoff sense, if there exists a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions, under which f is summable, with the corresponding sequence of diameters of integral ranges tending to zero. If f is B-integrable, then the intersection of all integral ranges is a singleton $\{J(f)\}$ (cf. [1, Th. 12], it suffices to apply Cantor's theorem for complete spaces). The element $J(f)$ of X is called B-integral of f.

LEMMA 5. *If for a partition $\Pi = \{E_i\}_{i\in\mathbb{N}}$ of Ω there exists $M > 0$ such that $\rho(J_\Pi(f)) \le M$ then each image $f(E_i)$, $i \in \mathbb{N}$, is bounded whenever $\mu(E_i) > 0$.*

Proof. Consider an arbitrary set $E_{i_0} \in \Pi$ such that $\mu(E_{i_0}) > 0$ and pick any $t_{i_0}, t'_{i_0} \in E_{i_0}$. Choose points $t_i \in E_i$ for $i \ne i_0$. Then,

$$\begin{aligned} &\left\| f(t_{i_0})\mu(E_{i_0}) - f(t'_{i_0})\mu(E_{i_0}) \right\| \\ &= \left\| \sum_{i \ne i_0} f(t_i)\mu(E_i) + f(t_{i_0})\mu(E_{i_0}) - \left(\sum_{i \ne i_0} f(t_i)\mu(E_i) + f(t'_{i_0})\mu(E_{i_0}) \right) \right\| \\ &\le \rho\left(\sum_{i=1}^{\infty} f(E_i)\mu(E_i) \right) = \rho\big(J_\Pi(f)\big) \le M\,. \end{aligned}$$

The last equality follows from Fact 1. Thus $\|f(t_{i_0}) - f(t'_{i_0})\| \le M/\mu(E_{i_0})$ and consequently, $\|f(t_{i_0})\| \le \|f(t'_{i_0})\| + M/\mu(E_{i_0})$. Since $t_{i_0} \in E_{i_0}$ is arbitrary, we have $\|f(E_{i_0})\| \le \|f(t'_{i_0})\| + M/\mu(E_{i_0})$ as desired. $\square$

Definition 6. (F r e m l i n [5]). A function $f\colon \Omega \to X$ is called B-integrable in the Fremlin sense, if there exists a point $x \in X$ such that for each $\varepsilon > 0$ there exists a partition $\Pi = \{E_i\}_{i\in\mathbb{N}}$ such that for every choice of points $t_i \in E_i$ we have $\left\|\sum_{i=1}^{\infty} f(t_i)\mu(E_i) - x\right\| \le \varepsilon$ and the series $\sum_{i=1}^{\infty} f(t_i)\mu(E_i)$ is unconditionally convergent.

Definition 7. (K a d e t s *et al.* [8]). A function $f\colon \Omega \to X$ is called unconditionally Riemann–Lebesgue (in short, RL-) integrable, if there exists a point $x \in X$ such that for each $\varepsilon > 0$ there exists a partition $\Pi = \{E_i\}_{i\in\mathbb{N}}$ such that for each partition $\Gamma \succ \Pi$, $\Gamma = \{D_i\}_{i\in\mathbb{N}}$, and for every choice of points $t_i \in D_i$ we have $\left\|\sum_{i=1}^{\infty} f(t_i)\mu(D_i) - x\right\| < \varepsilon$ and the series $\sum_{i=1}^{\infty} f(t_i)\mu(D_i)$ is unconditionally convergent.

Now, we are going to show that the notions of integrals from Definitions 4, 6, and 7 are equivalent. Note that K. M u s i a ł [10] observed that every RL-integrable function is B-integrable in the Birkhoff sense. The reverse implication was an open problem. Recently, I have learnt that this equivalence has been discovered independently by B. C a s c a l e s and J. R o d r i g u e z [3].

Theorem 8. *Let (Ω, Σ, μ) be a σ-finite measure space, let X be a Banach space and $f\colon \Omega \to X$. The following statements are equivalent:*

(B) *the function f is B-integrable in the Birkhoff sense,*

(RL) *the function f is unconditionally RL-integrable,*

(F) *the function f is B-integrable in the Fremlin sense.*

Additionally, all the considered integrals coincide.

P r o o f. $(B) \Rightarrow (RL)$ Let $\varepsilon > 0$. By assumption there exists a partition Π such that f is summable under Π and $\rho\big(J_\Pi(f)\big) \le \varepsilon$. Consider a finer partition $\Gamma = \{D_i\}_{i\in\mathbb{N}}$. Corollary 3 implies that f is summable under Γ and $\rho\big(J_\Gamma(f)\big) \le \rho\big(J_\Pi(f)\big) \le \varepsilon$. Consequently, a series $\sum_{i=1}^{\infty} f(t_i)\mu(D_i)$ is unconditionally convergent for every choice of points $t_i \in D_i$. Put $x = J(f)$, the Birkhoff integral of f. We have

$$\left\|\sum_{i=1}^{\infty} f(t_i)\mu(D_i) - x\right\| \le \left\|\sum_{i=1}^{\infty} f(D_i)\mu(D_i) - x\right\| = \big\|J_\Gamma(f) - x\big\| \le \rho\big(J_\Gamma(f)\big) \le \varepsilon\,.$$

$(RL) \Rightarrow (F)$ This implication is obvious.

$(F) \Rightarrow (B)$ Let x denote the B-integral of f in the Fremlin sense. For each $n \in \mathbb{N}$ pick a partition $\Pi_n = \{E_i^n\}_{i\in\mathbb{N}}$ such that for every choice of points

$t_i^n \in E_i^n$ we have $\left\| \sum\limits_{i=1}^{\infty} f(t_i^n)\mu(E_i^n) - x \right\| \le 1/(2n)$ and the series $\sum\limits_{i=1}^{\infty} f(t_i^n)\mu(E_i^n)$ is unconditionally convergent. It follows that $\left\| \sum\limits_{i=1}^{\infty} f\left(E_i^n\right)\mu(E_i^n) - x \right\| \le 1/(2n)$. Using Fact 1 we have

$$\rho\big(J_{\Pi_n}(f)\big) = \rho\left(\overline{Co\bigg(\sum_{i=1}^{\infty} f(E_i^n)\mu(E_i^n)\bigg)}\right)$$

$$= \rho\left(\sum_{i=1}^{\infty} f(E_i^n)\mu(E_i^n)\right) = \left\| \sum_{i=1}^{\infty} f(E_i^n)\mu(E_i^n) - \sum_{i=1}^{\infty} f(E_i^n)\mu(E_i^n) \right\|$$

$$\le \left\| \sum_{i=1}^{\infty} f(E_i^n)\mu(E_i^n) - x \right\| + \left\| x - \sum_{i=1}^{\infty} f(E_i^n)\mu(E_i^n) \right\| \le \frac{1}{n}$$

which implies that $\rho\big(J_{\Pi_n}(f)\big) \to 0$ if $n \to \infty$. Also, by Lemma 5, for each $E_i^n \in \Pi_n$, $f(E_i^n)$ is bounded whenever $\mu(E_i^n) > 0$. Thus f is B-integrable in the Birkhoff sense. It suffices to show that $J(f) = x$. For every $n \in \mathbb{N}$ we have $J(f) \in J_{\Pi_n}(f)$ and $\left\| J_{\Pi_n}(f) - x \right\| = \left\| \sum\limits_{i=1}^{\infty} f(E_i^n)\mu(E_i^n) - x \right\| \le 1/n$ by Fact 1. Then $\left\| J(f) - x \right\| \le 1/n$ for every $n \in \mathbb{N}$. Consequently, $J(f) = x$. □

Remark 9. In Definitions 4, 6, and 7 we may replace unconditional convergence of series by absolute convergence and we then obtain more restricted notion of integral which coincide since arguments for Theorem 8 can be repeated. We thus can speak about absolute Birkhoff (in short absB-) integrability (in [7], [8] it was called absolute Riemann-Lebesgue integrability).

Using Theorem 8 we can show the equivalence of Birkhoff integrability with several Cauchy-type conditions. Similar version can be produced for absolute Birkhoff integrability.

Theorem 10. *Let (Ω, Σ, μ) be a σ-finite measure space, let X be a Banach space and $f\colon \Omega \to X$. The following statements are equivalent:*

(B) f is B-integrable in the Birkhoff sense.

(FC) For each $\varepsilon > 0$ there is a partition $\Pi = \{E_i\}_{i\in\mathbb{N}}$ such that for any choices $t_i, t_i' \in E_i$ $(i \in \mathbb{N})$ we have $\left\| \sum\limits_{i=1}^{\infty} f(t_i)\mu(E_i) - \sum\limits_{i=1}^{\infty} f(t_i')\mu(E_i) \right\| \le \varepsilon$ where both the series are unconditionally convergent.

(RLC1) For each $\varepsilon > 0$ there is a partition Π such that for every partition $\Gamma = \{D_i\}_{i\in\mathbb{N}}$ finer than Π and for any choices $t_i, t_i' \in D_i$ $(i \in \mathbb{N})$ we have

$\left\| \sum_{i=1}^{\infty} f(t_i)\mu(D_i) - \sum_{i=1}^{\infty} f(t_i')\mu(D_i) \right\| \le \varepsilon$ *where both the series are unconditionally convergent.*

(*RLC2*) *For each* $\varepsilon > 0$ *there is a partition* Π *such that for any partitions* $\Gamma = \{D_i\}_{i\in\mathbb{N}}$, $\Gamma' = \{D_i'\}_{i\in\mathbb{N}}$ *finer than* Π *and for any choices* $t_i \in D_i$, $t_i' \in D_i'$ $(i \in \mathbb{N})$ *we have* $\left\| \sum_{i=1}^{\infty} f(t_i)\mu(D_i) - \sum_{i=1}^{\infty} f(t_i')\mu(D_i') \right\| \le \varepsilon$ *where both the series are unconditionally convergent.*

P r o o f. Implications $(RLC2) \Rightarrow (RLC1)$ and $(RLC1) \Rightarrow (FC)$ are obvious.

$(FC) \Rightarrow (B)$ For each $n \in \mathbb{N}$ there exists a partition $\Pi_n = \{E_i^n\}_{i\in\mathbb{N}}$ such that (FC) is satisfied for $\varepsilon = 1/n$. Then

$$\left\| \sum_{i=1}^{\infty} f(t_i)\mu(E_i^n) - \sum_{i=1}^{\infty} f(t_i')\mu(E_i^n) \right\| \le \frac{1}{n}$$

for any choices $t_i, t_i' \in E_i^n$ $(i \in \mathbb{N})$ where both the series are unconditionally convergent. Taking $\sup\limits_{t_i, t_i' \in E_i^n}$ and using Fact 1 we obtain $\rho\big(J_{\Pi_n}(f)\big) = \big\| J_{\Pi_n}(f) - J_{\Pi_n}(f) \big\| \le 1/n$. Thus we have a sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ such that the respective sequence of diameters of integral ranges tends to zero. By Lemma 5, for each $E_i^n \in \Pi_n$, $f(E_i^n)$ is bounded whenever $\mu(E_i^n) > 0$. Hence (B) has been proved.

$(B) \Rightarrow (RLC2)$ Theorem 8 implies that f is unconditionally RL-integrable. Let x stand for the integral of f and let $\varepsilon > 0$. So there exists a partition Π such that for every finer partition $\Gamma = \{D_i\}_{i\in\mathbb{N}}$ and every choice of points $t_i \in D_i$ we have $\left\| \sum_{i=1}^{\infty} f(t_i)\mu(D_i) - x \right\| < \varepsilon/2$ and the series $\sum_{i=1}^{\infty} f(t_i)\mu(D_i)$ is unconditionally convergent. Let $\Gamma = \{D_i\}_{i\in\mathbb{N}}$, $\Gamma' = \{D_i'\}_{i\in\mathbb{N}}$ be finer than Π and let $t_i \in D_i$, $t_i' \in D_i$ $(i \in \mathbb{N})$. Then

$$\left\| \sum_{i=1}^{\infty} f(t_i)\mu(D_i) - \sum_{i=1}^{\infty} f(t_i')\mu(D_i') \right\|$$

$$\le \left\| \sum_{i=1}^{\infty} f(t_i)\mu(D_i) - x \right\| + \left\| x - \sum_{i=1}^{\infty} f(t_i')\mu(D_i') \right\| \le \varepsilon .$$

□

2. A simple version of the Birkhoff integral

In this section we assume that (Ω, Σ, μ) is a finite measure space. We are going to discuss a simple version of the Birkhoff integral approximated by finite integral sums. Proofs presented here are more elementary and there is no need to use nontrivial Fact 2. We consider finite partitions of Ω, that is finite families $\{E_i\}_{i\le n}$ of pairwise disjoint measurable sets such that $\bigcup_{i=1}^{n} E_i = \Omega$. In this case, the integral range $J_\Pi(f)$ and the relation "to be a finer partition" are defined analogously as in Section 1.

Definition 11. A function $f\colon \Omega \to X$ is called BF-integrable, if there exists a sequence of finite partitions $(\Pi_n)_{n\in\mathbb{N}}$ such that the corresponding sequence of diameters of integral ranges tends to zero. An element of X which is the only common point of all integral ranges is called the BF-integral of f.

Let us notice that if f is BF-integrable then each image $f(E_i^n)$ is bounded whenever $\mu(E_i^n) > 0$, $E_i^n \in \Pi_n$ (Lemma 5).

Definition 12. A function $f\colon \Omega \to X$ is called FF-integrable, if there exists a point $x \in X$ such that for each $\varepsilon > 0$ there exists a partition $\Pi = \{E_i\}_{i\le n}$ such that for every choice of points $t_i \in E_i$ we have $\left\| \sum_{i=1}^{n} f(t_i)\mu(E_i) - x \right\| \le \varepsilon$. The point x is called the FF-integral of f.

Definition 13. A function $f\colon \Omega \to X$ is called RLF-integrable, if there exists a point $x \in X$ such that for each $\varepsilon > 0$ there exists a finite partition Π such that for every finer partition $\Gamma = \{D_i\}_{i\le m}$ and every choice of points $t_i \in D_i$ we have $\left\| \sum_{i=1}^{m} f(t_i)\mu(D_i) - x \right\| < \varepsilon$. The point x is called the RLF-integral of function f.

The next result is an analogue of Corollary 3 for finite sums. The proof is elementary.

Lemma 14. *Let $f\colon \Omega \to X$ and let Π be a finite partition of Ω. Then for every finer (finite) partition Γ we have $J_\Gamma(f) \subset J_\Pi(f)$.*

P r o o f. Let $\Pi = \{E_i\}_{i\le n}$ and $\Gamma = \{E_i^j\}_{i\le n, j\le m_i}$ where for a fixed $i \le n$, the sets E_i^j for $j \le m_i$ are measurable, pairwise disjoint and $\bigcup_{j\le m_i} E_i^j = E_i$. For each E_i such that $\mu(E_i) > 0$ we have

$$\begin{aligned}\sum_{j=1}^{m_i} f(E_i^j)\mu(E_i^j) \subset \sum_{j=1}^{m_i} f(E_i)\mu(E_i^j) = \mu(E_i)\sum_{j=1}^{m_i} f(E_i)\frac{\mu(E_i^j)}{\mu(E_i)} \\ \subset \mu(E_i)Co\big(f(E_i)\big) = Co\big(\mu(E_i)f(E_i)\big)\,.\end{aligned}$$

If $\mu(E_i) = 0$, the above inclusion is obvious. Consequently, using Fact 1 we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} f(E_i^j)\mu(E_i^j) \subset \sum_{i=1}^{n} Co\big(\mu(E_i)f(E_i)\big)Co\left(\sum_{i=1}^{n}\mu(E_i)f(E_i)\right).$$

Thus, $J_\Gamma(f) \subset J_\Pi(f)$. ◻

Theorem 15. *Let (Ω, Σ, μ) be a finite measure space, let X be a Banach space and $f\colon \Omega \to X$. The following conditions are equivalent:*

(BF) the function f is BF-integrable,

(RLF) the function f is RLF-integrable,

(FF) the function f is FF-integrable.

Additionally, all the considered integrals coincide.

Proof. The proof is analogous to the argument for countable sums but simpler, because we use Lemma 14 instead of Fact 2 and Corollary 3. ◻

The Cauchy-type conditions given in the previous section are also true for BF-integrable functions in the respectively modified versions. We also have the following characterization of BF-integrability:

Theorem 16. *Let (Ω, Σ, μ) be a finite measure space, let X be a Banach space and $f\colon \Omega \to X$. The following conditions are equivalent:*

(1) *the function f is BF-integrable,*

(2) *the function f is absB-integrable and bounded μ almost everywhere,*

(3) *the function f is B-integrable and bounded μ almost everywhere.*

Proof. (1) $\Rightarrow$ (2) By (1) pick a partition $\Pi = \{E_i\}_{i \le n}$ such that $\rho\big(J_\Pi(f)\big) \le 1$. Then Lemma 5 works and hence each image $f(E_i)$, $i \le n$, is bounded whenever $\mu(E_i) > 0$. This implies that f is bounded μ almost everywhere. By (1) we infer that f is absB-integrable.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Let $E \in \Sigma$ be such that $\mu(E) = \mu(\Omega)$ and $f(E)$ is bounded. There exists $M > 0$ such that $\|f(E)\| \le M$. Without loss of generality we may assume that $\Omega = E$. Let $\varepsilon > 0$. Since f is B-integrable (cf. Theorem 8), there is a partition $\Pi = \{E_i\}_{i\in\mathbb{N}}$ such that $\left\| \sum_{i=1}^{\infty} f(t_i)\mu(E_i) - x \right\| \le \varepsilon/3$ for every choice of points $t_i \in E_i$ where the series $\sum_{i=1}^{\infty} f(t_i)\mu(E_i)$ is unconditionally convergent.

Pick $m \in \mathbb{N}$ such that for $E_0 = \bigcup_{i=m+1}^{\infty} E_i$ we have $\mu(E_0) < \varepsilon/(3M)$. Put $\Gamma = \{E_i\}_{0 \le i \le m}$ and consider any choice t_i, $i \in \{0\} \cup \mathbb{N}$. Then

$$\begin{aligned}\left\| \sum_{i=0}^{m} f(t_i)\mu(E_i) - x \right\| &= \left\| f(t_0)\mu(E_0) - \sum_{i=m+1}^{\infty} f(t_i)\mu(E_i) + \sum_{i=1}^{\infty} f(t_i)\mu(E_i) - x \right\| \\ &\le \|f(t_0)\|\mu(E_0) + \sum_{i=m+1}^{\infty} \|f(t_i)\|\mu(E_i) + \left\| \sum_{i=1}^{\infty} f(t_i)\mu(E_i) - x \right\| \\ &\le M\frac{\varepsilon}{3M} + M \sum_{i=m+1}^{\infty} \mu(E_i) + \frac{\varepsilon}{3} \le \varepsilon\,.\end{aligned}$$

□

Since the Riemann integral is a special case of BF-integral then from Theorem 16 we conclude the following fact given in [7].

Corollary 17. *Every Riemann integrable function from* $[0,1]$ *into a Banach space is absRL-integrable.*

Let us finish with two examples concerning BF-integrals. As μ consider the Lebesgue measure on $\Omega = [0,1]$. As X take the Banach space $l_\infty([0,1])$ of all bounded functions from $[0,1]$ into $\mathbb{R}$ with the supremum norm. Put $e_t = \chi_{\{t\}}$, $t \in [0,1]$, and Θ—the zero function on $[0,1]$. Let $\mathbb{Q}$ stand for the set of all rationals.

The first example shows that BF-integrable function need not be bounded.

Example 18. Define $f\colon [0,1] \to X$ by the formula

$$f(t) = \begin{cases} \frac{e_t}{t} & \text{for } t \in \mathbb{Q} \cap (0,1]\,, \\ \Theta & \text{for } t \in \big([0,1] \setminus \mathbb{Q}\big) \cup \{0\}\,. \end{cases}$$

Then for the partition $\{E_1, E_2\}$, where $E_1 = \mathbb{Q} \cap (0,1]$, $E_2 = \big([0,1] \setminus \mathbb{Q}\big) \cup \{0\}$, and for every choice $t_1 \in E_1$, $t_2 \in E_2$ we have $f(t_1)\mu(E_1) + f(t_2)\mu(E_2) = \Theta$. Thus f is unbounded and BF-integrable on [0,1].

The next example shows that the class of BF-integrable functions is wider than the class of Riemann integrable functions.

Example 19. Define $f\colon [0,1] \to X$ by the formula

$$f(t) = \begin{cases} e_t & \text{for } t \in \mathbb{Q} \cap [0,1]\,, \\ \Theta & \text{for } t \in \big([0,1] \setminus \mathbb{Q}\big)\,. \end{cases}$$

Then for the partition $\{E_1, E_2\}$, where $E_1 = \mathbb{Q} \cap [0,1]$, $E_2 = [0,1] \setminus \mathbb{Q}$, and for every choice $t_1 \in E_1$, $t_2 \in E_2$ we have $f(t_1)\mu(E_1) + f(t_2)\mu(E_2) = \Theta$. Thus f

is BF-integrable. On the other hand, f is not Riemann integrable since for any partition $\{E_i\}_{i\le n}$ of $[0,1]$ into arbitrarily small intervals, choosing $t_i \in E_i \cap \mathbb{Q}$, we have $\|\sum_{i=1}^{n} f(t_i)\mu(E_i)\| = 1$.

Of course, the ideas of the above examples come from the function $1/t$ and the Dirichlet function, when X is replaced by $\mathbb{R}$.

Finally note that, for a finite measure space (Ω, Σ, μ) and a Banach space X, every bounded μ almost everywhere, Bochner integrable function $f\colon \Omega \to X$ is BF-integrable, but there exists a BF-integrable function (being Riemann integrable) which is not Bochner integrable [8].

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