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ON SOME CLASS OF FUNCTIONS INTERMEDIATE BETWEEN THE CLASS B_1^* AND THE FAMILY OF CONTINUOUS FUNCTIONS

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ABSTRACT. The considerations concentrate on the topological and algebraical properties of the functions whose restrictions to the set of discontinuity points are continuous.

Introduction.

In many papers (e.g., [12], [15], [13]) the question of the existence of certain algebraic structures (e.g., rings or complete systems) or topological ones (related to the generalization of homotopies, retracts and the like) was connected with the existence of some topology on the real line, generating various classes of functions.

In particular in 1994 A. Tomaszewska [15] distinguished the property (top): *We say that a Darboux and Świątkowski function* $f\colon \mathbb{R} \to \mathbb{R}$ *has the property* (top) *if there exists a topology* T *stronger than the natural topology in* $\mathbb{R}$ *such that the ring* $C(T)$ *consists of Darboux and Świątkowski functions, and* $f \in C(T)$. Unfortunately, when we consider these topologies, one could not, in general, obtain full characterization of the existence of algebraic structures. The analysis of this situation proved that the essential difficulty was, in this case, the fact that the neighbourhoods (in the new topology) of discontinuity points of the functions considered could not contain other points of discontinuity. On the other hand the studies connected with this topic seem to suggest that 2-continuous functions (i.e., functions for which their domain may be decomposed into two sets in such a way that the restriction of the function to each of them is a continuous mapping) are interesting for these studies. This observation, according to results [8] and [5], leads to the conclusion that classes of functions, intermediate between the family of continuous functions and the class B_1^* ([9]) will be interesting for this considerations. Consequently, we shall define the class B_1^{**} of functions:

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We say that $f\colon X \to Y$ (where X and Y are topological spaces)[1] belongs to the class B_1^{**} if either $D_f = \emptyset$ or $f \restriction D_f$ is a continuous function (where D_f denote the set of all discontinuity points of f).

Preliminaries.

We apply the classical symbols and notions ([1], [7], [4], [8], [9]). In particular, by $\mathbb{R}$ we denote the set of all real numbers (it will cause no confusion if we use the same letter to designate a topological space $\mathbb{R}$ with the natural topology). The symbols $\overline{A}$, $\mathrm{Int}(A)$ stand for the closure and interior of the set A, respectively.

We say that a topology $\mathscr{T}$ is compatible with the natural topology of the real line ($\mathscr{T}_0$) relative to the family of functions $\mathscr{F}$ if

1. $f\colon (\mathbb{R}, \mathscr{T}) \to \mathbb{R}$ is a continuous function for any $f \in \mathscr{F}$.
2. If $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies of the subspace $\mathbb{R} \setminus \bigcup_{f\in\mathscr{F}} \overline{D_f}$ of the space $(\mathbb{R}, \mathscr{T})$ and $\mathbb{R}$, respectively; then $\mathscr{T}_1 = \mathscr{T}_2$.
3. For an arbitrary $x \in \mathbb{R}$ and each $\mathscr{T}$-neighbourhood U of x, x is a bilateral accumulation point of $U \setminus \bigcup_{f\in\mathscr{F}} \overline{D_f}$ (in the topology $\mathscr{T}_0$).

By C_f (D_f) we shall denote the set of all continuity (discontinuity) points of f. The constant function (assuming value α) we shall denote by const_α; the characteristic function of a set A we denote by χ_A and the restriction of a function f to a set A we shall denote by $f \restriction A$.

By $\mathcal{C}$ (B_1) we shall denote the class of all continuous (Baire one) functions and by the symbol B_1^* we shall denote the class of all functions f such that for every perfect set P, there exists a portion T (i.e., nonempty set of form $T = P \cap (a,b)$) such that $f \restriction T$ is continuous.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is cliquish if for any $x \in \mathbb{R}$, $\varepsilon > 0$, $\delta > 0$ there exists a nonempty open set $G \subset (x-\delta, x+\delta)$ such that $|f(y) - f(z)| < \varepsilon$, for each $y, z \in G$. The family of all cliquish functions $f\colon \mathbb{R} \to \mathbb{R}$ we shall denote by $\mathscr{K}$.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is quasi-continuous at a point $x \in \mathbb{R}$ if for each neighbourhoods U of x and V of $f(x)$, there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be quasi-continuous if it is quasi-continuous at each point $x \in \mathbb{R}$. The family of all quasi-continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ we shall denote by $\mathscr{Q}$. We say that f is weakly quasi-continuous function if f is cliquish and $f \restriction \overline{A}$ is quasi-continuous, for each connected set $A \subset C_f$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be left (right) hand sided quasi-continuous at a point $x \in \mathbb{R}$ if for every $\delta > 0$ and for every open neighbourhood V of $f(x)$ there exists an open nonempty set $W \subset (x-\delta, x) \cap f^{-1}(V)$ ($W \subset (x, x+\delta) \cap f^{-1}(V)$). A function f is bilaterally quasi-continuous if it is both left and right sided quasi-continuous at each point of its domain.

[1] In this paper we restrict our considerations to the case of real functions of real variable.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be peripherally continuous if for each $x \in \mathbb{R}$ there exist sequences $x_n \nearrow x \swarrow y_n$ such that $\lim_{n\to\infty} f(x_n) = f(x) = \lim_{n\to\infty} f(y_n)$.

A set A is simply open if it is the union of an open set and a nowhere dense set. We recall that a function f is simply continuous if $f^{-1}(V)$ is a simply open set, for each open set V. The family of all simply continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ we denote by $\mathcal{S}$.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is almost continuous (in the sense of Stalligs) if for each open set $U \subset \mathbb{R} \times \mathbb{R}$ containing graph of f, U contains the graph of some continuous function $g\colon \mathbb{R} \to \mathbb{R}$.

If $\mathscr{F}, \mathscr{G}$ are some family of functions, then the intersection $\mathscr{F} \cap \mathscr{G}$ we shall denote briefly by $\mathscr{F}\mathscr{G}$.

The class Φ is an ordinary system (in the sense of Aumann) if

1. $f, g \in \Phi$ imply $\max\{f,g\} \in \Phi$, $\min\{f,g\} \in \Phi$;
2. Φ contains all constants;
3. $f, g \in \Phi$ imply $f + g \in \Phi$, $f \cdot g \in \Phi$ and $f/g \in \Phi$, for g such that $\{x\colon g(x) = 0\} = \emptyset$.

The class Φ is a complete system (in the sense of Aumann) if Φ is an ordinary system and the uniform limit of a sequence of functions from Φ belongs to Φ.

1. Comparison with other class of functions

First we shall prove the following lemmas

Lemma 1. *A function f belongs to the class B_1^{**} if and only if either $D_f = \emptyset$ or $f \restriction \overline{D_f}$ is the continuous function.*

Proof. Necessity. Suppose, contrary to our claim, that there exists $x_0 \in \overline{D_f}$ such that x_0 is the discontinuity point of $g = f \restriction \overline{D_f}$. Of course, $x_0 \in D_f$ and there exists a sequence $\{x_n\} \subset \overline{D_f}$ such that $x_n \to x_0$ and $f(x_n) \to \alpha \neq f(x_0)$ (it is possible that $\alpha = +\infty$ or $\alpha = -\infty$). Let, for instance, $\alpha > f(x_0)$. Without loss of generality we may assume that $\{x_n\} \subset C_f$. Let $\beta \in \bigl(f(x_0), \alpha\bigr)$. We may assume that $f(x_n) \in (\beta, +\infty)$.

Let y_n be an element of the intersection $\bigl(x_n - \frac{1}{n}, x_n + \frac{1}{n}\bigr) \cap D_f$ (for $n = 1, 2, \dots$) such that

$$f(y_n) \in (\beta, +\infty). \tag{1}$$

Then $\{y_n\} \subset D_f$ and $y_n \to x_0$. Since $f \in B_1^{**}$, then $\lim_{n\to\infty} f(y_n) = f(x_0)$, which contradicts (1).

Sufficiency is obvious. □

Of course, there exists a function $f \in B_1^{**}$ for which D_f is not a closed set.

Lemma 2. *If $f \in B_1^{**}$, then D_f is a nowhere dense set.*

Proof.[2] Suppose, on the contrary, that D_f is not a nowhere dense set. This means that there exists an open (in the space X) set $U \neq \emptyset$ such that

$$U \subset \overline{D_f}\,. \tag{2}$$

So, let $x_0 \in U \cap D_f$ and let $\sigma > 0$ be a number such that for any neighbourhood O of x_0 there exists $y_0 \in O$ such that $f(y_0) \notin \left[f(x_0) - \sigma,\ f(x_0) + \sigma\right]$. Since $f \in B_1^{**}$, we infer that there exists a neighbourhood $V \subset U$ of x_0 such that

$$f(V \cap D_f) \subset \left(f(x_0) - \sigma\,,\ f(x_0) + \sigma\right). \tag{3}$$

Simultaneously, there exists $y \in V$ such that $f(y) \notin \left[f(x_0) - \sigma,\ f(x_0) + \sigma\right]$. According to (3), we have $y \in C_f$ and so there exists a neighbourhood $S \subset V$ of y such that

$$f(S) \cap \left[f(x_0) - \sigma\,,\ f(x_0) + \sigma\right] = \emptyset\,.$$

The above equality, according to (3), contradicts (2). □

The starting point of our considerations will be the observations connected with the comparison with other class of functions.

Proposition 1. *$\mathcal{C} \subset B_1^{**} \subset B_1^{*} \subset B_1$, and these inclusions cannot be replaced by the equalities.*

Proof. The first inclusion $\mathcal{C} \subset B_1^{**}$ and the inequality $\mathcal{C} \neq B_1^{**}$ are obvious.

Now, we shall show the second inclusion: $B_1^{**} \subset B_1^{*}$. So, let $f \in B_1^{**}$ and let F be an arbitrary perfect set. If $F \subset \overline{D_f}$ then, by Lemma 1, $f \restriction F$ is a continuous function. If $F \setminus \overline{D_f} \neq \emptyset$, then there exists an interval (a,b) such that $(a,b) \cap F \neq \emptyset = (a,b) \cap \overline{D_f}$. From this we get $(a,b) \subset C_f$, and so $f \restriction (a,b) \cap F$ is a continuous function.

The function $\chi_{\{\frac{1}{n}:\ n=1,2,\dots\}}$ shows that $B_1^{**} \neq B_1^{*}$.

The facts: $B_1^{*} \subset B_1$ and $B_1^{*} \neq B_1$ are well known. □

[2]Since this lemma is also important for the considerations connected with the seeking for methods for improving some functions, we prove this lemma in a more general version — in the case of functions $f\colon X \to \mathbb{R}$, where X is an arbitrary topological space.

PROPOSITION 2. *$B_1^{**} \subset \mathcal{S}$, and this inclusion is proper.*

Proof. By Lemma 2 and Lemma 1 from the paper [2] we may infer that $B_1^{**} \subset \mathcal{S}$.

The example showed in the proof of Theorem 1 proves that the inclusion $B_1^{**} \subset \mathcal{S}$ is proper. □

COROLLARY 1. $B_1^{**} \subset \mathscr{K}$.

Proof. This fact follows (according to [11][3]) immediately from the Proposition 2. □

Now, we shall present the comparisons with the quasi-continuity and Darboux property.

THEOREM 1. *Let $f \in B_1^{**}$. Then the following conditions are equivalent:*

(i) *f is a weakly quasi-continuous function;*[4]

(ii) *f is a bilaterally quasi-continuous function;*

(iii) *f is a Darboux function;*

(iv) *f is an almost continuous function;*

(v) *f is a peripherally continuous function.*

Proof. First, we shall prove the inclusion (i) $\Rightarrow$ (ii). Let $x_0 \in \mathbb{R}$. We shall show that f is "right-side" quasi-continuous at x_0 (the proof that f is "left-side" quasi continuous at x_0 is similar). Of course, it is sufficient to consider the case if $x_0 \in D_f$. Let us consider the following cases:

1^o There exists y_0 such that $(x_0, y_0) \subset C_f$. This means that $f \restriction [x_0, y_0]$ is a quasi-continuous function, and consequently f is "right-side" quasi-continuous at x_0.

2^o There does not exist element y_0 such that $(x_0, y_0) \subset C_f$. Let $\varepsilon > 0$ and $\delta > 0$ and let $\delta_0 \in (0, \delta)$ be a number (Lemma 1) such that

$$f\big(\overline{D_f} \cap (x_0 - \delta_0\,,\, x_0 + \delta_0)\big) \subset \big(f(x_0) - \varepsilon,\ f(x_0) + \varepsilon\big) \tag{4}$$

and let (a, b) be an arbitrary component ($a < b$) of the set $\big(\mathbb{R} \setminus \overline{D_f}\big) \cap (x_0, x_0 + \delta_0)$ (by virtue of Lemma 2, such a component there exists). Applying (4) we obtain that $f(a) \in \big(f(x_0) - \varepsilon,\ f(x_0) + \varepsilon\big)$. Since $f \restriction [a, b]$ is a quasi-continuous function, then there exists an interval $(c, d) \subset (a, b) \subset (x_0, x_0 + \delta)$ such that $f\big((c, d)\big) \subset \big(f(x_0) - \varepsilon,\ f(x_0) + \varepsilon\big)$, which ends the proof of the first implication.

[3]If X is a Baire space, then every simply continuous function $f \colon X \to \mathbb{R}$ is cliquish.

[4]Observe, that the weak quasi-continuity is a very weak assumption and even in the Baire class one it implies neither quasi-continuity nor the Darboux property.

The implication (ii) $\Rightarrow$ (iii) follows immediately from the Remark 3.1 of the paper [3]. The implication (and even equivalence) (iii) $\Rightarrow$ (iv), for a Baire one functions, is well known (see, e.g., [14], [10]). The implication (iv) $\Rightarrow$ (v) follows immediately from the fact that f is Darboux and Baire one function (see, e.g., [14], [16], [4]).

(v) $\Rightarrow$ (i). Let A be an arbitrary connected subset of C_f. Suppose, for instance, that $A = (p, q]$ (in the other cases, the considerations are similar). Let $g = f \restriction [p, q]$. It is sufficient to show that p is a quasi-continuity point of g. Let $\varepsilon > 0$, $\delta > 0$. Since f is peripherally continuous function then there exists a sequence $\{p_n\} \subset (p, q)$ such that $p_n \searrow p$ and $f(p_n) \to f(p)$. Let n_0 be a positive integer such that $p_{n_0} \in (p, p + \delta)$ and $f(p_{n_0}) \in \big(f(p) - \varepsilon,\ f(p) + \varepsilon\big)$. From the fact that $p_{n_0} \in C_f$ we conclude that there exists an open neighbourhood U of p_{n_0} such that $U \subset (p, p + \delta)$ and $f(U) \subset \big(f(p) - \varepsilon,\ f(p) + \varepsilon\big)$.

By the Corollary 1, we infer that f is cliquish. $\square$

Remark. It is not difficult to verify that $\mathscr{Q} \setminus B_1^{**} \neq \emptyset \neq B_1^{**} \setminus \mathscr{Q}$.

2. Main results

Let's go to the main problem of our considerations. It is well known that even simple operation (addition, multiplication, lattice operations etc.) performed on Darboux functions of the first class of Baire lead out of this class, even if both the functions considered are discontinuous at the same point. The more interesting questions are those of when two Darboux functions belong to a joint complete system or ordinary system. The earlier considerations (e.g., [12], [13]) have shown that this problem is connected in a direct way with the constructing of a special topologies. However, even under quite strong assumptions, it was not possible to obtain, in general, a full characterization. The situation is better when we consider the class B_1^{**}.

Theorem 2. *Let $f, g \in DB_1^{**}$ be the functions such that $D_f = D_g$. Then the following conditions are equivalent:*

(a) *$f, g \in \Phi$, where $\Phi \in DB_1^{**}$ is some complete system containing $\mathcal{C}$;*

(b) *$f, g \in \Phi$, where $\Phi \in DB_1^{**}$ is some ordinary system containing $\mathcal{C}$;*

(c) *there exists a topology $\mathscr{T}$ finer than the natural topology of the real line $\mathscr{T}_0$ and compatible with the topology $\mathscr{T}_0$ relative to the family $\{f, g\}$.*

Proof. If $D_f = D_g = \emptyset$, then, of course, theorem is true. So, in the further considerations we shall assume that $D_f = D_g \neq \emptyset$. The implication (a) $\Rightarrow$ (b) is obvious. Now, we shall prove the implication (b) $\Rightarrow$ (c). Let us denote $D = \overline{D_f} = \overline{D_g}$. By Lemma 2, D is a nowhere dense set. Fix $x_0 \in D$. Put

$h = \left|f - \mathrm{const}_{f(x_0)}\right| + \left|g - \mathrm{const}_{g(x_0)}\right|$. Thus $h \in \Phi \subset DB_1^{**}$, where Φ is some ordinary system. It is obvious that $h(x_0) = 0$.

It is easy to see

$$h \restriction D \text{ is a function continuous at } x_0 . \tag{5}$$

From (5) we conclude:

$$\begin{aligned}&\text{for any positive integer } m \text{ there exists } \delta_m \in \left(0, \tfrac{1}{m}\right)\\ &\text{such that } h\big((x_0 - \delta_m ,\ x_0 + \delta_m) \cap D\big) \subset \left[0, \tfrac{1}{m}\right).\end{aligned} \tag{6}$$

Let us denote by K_m (for $m = 1, 2, \dots$) the family of all components of the set $(\mathbb{R} \setminus D) \cap (x_0 - \delta_m ,\ x_0 + \delta_m)$. By virtue of (6) we infer that if s is an endpoint of some component belonging to K_m (which is different from $x_0 - \delta_m$ and $x_0 + \delta_m$) then $h(s) \in \left[0, \frac{1}{m}\right)$. Since $h \in \Phi \subset DB_1^{**}$, Theorem 1 now shows that for each $C \in K_m$ there exists $w_C^m \in C$ such that $h(w_C^m) \in \left[0, \frac{1}{m}\right)$.

Let $\hat{B}_{x_0}^m = \{w_C^m : C \in K_m\}$ (for $m = 1, 2, \dots$). Observe that h is continuous at each point of $\hat{B}_{x_0}^m$ $(m = 1, 2, \dots)$, and so

$$\begin{aligned}\text{for any } z \in \hat{B}_{x_0}^m \ &\text{there exists } \sigma_z > 0 \text{ such that } (z - \sigma_z ,\ z + \sigma_z) \subset C_z \in K_m\\ &\text{and } h\big((z - \sigma_z ,\ z + \sigma_z)\big) \subset \left[0, \tfrac{1}{m}\right),\\ &\text{where } C_z \text{ is a component containing } z \ (m = 1, 2, \dots).\end{aligned}$$

Let $\tilde{B}_{x_0}^m = \bigcup\limits_{z \in \hat{B}_{x_0}^m} (z - \sigma_z ,\ z + \sigma_z)$ (for $m = 1, 2, \dots$). Put $B_{x_0}^m = \bigcup\limits_{k \geq m} \tilde{B}_{x_0}^k$. It is not difficult to verify that

$$B_{x_0}^m \neq \emptyset , \quad \text{for each positive integer } m \tag{7}$$

and

$$h(B_{x_0}^m) \subset \left[0, \tfrac{1}{m}\right) , \quad \text{for each positive integer } m .$$

The last inclusion gives

$$\begin{aligned}f(B_{x_0}^m) &\subset \left(f(x_0) - \tfrac{1}{m} ,\ f(x_0) + \tfrac{1}{m}\right) , \qquad \text{and}\\ g(B_{x_0}^m) &\subset \left(g(x_0) - \tfrac{1}{m} ,\ g(x_0) + \tfrac{1}{m}\right) ,\\ &\text{for each positive integer } m .\end{aligned} \tag{8}$$

Now, let us assume that $x_0 \notin D$. This means that $x_0 \in C_f = C_g$. In this case let $B_{x_0}^m$ (for a fixed positive integer m) be an open interval containing x_0 for which the inclusions (8) are fulfilled (additionally, we may assume that $B_{x_0}^{m_1} \subset B_{x_0}^{m_2}$, for $m_1 > m_2$ and $B_{x_0}^m \subset \left(x_0 - \frac{1}{m},\ x_0 + \frac{1}{m}\right)$, for any positive integer m).

In this way for any $x \in \mathbb{R}$ we have defined the family $\{B_x^m : m = 1, 2, \dots\}$. Now, we shall define a interior operator "$\mathscr{T} - \mathrm{int}$" in the following way:

$$\forall_{A\subset\mathbb{R}} \left(x \in \mathscr{T} - \mathrm{int}(A) \iff x \in A \wedge \exists_m B_x^m \subset A\right).$$

It is easy to see that by the $\mathscr{T} - \mathrm{int}$ operator we may generate the topology $\mathscr{T} = \{A \subset \mathbb{R} : A = \mathscr{T} - \mathrm{int}(A)\}$ (see, e.g., [7]).

It is not difficult to verify (using, for example, (7)) that $\mathscr{T}$ is finer than the natural topology of $\mathbb{R}$.

Now, we shall show that $f, g : (\mathbb{R}, \mathscr{T}) \to \mathbb{R}$ are continuous functions. We prove this fact for a function f (the proof in the case of the function g is similar).

Let $z \in \mathbb{R}$ and $\varepsilon > 0$. Fix a positive integer m_0 such that $\frac{1}{m_0} < \varepsilon$. If $z \notin D$, then $B_z^{m_0}$ is an open neighbourhood of z such that (by (8))

$$f(B_z^{m_0}) \subset \left(f(z) - \tfrac{1}{m_0}\,,\ f(z) + \tfrac{1}{m_0}\right) \subset \left(f(z) - \varepsilon\,,\ f(z) + \varepsilon\right). \tag{9}$$

Observe that if $z \in D$, (according to (8)) the condition (9) is true, too. Since $\{z\} \cup B_z^{m_0} \in \mathscr{T}$, the inclusion (9) ends the proof of the continuity of f at z.

Now, we shall show that the subspaces $\mathbb{R} \setminus D$ of the both spaces $(\mathbb{R}, \mathscr{T})$ and $\mathbb{R}$ possesses the identical topology.

Let U be an $\mathscr{T}$-open set in the subspace $\mathbb{R} \setminus D$ of the space $(\mathbb{R}, \mathscr{T})$. Then $U \in \mathscr{T}$. Let $u \in U$. Since $u \in \mathbb{R} \setminus D$, then there exists an open set (in the natural topology of the real line) B_u^m such that $B_u^m \subset U$ and so $u \in \mathrm{Int}(U)$ (in the natural topology). According to the arbitrariness of the choice of U we may infer that $U = \mathrm{Int}(U)$, which means that U is an open set in the subspace $\mathbb{R} \setminus D$ of $\mathbb{R}$.

So, let U be an open set in the subspace $\mathbb{R} \setminus D$ of $\mathbb{R}$. Then U is an open set in $\mathbb{R}$, and consequently $U \in \mathscr{T}$ (because $\mathscr{T}$ is finer than the natural topology of the real line). This proves that U is an open set in the subspace $\mathbb{R} \setminus D$ of $(\mathbb{R}, \mathscr{T})$.

From the construction of B_x^m immediately follows (using Lemma 2) that for each point $x \in \mathbb{R}$ and any $\mathscr{T}$-neighbourhood U of x, x is a bilateral accumulation point of $U \setminus D$.

At last, we shall prove the implication (c) $\Rightarrow$ (a).

Let $\mathscr{T}$ be a topology described in (c) and put, similarly as in previous part of the proof, $D = \overline{D_f} = \overline{D_g}$. Now, we define a new topology $\mathscr{O}$. Let $\{(a_n, b_n)\}$ be a sequence of all components of the set $\mathbb{R} \setminus D$ (in the natural topology). Let us denote by $\mathscr{B}_{\mathscr{T}}(x)$ a base of the space $(\mathbb{R}, \mathscr{T})$ at x. For any positive integers n and m we can choose sets $U_{a_n}^m \in \mathscr{B}_{\mathscr{T}}(a_n)$, $U_{b_n}^m \in \mathscr{B}_{\mathscr{T}}(b_n)$ such that

$$f(U_{a_n}^m) \subset \left(f(a_n) - \tfrac{1}{m}\,,\ f(a_n) + \tfrac{1}{m}\right),$$
$$f(U_{b_n}^m) \subset \left(f(b_n) - \tfrac{1}{m}\,,\ f(b_n) + \tfrac{1}{m}\right),$$
$$g(U_{a_n}^m) \subset \left(g(a_n) - \tfrac{1}{m}\,,\ g(a_n) + \tfrac{1}{m}\right),$$
$$g(U_{b_n}^m) \subset \left(g(b_n) - \tfrac{1}{m}\,,\ g(b_n) + \tfrac{1}{m}\right).$$

Since topology $\mathscr{T}$ is compatible with the natural topology of the real line relative to the family $\{f,g\}$, then from the above inclusions we may infer that there exist sequences $\{V_m^{(n)}\}_{m=1}^{\infty}$ and $\{W_m^{(n)}\}_{m=1}^{\infty}$ such that $V_m^{(n)}$, $W_m^{(n)}$ are open in the natural topology (for $m = 1, 2, \dots$) and

$$\begin{gathered} V_m^{(n)} \subset (a_n, b_n) \cap \big(a_n,\, a_n + \tfrac{1}{m}\big)\,,\ W_m^{(n)} \subset (a_n, b_n) \cap \big(b_n - \tfrac{1}{m}, b_n\big)\,, \\ f(V_m^{(n)}) \subset \big(f(a_n) - \tfrac{1}{m}\,,\ f(a_n) + \tfrac{1}{m}\big)\,,\ f(W_m^{(n)}) \subset \big(f(b_n) - \tfrac{1}{m}\,,\ f(b_n) + \tfrac{1}{m}\big)\,, \\ g(V_m^{(n)}) \subset \big(g(a_n) - \tfrac{1}{m}\,,\ g(a_n) + \tfrac{1}{m}\big)\,,\ g(W_m^{(n)}) \subset \big(g(b_n) - \tfrac{1}{m}\,,\ g(b_n) + \tfrac{1}{m}\big)\,. \end{gathered} \tag{10}$$

Fix $x_0 \in D$.

For an arbitrary m there exists $\delta_m \in \big(0, \frac{1}{m}\big)$ such that

$$\begin{aligned} f\big((x_0 - \delta_m, x_0 + \delta_m) \cap D\big) \subset \big(f(x_0) - \tfrac{1}{m}\,,\ f(x_0) + \tfrac{1}{m}\big)\,, \\ g\big((x_0 - \delta_m, x_0 + \delta_m) \cap D\big) \subset \big(g(x_0) - \tfrac{1}{m}\,,\ g(x_0) + \tfrac{1}{m}\big) \end{aligned} \tag{11}$$

(we may assume that $\delta_{m_1} < \delta_{m_2}$, for $m_1 > m_2$).

Put

$$\begin{aligned} T_m(x_0) = {} & \big[(x_0 - \delta_m,\, x_0 + \delta_m) \cap D\big] \\ & \cup \left[\left(\bigcup_n \bigcup_{k \ge m} (V_k^{(n)} \cup W_k^{(n)})\right) \cap \big(x_0 - \delta_m\,,\ x_0 + \delta_m\big)\right]. \end{aligned}$$

Observe that

$$T_{m_1}(x_0) \subset T_{m_2}(x_0)\,, \qquad \text{for} \quad m_1 > m_2\,. \tag{12}$$

In the sequel $B(x)$, for $x \notin D$, denotes a base at x in the natural topology of the real line. Let us define a interior operator $\mathscr{O} - \operatorname{int}$ in the following way:

$$t \in \mathscr{O} - \operatorname{int}(A) \iff \begin{cases} \exists_m\, T_m(t) \subset A\,, & \text{if } t \in D\,, \\ \exists_{U \in B(t)}\, U \subset A\,, & \text{if } t \notin D\,, \end{cases}$$

for an arbitrary set $A \subset \mathbb{R}$.

An easy verification shows that by the above defined operation we may generate the topology $\mathscr{O} = \{A \subset \mathbb{R} \colon A = \mathscr{O} - \operatorname{int}(A)\}$ (see, e.g., [7]). Let Φ be a complete system of all continuous functions $\varphi \colon (\mathbb{R}, \mathscr{O}) \to \mathbb{R}$. It is easy to check that $\mathscr{O}$ is finer than the natural topology of the real line. So, $\mathcal{C} \subset \Phi$.

From (10) and (11) we deduce that $f, g \in \Phi$.

Now, we shall show that

$$\Phi \subset DB_1^{**}.$$

In fact, let $\varphi \in \Phi$. In the sequel we shall consider the function φ as a function mapping real line (with natural topology) in the real line. It is not difficult to verify that $D_\varphi \subset D$. From the definition of T_m we conclude that $\varphi \restriction D$ is a continuous function and so $\varphi \restriction D_\varphi$ is continuous and consequently $\varphi \in B_1^{**}$. By (10) and the definition of the topology $\mathscr{O}$ (and of course, according to the nowhere density of D) we infer that φ is a bilateral quasi-continuous function. This fact, by virtue of the previous theorem, means that the inclusion $\Phi \subset DB_1^{**}$ is true. $\square$

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