CONVERGENCE OF THE SOLUTIONS FOR A NEUTRAL DIFFERENCE EQUATION WITH NEGATIVE COEFFICIENTS

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ABSTRACT. In this paper, we investigate the asymptotic behavior of the solutions of a neutral type difference equation of the form
\[ \Delta \left[ x(n) + \sum_{j=1}^{w} c_j x(\tau_j(n)) \right] + (-p(n))x(\sigma(n)) = 0, \quad n \geq 0, \]
where \( \tau_j(n), j = 1, \ldots, w \) are general retarded arguments, \( \sigma(n) \) is a general deviated argument, \( c_j \in \mathbb{R}, j = 1, \ldots, w, \) \( (p(n))_{n \geq 0} \) is a sequence of positive real numbers such that \( p(n) \geq p, p \in \mathbb{R}_+ \), and \( \Delta \) denotes the forward difference operator \( \Delta x(n) = x(n + 1) - x(n) \).

1. Introduction

A neutral difference equation is a difference equation in which the higher order difference of the unknown sequence appears in the equation both with and without delays or advances. See, for example, [1], [4], [5], [12] and the references cited therein. We should note that, the theory of neutral difference equations presents complications, and results which are true for non-neutral difference equations may not be true for neutral equations [19].

The study of the asymptotic and oscillatory behavior of the solutions of neutral difference equations presents a strong theoretical interest. Aside from the mathematical interest, the study of those equations is motivated by their applications. Neutral difference equations arise in several areas of applied mathematics, including circuit theory, bifurcation analysis, population dynamics, stability theory, the dynamics of delayed network systems and others. Neutral difference equations are used in the analysis of computer networks containing lossless
transmission lines, as in high speed networks where lossless transmission lines serve to connect switching circuits in the network. Neutral difference equations also come up in the study of vibrating masses attached to an elastic bar, as for example, the Euler equation is used in some variational problems and in the theory of automatic control. As a result of the wide range of applications, neutral difference equations have attracted a great interest in the literature.

Consider the neutral difference equation in which the difference of the unknown sequence appears in the equation both with and without more than one delays

\[ \Delta \left[ x(n) + \sum_{j=1}^{w} c_{j} x(\tau_{j}(n)) \right] + \left( -p(n) \right) x(\sigma(n)) = 0, \quad n \geq 0, \]  

(\text{E})

where \( (p(n))_{n \geq 0} \) is a sequence of positive real numbers such that \( p(n) \geq p, \quad p \in \mathbb{R}_{+} \), \( c_{j} \in \mathbb{R}, \quad j = 1, \ldots, w \), \( (\tau_{j}(n))_{n \geq 0}, \quad j = 1, \ldots, w \)

are increasing sequences of integers that satisfy

\[ \tau_{j}(n) \leq n - 1, \quad j = 1, \ldots, w \quad \forall n \geq 0, \quad \lim_{n \to \infty} \tau_{j}(n) = +\infty \]  

and

\[ \tau_{\ell}(n) < \tau_{m}(n + 1), \quad \forall \ell, m \in [1, w] \cap \mathbb{N} \]  

(1.1)

and \( (\sigma(n))_{n \geq 0} \) is an increasing sequence of integers such that

\[ \sigma(n) \leq n - 1 \quad \forall n \geq 0, \quad \lim_{n \to \infty} \sigma(n) = +\infty, \]  

or

\[ \sigma(n) \geq n + 1 \quad \forall n \geq 0. \]  

(1.2)

Define

\[ k_{1} = - \min_{1 \leq j \leq w} \tau_{j}(n), \quad k_{2} = - \min_{n \geq 0} \sigma(n) \quad \text{and} \quad k = \max \{ k_{1}, k_{2} \}. \]

(Clearly, \( k \) is a positive integer.)

By a solution of the neutral difference equation (\text{E}) we mean a sequence of real numbers \( (x(n))_{n \geq -k} \) which satisfies (\text{E}) for all \( n \geq 0 \). It is clear that, for each choice of real numbers \( c_{-k}, \ c_{-k+1}, \ldots, \ c_{-1}, \ c_{0} \), there exists a unique solution \( (x(n))_{n \geq -k} \) of (\text{E}) which satisfies the initial conditions

\[ x(-k) = c_{-k}, \quad x(-k + 1) = c_{-k+1}, \ldots, \ x(-1) = c_{-1}, \quad x(0) = c_{0}. \]

A solution \( (x(n))_{n \geq -k} \) of the neutral difference equation (\text{E}) is called oscillatory if the terms \( x(n) \) of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.
ASYMPTOTIC BEHAVIOR

In the special case, where \( \tau_j(n) = n - a_j \) and \( \sigma(n) = n \pm b \), \( a_j, b \in \mathbb{N} \), the equation (E) takes the form

\[
\Delta \left[ x(n) + \sum_{j=1}^{w} c_j x(n - a_j) \right] + \left( -p(n) \right) x(n \pm b) = 0, \quad n \geq 0. \quad (E_1)
\]

In the last few decades the asymptotic behavior of neutral difference equations has been extensively researched and developed. Hence, a large number of related papers have been published. See [2], [3], [6]–[11], [13]–[27], and the references cited therein. The objective in this paper is to investigate the convergence and divergence of the solutions of the equation (E) in the case of general delay arguments \( \tau_j(n), j = 1, 2, \ldots, w \) and a general deviated argument \( \sigma(n) \), depending on real constants \( c_j, j = 1, \ldots, w \).

2. Some preliminaries

Assume that \( (x(n))_{n \geq -k} \) is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As \( (-x(n))_{n \geq -k} \) is also a solution of (E), we can restrict ourselves only to the case where \( x(n) > 0 \) for all large \( n \). Let \( n_1 \geq -k \) be an integer such that \( x(n) > 0, \forall n \geq n_1 \). Then, there exists \( n_0 \geq n_1 \) such that

\[
x(\sigma(n)) > 0, \quad x(\tau_j(n)) > 0, \quad j = 1, 2, \ldots, w, \quad \forall n \geq n_0.
\]

Set

\[
z(n) = x(n) + \sum_{j=1}^{w} c_j x(\tau_j(n)). \quad (2.1)
\]

In view of (2.1), the equation (E) becomes

\[
\Delta z(n) - p(n) x(\sigma(n)) = 0. \quad (2.2)
\]

Taking into account that \( p(n) \geq p > 0 \), we have

\[
\Delta z(n) = p(n) x(\sigma(n)) \geq px(\sigma(n)) > 0, \quad \forall n \geq n_0,
\]

which means that the sequence \( (z(n)) \) is eventually strictly increasing, regardless of the values of the real constants \( c_j \).

Let the domain of \( \tau_j \) be the set

\[
D(\tau_j) = \mathbb{N}_{n^*_j} = \{ n^*_j, n^*_j + 1, n^*_j + 2, \ldots \},
\]

where \( n^*_j \) is the smallest natural number such that \( \tau_j \) is defined with. Set

\[
n^*_j = \max_{1 \leq j \leq w} n^*_j.
\]

Then \( \tau_j, j = 1, 2, \ldots, w \) is defined in the set \( \mathbb{N}_{n^*_j} = \{ n^*_j, n^*_j + 1, n^*_j + 2, \ldots \} \).
Let the subsequence
\[ x(\tau_{\rho(n)}(n)) = \max \{ x(\tau_1(n)), x(\tau_2(n)), \ldots, x(\tau_w(n)) \}, \tag{2.3} \]
where \( \rho(n) \) is a sequence that takes values in the set \( \{1, 2, \ldots, w\} \). Clearly, condition (1.1) guarantees that \( (x(\tau_{\rho(n)}(n))) \) is a subsequence of \( (x(n)) \).

Notice that
\[ \tau_{j_1}(\tau_{j_2}(\cdots \tau_{j_\ell}(n))) = \tau_{j_1}(n_s), \quad \text{where} \quad n_s = \tau_{j_2}(\cdots \tau_{j_\ell}(n)), \quad 1 \leq j_i \leq w. \tag{2.4} \]

The following lemma provides us with some useful tools for establishing the main results.

**Lemma 2.1.** Assume that \((x(n))_{n \geq -k}\) is a positive solution of \((E)\). Then the following statements hold:

(i) If
\[ \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty, \]
then
\[ \lim_{n \to \infty} z(n) = A = \lim_{n \to \infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))), \quad A \in \mathbb{R}. \tag{2.5} \]

(ii) If
\[ \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty, \]
then
\[ z(n) > 0, \quad \text{eventually.} \tag{2.6} \]

**Proof.** Summing up (2.2) from \( n_0 \) to \( n, \ n \geq n_0 \), we obtain
\[ z(n + 1) = z(n_0) + \sum_{i=n_0}^{n} p(i)x(\sigma(i)). \tag{2.7} \]

For the above relation, exactly one of the following can be true:
\[ \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty, \tag{2.7.a} \]
or
\[ \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty. \tag{2.7.b} \]
ASYMPTOTIC BEHAVIOR

Assume that (2.7.a) holds. Since \( p(n) \geq p > 0 \), we have

\[ +\infty > S_0 = \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) \geq p \sum_{i=n_0}^{\infty} x(\sigma(i)). \]

The last inequality guarantees that

\[ \sum_{i=n_0}^{\infty} x(\sigma(i)) < +\infty \]

and consequently,

\[ \lim_{n \to \infty} x(\sigma(n)) = 0. \quad (2.8) \]

Also, (2.7.a) guarantees that \( \lim_{n \to \infty} z(n) \) exists as a real number. Set

\[ \lim_{n \to \infty} z(n) = A \in \mathbb{R}. \]

Since \( (z(\sigma(n))) \) is a subsequence of \( (z(n)) \), we have

\[ \lim_{n \to \infty} z(\sigma(n)) = A, \]

or

\[ \lim_{n \to \infty} \left[ x(\sigma(n)) + \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))) \right] = A. \]

Using (2.8), we obtain

\[ \lim_{n \to \infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))) = A. \]

Thus

\[ \lim_{n \to \infty} z(n) = A = \lim_{n \to \infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))). \]

The proof of the part (i) of the lemma is complete.

Assume that (2.7.b) holds. Then, by taking limits on both sides of (2.7), we obtain

\[ \lim_{n \to \infty} z(n) = +\infty \]

which in conjunction with that fact that the sequence \( (z(n)) \) is eventually strictly increasing, means that

\[ z(n) > 0, \quad \text{eventually.} \]

The proof of the part (ii) of the lemma is complete.

The proof of Lemma 2.1 is complete. \( \square \)
3. Main results

Throughout this section we are going to use the following notation
\[ c = \sum_{j=1}^{w} c_j. \]  
(3.1)

The asymptotic behavior of the solutions of the neutral difference equation (E) is described by the following theorem:

**Theorem 3.1.** For every nonoscillatory solution \((x(n))\) of the equation (E) the following statements hold:

(I) If the constants \(c_j\) are all nonpositive and \(c < -1\), then \((x(n))\) either has at least one real accumulation point which is zero or tends to infinity.

(II) If the constants \(c_j\) are all nonpositive and \(c = -1\), then \((x(n))\) either tends to zero or it is bounded with more than one real accumulation point besides zero or tends to infinity.

(III) If the constants \(c_j\) are all nonpositive and \(-1 < c < 0\), then \((x(n))\) either tends to zero or tends to infinity.

(IV) If the constants \(c_j\) are all equal to zero, then \((x(n))\) tends to infinity.

(V) If the constants \(c_j\) are all nonnegative and \(0 < c < 1\), then \((x(n))\) is unbounded.

(VI) If the constants \(c_j\) are all nonnegative and \(c \geq 1\), then \((x(n))\) does not converge in \(\mathbb{R}\).

**Proof.** Assume that \((x(n))_{n \geq -k}\) is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As \((-x(n))_{n \geq -k}\) is also a solution of (E), we can restrict ourselves only to the case, where \(x(n) > 0\) for all large \(n\). We define the sequence \((z(n))\) as in (2.1) and reformulate the equation (E) as in (2.2), in the preliminaries. From the preliminaries, we also have that since \(p(n) \geq p > 0\), the sequence \((z(n))\) is eventually strictly increasing, regardless of the values of the real constants \(c_j\).

Assume that the constants \(c_j\) are all nonpositive and \(c < -1\).

If \((2.7.a)\) holds, then, in view of part (i) of Lemma 2.1, we have
\[ \lim_{n \to \infty} z(n) = A = \lim_{n \to \infty} \sum_{j=1}^{w} c_j x\left(\tau_j(\sigma(n))\right), \quad A \in \mathbb{R}, \]
which means, the sequence \((x(n))\) has at least one accumulation point, which is zero, since \((2.8)\) is satisfied.
ASYMPTOTIC BEHAVIOR

If (2.7.b) holds, then, in view of (2.7), we have
\[
\lim_{n \to \infty} z(n) = +\infty,
\]
which guarantees that
\[
\lim_{n \to \infty} x(n) = +\infty.
\]
The proof of the part (I) of the theorem is complete.

Assume that the constants \( c_j \) are all nonpositive and \( c = -1 \).
If (2.7.a) holds, then, in view of part (i) of Lemma 2.1 we have
\[
\lim_{n \to \infty} z(n) = A = \lim_{n \to \infty} \sum_{j=1}^{w} c_j x (\tau_j (\sigma(n))), \quad A \in \mathbb{R}
\]
which guarantees that \( A \leq 0 \).

Since \((z(n))\) is eventually strictly increasing, we have
\[
z(n) = x(n) + \sum_{j=1}^{w} c_j x (\tau_j(n)) < A \leq 0.
\]

Using (2.3), (2.4) and (3.1), the last inequality becomes
\[
x(n) + \left( \sum_{j=1}^{w} c_j \right) x (\tau_{p_1}(n)) < 0,
\]
or
\[
x(n) < x (\tau_{p_1}(n)),
\]
where
\[
x (\tau_{p_1}(n)) = \max_{1 \leq j \leq w} \left\{ x (\tau_j(n)) \right\}.
\]
Thus
\[
x(n) < x (\tau_{p_1}(n)) < \cdots < x (\tau_{p_m}(n_*)),
\]
where \( m(n) \) is a natural number which determines the number of steps we make in order to reach \( n_* \). This means that the sequence \((x(n))\) is bounded.

Let \( A < 0 \). Set
\[
\limsup_{n \to \infty} x(n) = M.
\]
Then there exists a subsequence \((x(\theta(n)))\) of \((x(n))\) such that
\[
\lim_{n \to \infty} x (\theta(n)) = M.
\]
Therefore
\[ \lim_{n \to \infty} \left[ x(\theta(n)) + \sum_{j=1}^{w} c_j x(\tau_j(\theta(n))) \right] = A, \]
or
\[ -\lim_{n \to \infty} \left[ \sum_{j=1}^{w} c_j x(\tau_j(\theta(n))) \right] = M - A, \]
or
\[ \lim_{n \to \infty} \left[ \sum_{j=1}^{w} (-c_j) x(\tau_j(\theta(n))) \right] = M - A. \]
Consequently,
\[ \limsup \left[ \sum_{j=1}^{w} (-c_j) x(\tau_j(\theta(n))) \right] = M - A, \]
or
\[ \sum_{j=1}^{w} (-c_j) \limsup x(\tau_j(\theta(n))) \geq M - A, \]
or
\[ \sum_{j=1}^{w} (-c_j) M \geq M - A. \]
Hence
\[ M \sum_{j=1}^{w} (-c_j) \geq M - A, \]
or
\[ M \geq M - A, \quad \text{since} \quad \sum_{j=1}^{w} (-c_j) = 1 \]
which contradicts to our assumption that \( A < 0 \). Therefore
\[ A = 0, \quad i.e., \quad \lim_{n \to \infty} z(n) = 0. \]
This means that \( (x(n)) \) has at least one real accumulation point which is zero.
If \( \text{(2.7.b)} \) holds, then, in view of \( \text{(2.7)} \), we have
\[ \lim_{n \to \infty} z(n) = +\infty, \]
which guarantees that
\[ \lim_{n \to \infty} x(n) = +\infty. \]
The proof of the part (II) of the theorem is complete.
ASYMPTOTIC BEHAVIOR

Assume that the constants $c_j$ are all nonpositive and $-1 < c < 0$. If (2.7.a) holds, then, in view of part (i) of Lemma 2.1, we have

$$
\lim_{n \to \infty} z(n) = A = \lim_{n \to \infty} \sum_{j=1}^{w} c_j x \left( \tau_j (\sigma(n)) \right), \quad A \in \mathbb{R}
$$

which guarantees that $A \leq 0$.

Since $(z(n))$ is eventually strictly increasing, we have

$$
z(n) = x(n) + \sum_{j=1}^{w} c_j x \left( \tau_j (n) \right) < A \leq 0.
$$

Using (2.3), (2.4) and (3.1) the last inequality becomes

$$
x(n) + \left( \sum_{j=1}^{w} c_j \right) x \left( \tau_{p1}(n) \right) < 0,
$$

or

$$
x(n) < -cx \left( \tau_{p1}(n) \right).
$$

Thus

$$
x(n) < -cx \left( \tau_{p1}(n) \right) < \cdots < (-c)^{m(n)} x \left( \tau_{p_{m(n)}(n_*)} \right) \to 0 \quad \text{as} \quad n \to \infty
$$

and consequently,

$$
\lim_{n \to \infty} x(n) = 0.
$$

If (2.7.b) holds, then, in view of (2.7), we have

$$
\lim_{n \to \infty} z(n) = +\infty,
$$

which guarantees that

$$
\lim_{n \to \infty} x(n) = +\infty.
$$

The proof of the part (III) of the theorem is complete.

Assume that the constants $c_j$ are all nonnegative. Then $c \geq 0$. By (2.7) we have

$$
z(n + 1) = z(n_0) + \sum_{i=n_0}^{n} p(i) x(\sigma(i)) > 0.
$$

Therefore

$$
\lim_{n \to \infty} z(n) > 0.
$$

Assume that the constants $c_j$ are all equal to zero. Then $c = 0$, and consequently $z(n) = x(n)$. Using (2.7), we take

$$
x(n + 1) = x(n_0) + \sum_{i=n_0}^{n} p(i) x(\sigma(i)) > 0,
$$

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which guarantees that
\[ \lim_{n \to \infty} x(n) > 0. \]
Thus \((x(\sigma(n)))\) cannot tend to zero, and therefore
\[ \lim_{n \to \infty} x(n) = +\infty. \]
The proof of the part (IV) of the theorem is complete.

Assume that the constants \(c_j\) are all nonnegative and \(0 < c < 1\).
If (2.7a) holds, then, in view part (i) of Lemma 2.1, we have
\[ \lim_{n \to \infty} z(n) = A = \lim_{n \to \infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))), \quad A \in \mathbb{R}. \]
Clearly, \((z(n))\) is bounded and therefore \((x(n))\) is bounded. Set
\[ \lim \sup x(n) = M. \]
Then there exists a subsequence \((x(\theta(n)))\) of \((x(n))\) such that
\[ \lim_{n \to \infty} x(\theta(n)) = M. \]
Therefore
\[ \lim_{n \to \infty} \left[ x(\theta(n)) + \sum_{j=1}^{w} c_j x(\tau_j(\theta(n))) \right] = A, \]
or
\[ \lim_{n \to \infty} \left[ \sum_{j=1}^{w} c_j x(\tau_j(\theta(n))) \right] = A - M \geq 0, \]
i.e.,
\[ M \leq A. \quad (3.2) \]
On the other hand,
\[ \lim_{n \to \infty} z(\sigma(n)) = A, \]
or
\[ \lim_{n \to \infty} \left[ x(\sigma(n)) + \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))) \right] = A. \]
ASYMPTOTIC BEHAVIOR

Using (2.8), the last relation becomes

\[
\lim_{n \to \infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))) = A
\]

and consequently

\[
\limsup \left[ \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))) \right] = A.
\]

Hence

\[
\sum_{j=1}^{w} c_j \limsup x(\tau_j(\sigma(n))) \geq A,
\]

or

\[
M \sum_{j=1}^{w} c_j \geq A,
\]

or

\[
M > cM \geq A
\]

which contradicts (3.2). Therefore \((x(n))\) is unbounded. Thus (2.7.a) is not satisfied, and therefore (2.7.b) holds. By (2.7), we have

\[
\lim_{n \to \infty} z(n) = +\infty,
\]

which guarantees that

\[
\lim_{n \to \infty} x(n) = +\infty.
\]

The proof of the part (V) of the theorem is complete.

Assume that the constants \(c_j\) are all nonnegative and \(c \geq 1\).

If (2.7.a) holds, then

\[
\lim_{n \to \infty} z(n) = A \geq 0, \quad A \in \mathbb{R}.
\]

Since \(c > 0\), then, in view of part (IV) of theorem, we have

\[
\lim_{n \to \infty} z(n) > 0,
\]

which means that \(A > 0\). Combined with the fact that \(\lim_{n \to \infty} x(\sigma(n)) = 0\), we conclude that \((x(n))\) has more than one real accumulation point. Thus \((x(n))\) does not converge in \(\mathbb{R}\).

If (2.7.b) holds, clearly \(\lim_{n \to \infty} z(n) = +\infty\), which means that \((x(n))\) is unbounded, and therefore \((x(n))\) does not converge in \(\mathbb{R}\).

The proof of the part (VI) of the theorem is complete.

The proof of Theorem 3.1 is complete.

\(\Box\)
As a consequence of Theorem 3.1, we postulate the following corollary.

**Corollary 3.1.** For every nonoscillatory solution \((x(n))\) of the equation \([E]\) the following statements hold:

(i) If the constants \(c_j\) are all nonpositive and \(c < 0\), then \((x(n))\) either tends to zero or tends to infinity.

(ii) If the constants \(c_j\) are all equal to zero, then \((x(n))\) tends to infinity.

(iii) If the constants \(c_j\) are all nonnegative and \(c > 0\), then \((x(n))\) is unbounded.

**REFERENCES**


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