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ON VERTEX-DEGREE RESTRICTED PATHS IN 4-CONNECTED PLANAR GRAPHS

Erhard Hexel — Hansjoachim Walther

ABSTRACT. Every 4-connected planar graph G is proved either to contain no k-path or to have a k-path containing no vertex of degree $\geq 2k+4$ if $\delta(G) \geq 4$ or of degree $\geq k+5$ if $\delta(G) = 5$, respectively. There are 4-connected planar graphs such that each k-path contains a vertex of degree $\geq 2k+2$ if $\delta(G) \geq 4$ or $\geq k+1$ if $\delta(G) = 5$ and $k \geq 15$. There are 5-connected planar graphs such that each k-path contains a vertex of degree $\geq \lfloor (2k+8)/3 \rfloor$ if $k \geq 20$.

1. Introduction

Throughout this paper we shall consider planar graphs without loops and multiple edges. For a plane graph $G(V,E,F)$, that is an embedding of a planar graph, let $V(G)$, $E(G)$, $F(G)$ denote the vertex-set, edge-set and the face-set of G, respectively. By $n_i = n_i(G)$, $f_i = f_i(G)$, $n = n(G)$, $f = f(G)$ and $e = e(G)$ we denote the number of vertices of degree i, the number of faces bounded by i edges, the number of vertices, the number of faces and the number of edges in G, respectively. The *vertex degree* of $x \in V(G)$, i.e., the number of edges incident with x, is denoted by $\deg_G(x)$ and the *face size* of $\alpha \in F(G)$, i.e., the number of edges incident with α (where each cut-edge is counted twice) is denoted by $\mathrm{size}_G(\alpha)$. $\delta(G) := \min\{\deg_G(x) \colon x \in V(G)\}$ is called the *minimum degree* and $\rho(G) := \min\{\mathrm{size}_G(\alpha) \colon \alpha \in F(G)\}$ the *minimum size* of G.

A path on k vertices is called a *k-path* and is denoted by P_k.

For given integers c, δ, ρ, where $3 \leq c \leq \delta \leq 5$ and $3 \leq \rho$, let $\mathcal{P}_c(\delta,\rho)$ be the family of all c-connected planar graphs G with $\delta(G) \geq \delta$ and $\rho(G) \geq \rho$. More specially, let $\mathcal{P}_c(\delta,\rho;H)$ for any given connected planar graph H be the family of all $G \in \mathcal{P}_c(\delta,\rho)$ containing a subgraph isomorphic to H.

A. Kotzig [7] has proved that any 3-connected planar graph G contains an edge $xy \in E(G)$ with $\deg_G(x) + \deg_G(y) \leq 13$, i.e., each *polyhedral graph* (planar and 3-connected) has a path P_2 whose both vertices have a degree ≤ 10.

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For k-paths, $k \in \{3, 4, 5\}$, S. Jendroľ [6] has proved results analogous to A. Kotzig. Moreover, S. Jendroľ proved in [6] that

> *For any two integers $k, m \geq 3$ there exists a 2-connected planar graph in which every k-path contains a vertex of degree $\geq m$.*

It means that the assumption for a planar graph to be 3-connected is of importance for investigations of this kind.

For every non-empty family $\mathcal{P}_c(\delta, \rho; H)$, let $\phi_c(\delta, \rho; H)$ denote the minimum integer such that every graph $G \in \mathcal{P}_c(\delta, \rho; H)$ contains a subgraph K isomorphic to H for which

$$\max_{x \in V(K)} \deg_G(x) \leq \phi_c(\delta, \rho; H)$$

is satisfied. A graph H is called *light* in $\mathcal{P}_c(\delta, \rho)$ if $\phi_c(\delta, \rho; H) < \infty$ holds, otherwise H is called *heavy* in $\mathcal{P}_c(\delta, \rho)$.

I. Fabrici and S. Jendroľ [2] proved the

THEOREM 1.

(i) *H is light in $\mathcal{P}_3(3, 3)$ if and only if $H = P_k$, $k \geq 1$;*

(ii) *$\phi_3(3, 3; P_k) = 5k$.*

In [4] it has been proved the

THEOREM 2.

(i) *H is light in $\mathcal{P}_3(4, 3)$ if and only if $H = P_k$, $k \geq 1$;*

(ii) *$\phi_3(4, 3; P_k) = 5k - 7$, $k \geq 8$;*

(iii) *$\frac{15}{11}k \leq \phi_4(4, 3; P_k) \leq 3k + 1$, $k \geq 4$.*

By the following result we are able to improve relation (iii) of Theorem 2.

THEOREM 3.

$2k + 2 \leq \phi_4(4, 3; P_k) \leq 2k + 3$, $k \geq 2$.

Moreover, we will prove the next two results.

THEOREM 4.

(i) *$\phi_4(5, 3; P_k) \leq k + 4$, $k \geq 2$;*

(ii) $$\phi_4(5, 3; P_k) \geq \begin{cases} k + 4 & \text{for } k = 2, \\ k + 3 & \text{for } k = 3, 4, 10, 11, \\ k + 2 & \text{for } k = 5, 6, 7, 8, 9, 12, 13, 14, \\ k + 1 & \text{for } k \geq 15. \end{cases}$$

Theorem 5.

$$\phi_5(5,3;P_k) \geq \begin{cases} 6 & \textit{for } k = 2,3\,, \\ 7 & \textit{for } k = 4,5\,, \\ 8 & \textit{for } k = 6\,, \\ 9 & \textit{for } k = 7\,, \\ 10 & \textit{for } k = 8,9,10\,, \\ 11 & \textit{for } k = 11\,, \\ 12 & \textit{for } k = 12,13,14\,, \\ 13 & \textit{for } k = 15\,, \\ 14 & \textit{for } k = 16,17\,, \\ 15 & \textit{for } k = 18\,, \\ 16 & \textit{for } k = 19\,, \\ \lfloor \frac{2k+8}{3} \rfloor & \textit{for } k \geq 20\,. \end{cases}$$

Remark. Let $\tau(c,\delta;k)$ denote the smallest integer such that every graph $G \in \mathcal{P}_c(\delta,3)$ of order at least k contains a connected subgraph H of order k with

$$\deg_G(x) \leq \tau(c,\delta;k)$$

for every vertex $x \in V(H)$.

In [3] and [1] it is shown that $\tau(3,3;k) = 4k+3$ for any $k \geq 3$ and $\tau(3,4;k) = 4k-1$ for any $k \geq 4$, respectively.

In fact, it is not hard to see that the given lower bounds for $\phi_c(\delta,3;P_k)$ where $4 \leq c \leq \delta \leq 5$ are also lower bounds for $\tau(c,\delta;k)$, and that

$$\tau(c,\delta;k) \leq \phi_c(\delta,3;P_k)$$

for $c \geq 4$. This together with the upper bounds for $\phi_c(\delta,3;P_k)$ implies

(i) $\phi_4(4,3;P_k) - \tau(4,4;k) \leq 1$ for any $k \geq 2$;

(ii) $\phi_4(5,3;P_k) - \tau(4,5;k) \leq 3$ for any $k \geq 2$;

(iii) $\phi_5(5,3;P_k) - \tau(5,5;k) \leq \frac{k+7}{3}$ for any $k \geq 2$.

2. Proofs

Proof of Theorem 3.

1. Proving the upper bound, suppose that $\phi := \phi_4(4,3;P_k) \leq 2k+3$ is not true. Then $\mathcal{P}_4(4,3;P_k)$ contains an example G such that each k-path of G has a vertex of degree $\geq \phi \geq 2k+4$. Among all these examples in $\mathcal{P}_4(4,3;P_k)$ with $n = n(G)$ vertices let G have a maximum number of edges. Put $S = \{x \in$

$V(G)\colon \deg_G(x) \geq \phi\}$ and $B = \{x \in V(G)\colon \deg_G(x) \leq \phi - 1\}$. Obviously, (S,B) is a partition of $V(G)$.

The following properties of G have been recognized in [2] and [4].

Property 1. *Every face α of G containing a vertex $x \in S$ is a triangle.*

Property 2. *Let S^* be the subgraph of G induced by S. Let $\alpha^* \in F(S^*)$ and $x \in V(S^*)$ be incident with α^* in S^* and let $N_B(x\colon \alpha^*)$ be the set of all vertices in B which are in G incident with x in the interior of α^*. Then these $m := \left|N_B(x\colon \alpha^*)\right|$ vertices form a P_m in G with $m \leq k-1$.*

Lemma 1.

$$|B| \leq \big(|S| - 2\big)(k-1)\,.$$

Proof. For $x \in S = V(S^*)$ we have $\deg_{S^*}(x) \geq 3$ because of Property 2 and $\deg_G(x) \geq 2k+4$. That implies $|S| \geq 4$ and $\left|E(S^*)\right| \geq 4$.

Improving the well-known Tutte Theorem that every planar 4-connected graph is hamiltonian, D. Sanders [8] proved:

Let $\widetilde{G}$ be a planar 4-connected graph. For any two vertices x, y and any edge $uv \neq xy$ of $\widetilde{G}$ there is a hamiltonian path P connecting x and y and passing uv.

In $\widetilde{G} = G$, consider a hamiltonian path P connecting vertices $x, y \in S$ and passing $uv \in E(S^*)$, where $uv \neq xy$. Between any two vertices of S consecutive in P there are at most $k-1$ vertices of B, which together with $uv \in E(P)$ proves the above inequality. □

Lemma 2.

$$\frac{(\phi - 6)|S|}{2} + 6 \leq |B|\,.$$

Proof. Because of the planarity of G we have

$$3f \leq 3f_3 + 4f_4 + \cdots = 2e,$$

implying $f \leq \frac{2}{3}e$. By $\delta(G) \geq 4$ we have

$$4n_4 + 5n_5 + \cdots + (\phi-1)n_{\phi-1} + \phi n_\phi + (\phi+1)n_{\phi+1} + \cdots = 2e\,,$$

which yields $4|B| + \phi|S| \leq 2e$ and further $\frac{4}{6}|B| + \frac{\phi}{6}|S| \leq \frac{e}{3}$. Together with the Eulerian formula

$$n + f = e + 2 = \frac{2}{3}e + \frac{1}{3}e + 2$$

we have

$$|S| + |B| \geq \frac{4}{6}|B| + \frac{\phi}{6}|S| + 2\,,$$

which proves Lemma 2. □

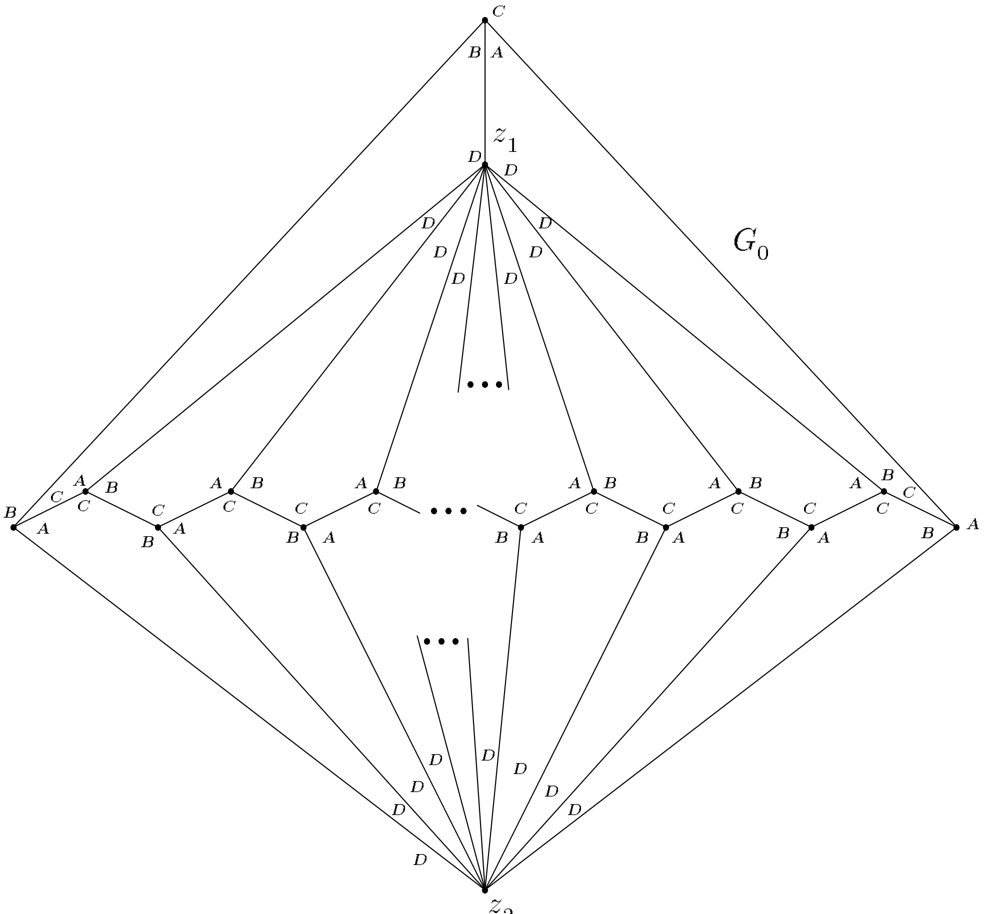

FIGURE 1

The correctness of the upper bound can be derived by combining the inequalities of Lemma 1 and 2.

2. The lower bound will be proved by constructing a member of $\mathcal{P}_4(4,3;P_k)$ in which any k-path has a vertex of degree at least $2k+2$.

Let G be the plane graph obtained by replacing each quadrangle $\alpha = x_1x_2x_3x_4$ of the graph G_0 of Fig. 1 by a copy of Fig. 2, such that the given numbers A, B, C, D correspond to the vertex degrees of x_i $(x_i = 1, 2, 3, 4)$ in G inside of α.

Obviously, G is a member of $\mathcal{P}_4(4,3;P_k)$. The replacing procedure yields

$$\deg_G(x) = \deg_{G_0}(x) + A + B + C = 3 + 2(k-1) + 1 = 2k+2$$

for every $x \in V(G_0)$ different from z_1 and z_2. Choosing the degree of z_i $(i = 1, 2, 3, 4)$ large enough, i.e., $\deg_{G_0}(z_i) \geq k+1$, we obtain

$$\deg_G(z_i) = \deg_{G_0}(z_i) + D \cdot \deg_{G_0}(z_i) \geq 2(k+1)\,,$$

which proves the lower bound and completes the proof of Theorem 3.

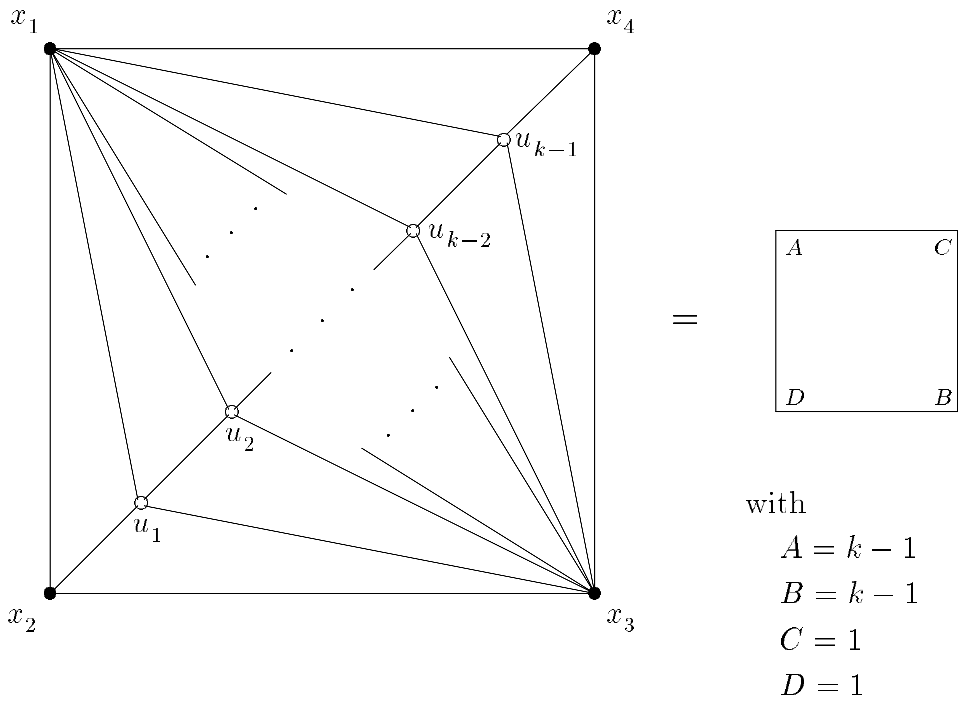

FIGURE 2

Proof of Theorem 4.

1. Proving the upper bound (i), suppose that $\phi := \phi_4(5,3;P_k) \le k+4$ is not true. I.e., $\mathcal{P}_4(5,3;P_k)$ contains an example G such that each k-path of G has a vertex of degree $\ge \phi \ge k+5$. Among all these examples in $\mathcal{P}_4(5,3;P_k)$ with $n = n(G)$ vertices let G have a maximum number of edges.

Put $S = \{x \in V(G)\colon \deg_G(x) \ge \phi\}$ and $B = \{x \in V(G)\colon \deg_G(x) \le \phi - 1\}$. (S,B) is again a partition of $V(G)$.

LEMMA 3.

$$(\phi-6)|S| + 12 \le |B|\,.$$

P r o o f. $\delta(G) \ge 5$ implies

$$5n_5 + \dots + (\phi-1)n_{\phi-1} + \phi n_\phi + (\phi+1)n_{\phi+1} + \dots = 2e\,,$$

and further $5|B| + \phi|S| \le 2e$, i.e., $\frac{5}{6}|B| + \frac{\phi}{6}|S| \le \frac{e}{3}$.

Together with the Eulerian formula and using $f \le \frac{2}{3}e$ we have

$$|S| + |B| \ge \frac{5}{6}|B| + \frac{\phi}{6}|S| + 2\,,$$

which proves Lemma 3. $\square$

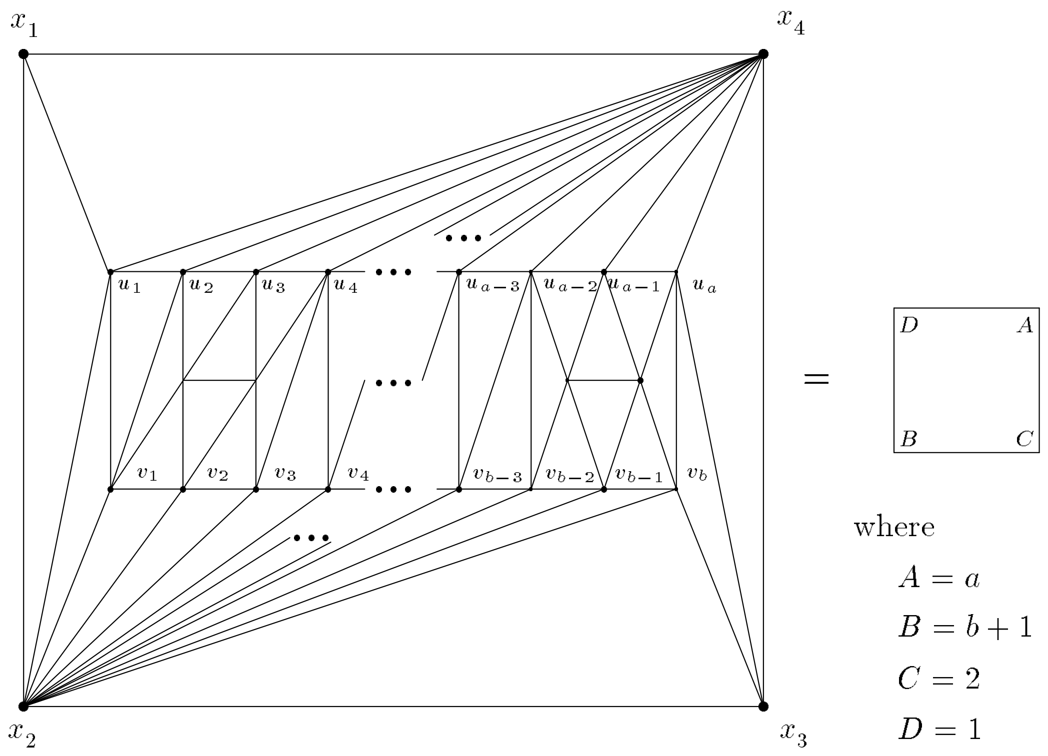

FIGURE 3

The correctness of the upper bound can be derived by combining the inequalities of Lemma 1 and 3.

2. To prove the lower bound (ii) it is sufficient to construct a suitable graph of $\mathcal{P}_4(5,3;P_k)$.

First, consider $k \geq 16$. Let G be the plane graph obtained by replacing each quadrangle $\alpha = x_1x_2x_3x_4$ of the graph G_0 of Fig. 1 by a copy of Fig. 3 such that for $a+b+4 = k-1$ with $a = \lceil (k-5)/2 \rceil$ and $b = \lfloor (k-5)/2 \rfloor$ the given numbers A, B, C, D are in accordance with the vertex degrees of x_i $(i = 1,2,3,4)$ in G inside of α.

Fig. 3 shows the replacing figure for odd k, especially. The corresponding replacing figure for even k can be derived from Fig. 3 by identifying the vertices v_{b-3} and v_{b-2}. Note that G belongs to $\mathcal{P}_4(5,3;P_k)$. The replacing procedure yields

$$\deg_G(x) = \deg_{G_0}(x) + A + B + C = 3 + \left\lceil \tfrac{k-5}{2} \right\rceil + \left(\left\lfloor \tfrac{k-5}{2} \right\rfloor + 1 \right) + 2 = k + 1$$

for every $x \in V(G_0)$ different from z_1 and z_2. Choosing $\deg_{G_0}(z_i) \geq \lceil (k+1)/2 \rceil$ for $i = 1,2$ we obtain

$$\deg_G(z_i) = \deg_{G_0}(z_i) + D \cdot \deg_{G_0}(z_i) \geq 2 \left\lceil \tfrac{k+1}{2} \right\rceil \geq k + 1 .$$

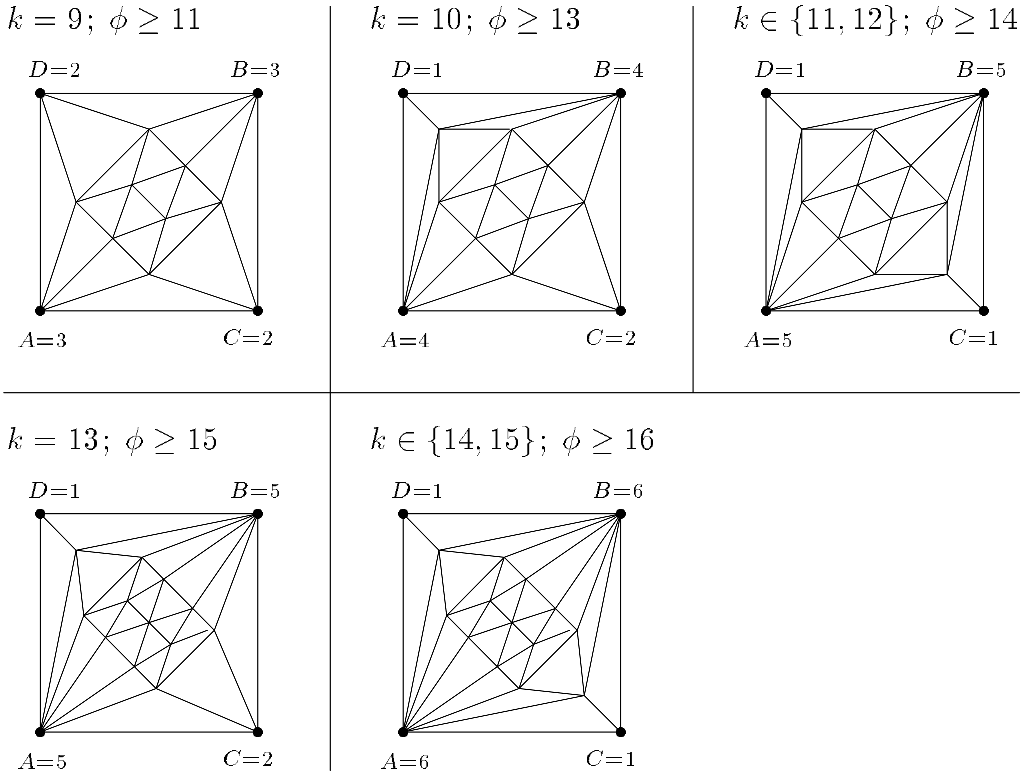

Figure 4

Thus, every k-path of G has a vertex of degree at least $k+1$, which proves the lower bound for $k \geq 16$.

For $2 \leq k \leq 8$, the lower bound results from Theorem 5 and considering that $\phi_4(5,3;P_k) \geq \phi_5(5,3;P_k)$.

To prove relation (ii) for $9 \leq k \leq 15$, we construct a graph $G \in \mathcal{P}_4(5,3;P_k)$ by replacing each quadrangle of the graph G_0 of Fig. 1 by the corresponding configuration of Fig. 4. Note that every k-path of G contains a vertex of $V(G_0)$. By choosing $\deg_{G_0}(z_i)$ large enough for $i = 1, 2$ a simple calculation shows that $\deg_G(x)$ for each $x \in V(G_0)$ is not smaller than the claimed number in relation (ii).

That completes the proof of Theorem 4.

Proof of Theorem 5.

It is sufficient to construct a suitable graph G of $\mathcal{P}_5(5,3;P_k)$.

First, consider $k \geq 20$. Let G be the plane graph obtained by replacing each pentagon $\alpha = x_1x_2x_3x_4x_5$ of the graph G_1 of Fig. 5 by a copy of Fig. 6 such that for $a+b+4 = k-1$ with $a = \lceil (k-5)/2 \rceil$, $b = \lfloor (k-5)/2 \rfloor$ and $c = \lfloor (k-4)/3 \rfloor$,

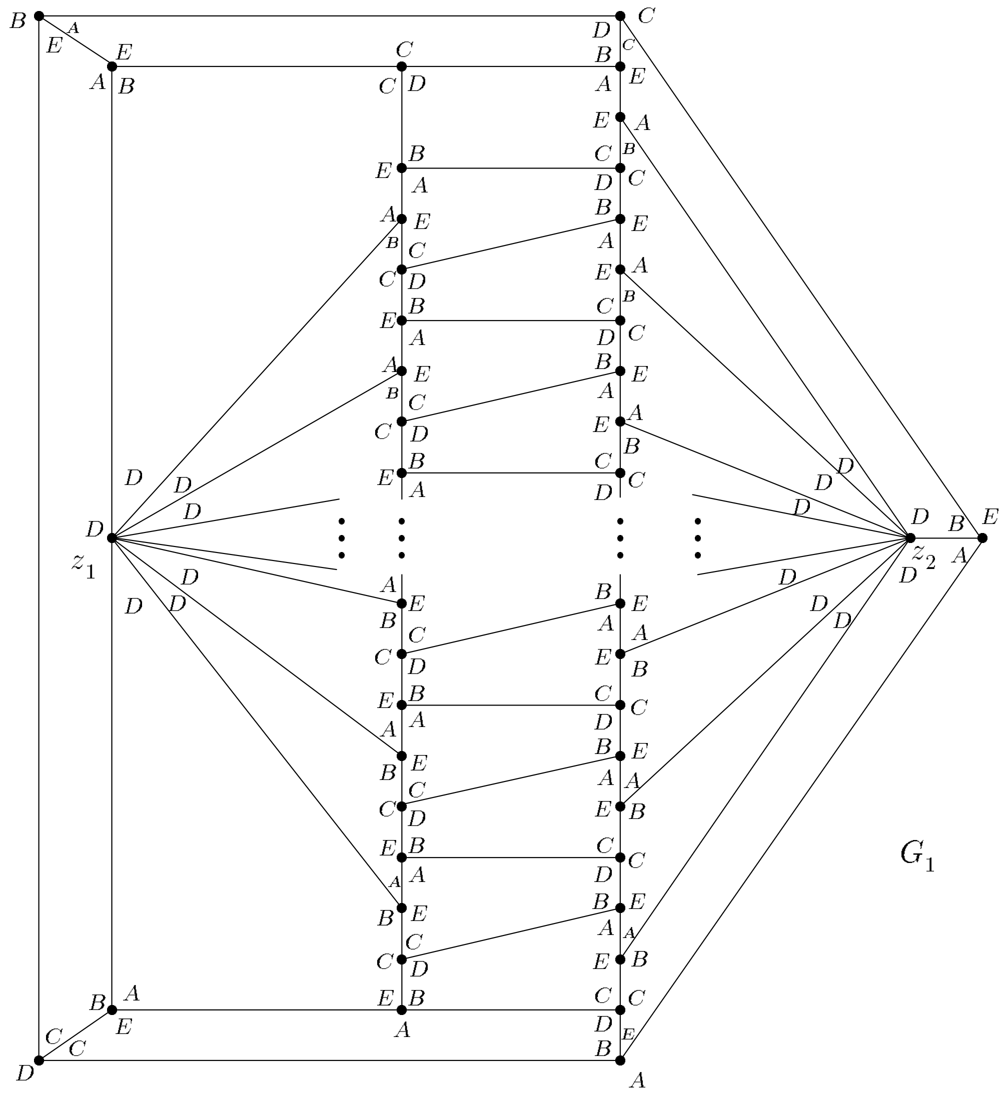

FIGURE 5

the given numbers A, B, C, D, E are in accordance with the vertex degrees of x_i $(i = 1, 2, 3, 4, 5)$ in G inside of α.

Fig. 6 shows the replacing figure for odd k, especially. The corresponding replacing figure for even k can be derived from Fig. 6 by identifying the vertices v_{b-3} and v_{b-2}. Note that G is a member of $\mathcal{P}_5(5, 3; P_k)$. The replacing procedure yields for every vertex $x \in V(G_1)$ different from z_1 and z_2 either

$$\deg_G(x) = \deg_{G_1}(x) + A + B + E = 6 + a + b - c = 6 + \left\lceil \tfrac{k-5}{2} \right\rceil + \left\lfloor \tfrac{k-5}{2} \right\rfloor - \left\lfloor \tfrac{k-4}{3} \right\rfloor$$

or

$$\deg_G(x) = \deg_{G_1}(x) + C + C + D = 6 + 2c = 6 + 2\left\lfloor \tfrac{k-4}{3} \right\rfloor.$$

A simple calculation confirms in both cases $\deg_G(x) \geq \lfloor (2k + 8)/3 \rfloor$.

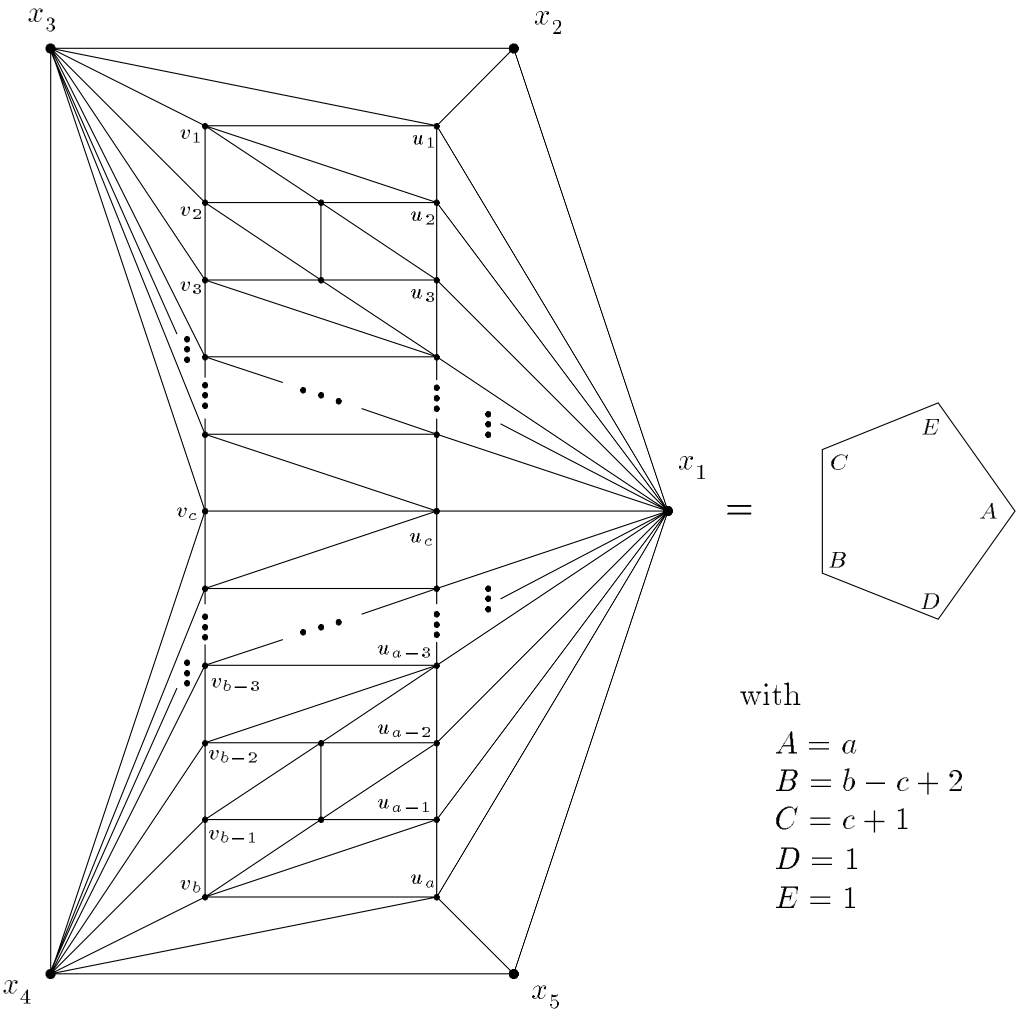

FIGURE 6

Choosing now $\deg_{G_1}(z_i)$ large enough for $i = 1, 2$, we can guarantee that every k-path of G has a vertex of degree at least $\lfloor (2k+8)/3 \rfloor$, which proves the assertion for $k \geq 20$.

For $k = 2$ or 3 let $G \in \mathcal{P}_5(5, 3; P_k)$ be the graph obtained by replacing each pentagon of the dodecahedron graph D by the configuration Fig. 7(a). Obviously, every k-path of G contains a vertex $x \in V(D)$, and $\deg_G(x) = 6$.

Let F_m, where $m \in \{20, 80\}$, denote the 3-regular planar graph consisting of 12 pentagons and of m hexagons, also called *Fullerene graph* (see, e.g., [5]). Then, for $k = 4$ or 5 let $G \in \mathcal{P}_5(5, 3; P_k)$ be the graph obtained from F_{20} by replacing each pentagon and each hexagon by the configuration Fig. 7(a) and Fig. 7(b), respectively. Every k-path of G contains a vertex of $V(F_{20})$, and the replacing procedure can be organized such that $\deg_G(x) \geq 7$ holds

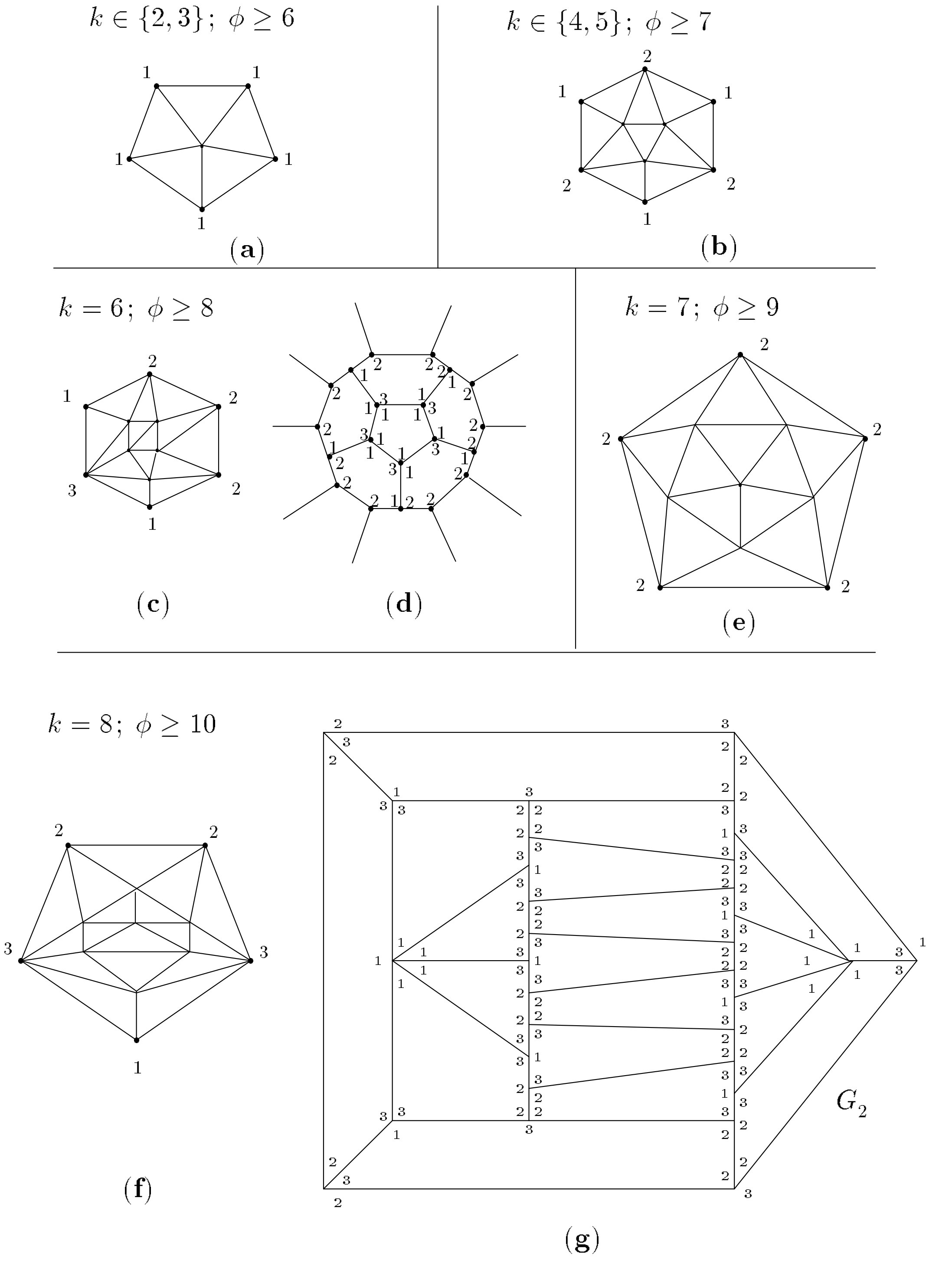
$k \in \{2,3\};\ \phi \geq 6$
(a)
$k \in \{4,5\};\ \phi \geq 7$
(b)
$k = 6;\ \phi \geq 8$
(c)
(d)
$k = 7;\ \phi \geq 9$
(e)
$k = 8;\ \phi \geq 10$
(f)
G_2
(g)

Figure 7

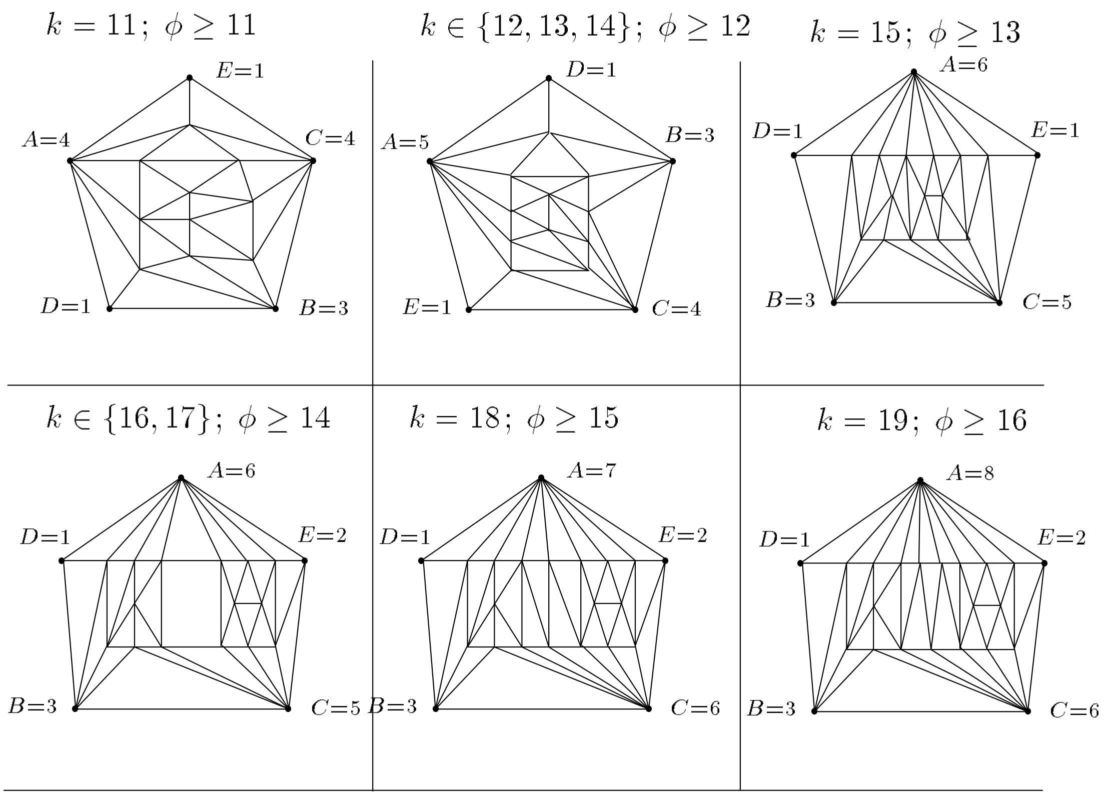

FIGURE 8

for each $x \in V(F_{20})$.

For $k=6$ let $G \in \mathcal{P}_5(5,3;P_6)$ be the graph obtained from F_{80} by replacing each pentagon and each hexagon by the configuration Fig. 7(a) and Fig. 7(c), respectively, where the 5 neighboured hexagons of every pentagon are to be replaced in accordance with Fig. 7(d). Then, the remaining 20 hexagons can be replaced in such a way that the degree of any $x \in V(F_{80})$ in G is $\deg_{F_{80}}(x) + 3+1+1$ or $\deg_{F_{80}}(x)+3+2+1$ or $\deg_{F_{80}}(x)+2+2+1$ or $\deg_{F_{80}}(x)+2+2+2$, which implies $\deg_G(x) \geq 8$. Any 6-path P_6 of G contains a vertex of $V(F_{80})$ and, consequently, P_6 contains a vertex of degree not smaller than 8.

For $k=7$ let $G \in \mathcal{P}_5(5,3;P_7)$ be the graph obtained by replacing each pentagon of the dodecahedron graph D by the configuration Fig. 7(e). Obviously, every 7-path of G contains a vertex $x \in V(D)$, and $\deg_G(x) = 9$.

For $8 \leq k \leq 10$ let $G \in \mathcal{P}_5(5,3;P_k)$ be the graph obtained by replacing each pentagon of the graph G_2 of Fig. 7(g) by the configuration of Fig. 7(f) in accordance with the given numbers. Every k-path of G contains a vertex x of G_2 and, clearly, $\deg_G(x) = 10$.

For $11 \le k \le 19$ we construct a graph $G \in \mathcal{P}_5(5,3;P_k)$ by replacing each pentagon of the graph G_1 of Fig. 5 by the corresponding configuration of Fig. 8.

Note that every k-path of G contains a vertex of $V(G_1)$. By choosing $\deg_{G_1}(z_i)$ for $i = 1, 2$ large enough, a simple calculation shows that $\deg_G(x)$ for each $x \in V(G_1)$ is not smaller than the claimed number in Theorem 5, which completes the proof.

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Technical University Ilmenau
Department of Mathematics
PF 0565
D–98684 Ilmenau
GERMANY

E-mail: hexel@mathematik.tu-ilmenau.de
walther@mathematik.tu-ilmenau.de