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ON SUP-MEASURABILITY OF MULTIVALUED FUNCTIONS WITH THE (Z) PROPERTY IN SECOND VARIABLE

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ABSTRACT. The sup-measurability of multivalued function F of two variables in metric spaces with differentiation basis is studied. It is shown that if all x-sections of F have (Z) property and all y-sections of F are measurable, then F is sup-measurable.

1. Introduction

Roughly speaking, sup-measurability means measururability of Carathéodory superposition $H(x) = F\big(x, G(x)\big) = \bigcup_{y \in G(x)} F(x, y)$, where $F : X \times Y \to Z$ is a multivalued function and $G : X \to Y$ is a measurable multivalued function. In the single valued version the problem of sup-measurability has been studied very substantially for the last 40 years, most of all by Z. Grande (some exposition may be found in [6]). However, there is no much knowledge about the multivalued case.

The problem of sup-measurability was considered by Carathéodory in his book [2] for the first time. He proved that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $f_x(\cdot) = f(x, \cdot)$ is continuous for every $x \in \mathbb{R}$ and $f^y(\cdot) = f(\cdot, y)$ is measurable for every $y \in \mathbb{R}$, then f is sup-measurable, i.e., the function $h(x) = f\big(x, g(x)\big)$ is measurable for any measurable function $g : \mathbb{R} \to \mathbb{R}$. Several results on sup-measurability of real functions are given by Grande in [5] and [6]. Certain condition of sup-measurability of functions in abstract spaces has been presented by Shragin [15]. The purpose of this paper is to prove some new sup-measurability result concerning the multivalued functions.

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Let $\phi : [0,1] \to \mathbb{R}$ be a quasi-continuous function that is not measurable (see S. Marcus [10]). Then the function $f(x,y) = \phi(y)$ is not sup-measurable, since the function $h(x) = f\big(x, g(x)\big)$, where $g(x) = x$, is not measurable. But the functions f_x are quasi-continuous for all x and the functions f^y are constant for all y. Thus the continuity of f_x in Carathéodory theorem cannot be replaced with quasi-continuity. We will define some property, termed (Z) property of multivalued functions (introduced earlier in [5] for real functions of real variable), stronger than quasi-continuity but more general than continuity, and we will show that this property assumed instead of the continuity in Carathéodory theorem assures sup-measurability of multivalued functions.

2. Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively. Let $\mathcal{L}(\mathbb{R})$ denote the σ-field of Lebesgue measurable subsets of $\mathbb{R}$ (in case of need the Lebesgue measure m on $\mathcal{L}(\mathbb{R})$ will be understood).

Let S and Z be two arbitrary nonempty sets. A multivalued function Φ from S to Z is such a mapping that $\Phi(s)$ is a non-empty subset of Z for each $s \in S$. We will write simply $\Phi : S \to Z$ is a multivalued function.

If $\phi : S \to Z$ is a point valued function such that $\phi(s) \in \Phi(s)$ for every $s \in S$, then ϕ is called a selection of Φ.

If $\Phi : S \to Z$ is a multivalued function, then two inverse images of subset $G \subset Z$ may be defined:

$$\Phi^+(G) = \big\{s \in S : \Phi(s) \subset G\big\}, \quad \text{and} \quad \Phi^-(G) = \big\{s \in S : \Phi(s) \cap G \neq \emptyset\big\}.$$

The following relations between these inverse images are true:

(i) $\Phi^-(G) = S \setminus \Phi^+(Z \setminus G)$ and $\Phi^+(G) = S \setminus \Phi^-(Z \setminus G)$.

(ii) If $\mathcal{I}$ is a set of indexes and $B_i \subset Z$ for $i \in \mathcal{I}$, then $\Phi^-\big(\bigcup_{i\in\mathcal{I}} B_i\big) = \bigcup_{i\in\mathcal{I}} \Phi^-(B_i)$.

Let $\big(S, \mathcal{T}(S)\big)$ and $\big(Z, \mathcal{T}(Z)\big)$ be topological spaces. If $s_0 \in S$, then we will use $\mathcal{B}(s_0)$ to denote the filterbase of open neighbourhoods of s_0. The family $\mathcal{B}(S)$ will denote the σ-field of Borel subsets of S. The closure of A will be denoted by $\mathrm{Cl}(A)$ and the interior of A by $\mathrm{Int}(A)$.

A multivalued function $\Phi : S \to Z$ is called *lower* (*upper*) *semicontinuous* at a point $s_0 \in S$ if, for each set $G \in \mathcal{T}(Z)$ such that $\Phi(s_0) \cap G \neq \emptyset$ $\big(\Phi(s_0) \subset G\big)$, there is a $U(s_0) \in \mathcal{B}(s_0)$ such that $\Phi(s) \cap G \neq \emptyset$ (resp. $\Phi^+(s) \subset G$) for each $s \in U(s_0)$.

We say that Φ is *lower* (*upper*) *semicontinuous* if it is lower (upper) semicontinuous at each point $s \in S$.

Φ is called *continuous* if it is both lower and upper semicontinuous.

The quasi-continuity introduced by K e m p i s t y [8] for a real function has been intensively studied. For multivalued functions this notion was introduced by P o p a [14] and widely considered by many authors, particularly by N e u b r u n n [11], [12] and E w e r t [4].

Let $(S, \mathcal{T}(S))$ and $(Z, \mathcal{T}(Z))$ be topological spaces. Following N e u b r u n n [11] we say that a multivalued function $\Phi : S \to Z$ is *lower* (*upper*) *quasi-continuous* at a point $s_0 \in S$ if, for every set $G \in \mathcal{T}(Z)$ such that $s_0 \in \Phi^-(G)$ $(s_0 \in \Phi^+(G))$ and for every set $U \in \mathcal{B}(s_0)$, there exists a nonempty open set $V \subset U$ such that $V \subset \Phi^-(G)$ (resp. $V \subset \Phi^+(G)$); Φ is said to be lower (upper) quasi-continuous if it is lower (upper) quasi-continuous at every point $s \in S$.

Note that in the case of single valued functions the notions of lower quasi-continuity and upper quasi-continuity coincide with quasi-continuity.

It is clear that every lower (upper) semicontinuous multivalued function is lower (upper) quasi-continuous.

A multivalued function $\Phi : S \to Z$ is said to be *quasi-continuous* at a point $s_0 \in S$ if, for arbitrary open sets $G \subset Z$ and $H \subset Z$ such that $s_0 \in \Phi^-(G) \cap \Phi^+(H)$ and for every set $U \in \mathcal{B}(s_0)$, there exists a nonempty open set $V \subset U$ such that $V \subset \Phi^-(G) \cap \Phi^+(H)$.

It is evident that a quasi-continuous multivalued function is both lower and upper quasi-continuous. The converse is not true (see [12, Example 1.2.7]).

A set $A \subset S$ is said to be *quasi-open* if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ [9]. It is known (see [12, 1.2.5]) that

(1) A multivalued function $\Phi : S \to Z$ is lower (upper) quasi-continuous if and only if for every set $G \in \mathcal{T}(Z)$ the inverse image $\Phi^-(G)$ $(\Phi^+(G))$ is quasi-open.

Let $(S, \mathcal{M}(S))$ be a measurable space and $(Z, \mathcal{T}(Z))$ a topological space. A multivalued function $\Phi : S \to Z$ is called $\mathcal{M}(S)$-*measurable* if the inverse image $\Phi^-(G) \in \mathcal{M}(S)$ for each set $G \in \mathcal{T}(Z)$.

There are various equivalent conditions of $\mathcal{M}(S)$-measurability of multivalued functions, particularly in metric spaces.

Let (Z, d) be a metric space. If a number $r > 0$ and $z_0 \in Z$ are given, then $B(z_0, r)$ will denote the open ball centered at z_0 with radius r. The diameter of $A \subset Z$ will be denoted by $\mathrm{diam}(A)$.

For a fixed point $z \in Z$ and a multivalued function $\Phi : S \to Z$ we define the function $g_z : S \to \mathbb{R}$ by

$$g_z(s) = d(z, \Phi(s)).$$

Let us consider the following properties:

(a) g_z is $\mathcal{M}(S)$-measurable for each $z \in Z$;

(b) Φ admits a sequence of $\mathcal{M}(S)$-measurable selections $(\phi_n)_{n\in\mathbb{N}}$ such that $\Phi(s) = \mathrm{Cl}\big(\{\phi_n(s) : n \in \mathbb{N}\}\big)$ for each $s \in S$.

It is known that

(2) If (Z, d) is separable, then
   (i) $\mathcal{M}(S)$-measurability of multivalued function $\Phi : S \to Z$ is equivalent to (a) [7, Th. 3.3].
   (ii) If Φ is complete valued, then $\mathcal{M}(S)$-measurability of Φ is equivalent to (b) [3, Th. III.9].

Now, we assume that $\big(S, \varrho, \mathcal{M}(S), \mu\big)$ is a measure metric space with a metric ϱ, with a σ-finite regular and complete measure μ defined on a σ-field $\mathcal{M}(S)$ of subsets of S containing $\mathcal{B}(S)$. Let μ^* be an outer measure corresponding to μ.

(3) Let $\mathcal{F}(S) \subset \mathcal{M}(S)$ be a family of μ-measurable sets with nonempty interiors of positive and finite measure μ, the boundaries of which are of μ-measure zero.

   Let $\{I_n\}_{n\in N} \subset \mathcal{F}(S)$ and $s \in S$. We take $I_n \to s$ to mean that $s \in \mathrm{Int}(I_n)$ for $n \in N$ and the sequence $\big(\mathrm{diam}(I_n)\big)_{n\in\mathbb{N}}$ converges to zero if n approaches the infinity . We assume that the family $\mathcal{F}(S)$ is countable and for every $s \in S$ there exists a sequence $(I_n)_{n\in N}$ such that $I_n \in \mathcal{F}(S)$ for $n \in \mathbb{N}$ and $I_n \to s$. Then the pair $\big(\mathcal{F}(S), \to\big)$ forms a differentiation basis of the space $\big(S, \varrho, \mathcal{M}(S), \mu\big)$, in accordance with Bruckner's terminology (see [1], p. 30).

Now, we assume that $\big(\mathcal{F}(S), \to\big)$ is a differentiation basis in the space $(S, \varrho, \mathcal{M}(S), \mu)$. E. Grande and Z. Grande in [5] introduced (Z) property of a set $A \subset \mathbb{R}$ and (Z_1) property of a real function of real variable. Now, we give some generalization of these notions for the case of a set $A \subset S$ and a multivalued function from the space S to a topological space, respectively.

Definition 1. A set $A \subset S$ has (Z) property with respect to $\mathcal{F}(S)$ if, for each $s \in A$, there is an open set $G \subset A$ and a number $\delta > 0$ such that

$$\mu(G \cap I) > \frac{1}{2}\,\mu(I),$$

for every set $I \in \mathcal{F}(S)$ such that $s \in \mathrm{Int}(I)$ and $\mathrm{diam}(I) < \delta$.

We claim that

(4) If a set $A \subset S$ has (Z) property with respect to $\mathcal{F}(S)$, then A is quasi-open.

Indeed, let us suppose that A has (Z) property with respect to $\mathcal{F}(S)$ and let $s \in A$. Let $U(s)$ be an open neighbourhood of s. Let $I \in \mathcal{F}(S)$, $s \in \mathrm{Int}(I) \subset U(s)$ and $\mathrm{diam}(I) < \delta$ (δ is given by (Z) property of the set A). Then

$$\mu\big(\mathrm{Int}(I) \cap \mathrm{Int}(A)\big) > \frac{1}{2}\,\mu\big(\mathrm{Int}(I)\big) > 0.$$

Hence there is an $s' \in \mathrm{Int}(I) \cap \mathrm{Int}(A) \subset U(s) \cap \mathrm{Int}(A) \neq \emptyset$, and the set A is quasi-open.

But if we put $S = \mathbb{R}$ with $\mathcal{L}(\mathbb{R})$ and m, and if $\mathcal{F}(S)$ is the family of intervals with nonempty interiors and rational end-points, then any closed interval is a quasi-open set, however, it does not have (Z) property with respect to $\mathcal{F}(S)$.

Let $\big(Z, \mathcal{T}(Z)\big)$ be a topological space.

Definition 2. We say that a multivalued function $\Phi : S \to Z$ has the lower (upper) (Z) property with respect to $\mathcal{F}(S)$ if the inverse image $\Phi^-(G)$ $(\Phi^+(G))$ has the (Z) property with respect to $\mathcal{F}(S)$ for each set $G \in \mathcal{T}(Z)$; Φ has (Z) property with respect to $\mathcal{F}(S)$ if it has both the lower and upper (Z) property with respect to $\mathcal{F}(S)$.

Note that by (4) and (1) we have the following corollary.

Corollary 1. *If a multivalued function $\Phi : S \to Z$ has the lower (upper) (Z) property with respect to $\mathcal{F}(S)$, then it is lower (upper) quasi-continuous. If Φ has (Z) property, then it is lower and upper quasi-continuous.*

Let X and Y be nonempty sets and let $F : X \times Y \to Z$ be a multivalued function. If $x \in X$ is fixed, then the multivalued function $F_x : Y \to Z$ given by formula $F_x(y) = F(x, y)$ is called x-section of F. Analogously, for fixed $y \in Y$, the y-section F^y of F is defined.

Now, let $\big(X, \mathcal{M}(X)\big)$ be a measurable space, $\big(Y, \mathcal{T}(Y)\big)$ and $\big(Z, \mathcal{T}(Z)\big)$ topological spaces.

A multivalued function $F : X \times Y \to Z$ is called $\mathcal{M}(X)$-*sup-measurable* if for each $\mathcal{M}(X)$-measurable multivalued function $G : X \to Y$ with closed values the Carathéodory superposition $H : X \to Z$ given by the formula

$$H(x) = F\big(x, G(x)\big) = \bigcup_{y \in G(x)} F(x, y)$$

is a $\mathcal{M}(X)$-measurable multivalued function.

It is known (see [13], [16] and [17]) that on the suitable assumption of the spaces X, Y and Z, a multivalued function $F : X \times Y \to Z$ with compact values such that its y-sections are $\mathcal{M}(X)$-measurable for all $y \in Y$ and its x-sections are continuous for all $x \in X$, is $\mathcal{M}(X)$-sup-measurable (all these results are some generalizations of Carathéodory theorem to the case of multivalued functions in some abstract spaces).

3. Main results

In the remainder of this section we assume $\big(X, \mathcal{M}(X), \mu\big)$ to be a measure space and $\big(Y, \rho, \mathcal{M}(Y), \nu\big)$ a measure metric spaces, where ν is σ-finite regular complete and $\mathcal{B}(Y) \subset \mathcal{M}(Y)$, with the differentiation basis $\big(\mathcal{F}(Y), \to\big)$ defined in the same way as in (3). We begin with some result on sup-measurability of real functions.

Theorem 1. *If $f : X \times Y \to \mathbb{R}$ is such a function that for each $y \in Y$ its y-section is $\mathcal{M}(X)$-measurable and for each $x \in X$ the sets $f_x^{-1}((-\infty, a))$ and $f_x^{-1}((a, \infty))$ have (Z) property with respect to $\mathcal{F}(Y)$ for every $a \in \mathbb{R}$, then the function f is $\mathcal{M}(X)$-sup-measurable.*

P r o o f. Suppose, contrary to our claim, that f is not $\mathcal{M}(X)$-sup-measurable. Then we could find an $\mathcal{M}(X)$-measurable function $g : X \to Y$ such that the Carathéodory's superposition $h(x) = f\big(x, g(x)\big)$ is not $\mathcal{M}(X)$-measurable. Then there is an $a \in \mathbb{R}$ such that the set $h^{-1}((-\infty, a)) \notin \mathcal{M}(X)$. Let S be the union of all sets $M \in \mathcal{M}(X)$ with $\mu(M) > 0$ such that

$$\mu\big(h^{-1}((-\infty, a)) \cap M\big) = 0 \quad \text{or} \quad \mu\Big(h^{-1}\big([a, \infty)\big)\Big) \cap M\big) = 0,$$

and let $T = X \setminus S$. Since $h^{-1}((-\infty, a)) \notin \mathcal{M}(X)$, there is an $n_0 \in \mathbb{N}$ such that

$$\mu^*\left(T \cap h^{-1}\left(\left(-\infty, a - \frac{1}{n_0}\right)\right)\right) > 0 \quad \text{and} \quad \mu^*\left(T \cap h^{-1}\left(\left(a - \frac{1}{2n_0}, \infty\right)\right)\right) > 0.$$

Thus

(5) there is an $n_0 \in \mathbb{N}$ and a set $A \in \mathcal{M}(X)$ such that $\mu(A) > 0$ and for every set $B \in \mathcal{M}(X)$ with $B \subset A$ and $\mu(B) > 0$ we have

$$B \cap h^{-1}\left(\left(-\infty, a - \frac{1}{n_0}\right)\right) \neq \emptyset \quad \text{and} \quad \left(B \cap h^{-1}\left(\left(a - \frac{1}{2n_0}, \infty\right)\right)\right) \neq \emptyset.$$

Let $x \in A \cap h^{-1}\big((-\infty, a - \frac{1}{n_0})\big)$. Then $g(x) \in f_x^{-1}\big((-\infty, a - \frac{1}{n_0})\big)$. By (Z) property of the set $f_x^{-1}\big((-\infty, a - \frac{1}{n_0})\big)$ with respect to $\mathcal{F}(Y)$, for $g(x)$ there is an open set $G \subset f_x^{-1}\big((-\infty, a - \frac{1}{n_0})\big)$ and a $\delta > 0$ such that $\nu(G \cap J) > \frac{1}{2}\,\nu(J)$ whenever $J \in \mathcal{F}(Y)$ is such a set that $g(x) \in \operatorname{Int}(J)$ and $\operatorname{diam}(J) < \delta$. Then there is an $I(x) \in \mathcal{F}(Y)$ such that $g(x) \in \operatorname{Int}\big(I(x)\big)$, $\operatorname{diam}\big(I(x)\big) < \delta$ and $\nu\big(G \cap I(x)\big) > \frac{1}{2}\,\nu\big(I(x)\big)$. Thus, if $x \in A \cap h^{-1}\big((-\infty, a - \frac{1}{n_0})\big)$, then there is an $I(x) \in \mathcal{F}(Y)$ such that

(6) $g(x) \in \operatorname{Int}\big(I(x)\big)$ and $\nu\big(\operatorname{Int}(f_x^{-1}((-\infty, a - \frac{1}{n_0}))) \cap J\big) > \frac{1}{2}\nu(J)$

for each $J \in \mathcal{F}(Y)$ such that $g(x) \in \operatorname{Int}(J) \subset J \subset I(x)$.

Since the family $\mathcal{F}(Y)$ is countable, there is an $I \in \mathcal{F}(Y)$ for which the set

$$B = \bigl\{x \in A : I(x) = I\bigr\}$$

is of positive outer measure μ^*. Let $D = g^{-1}\bigl(\operatorname{Int}(I)\bigr)$. By $\mathcal{M}(X)$-measurability of the function g, the set $D \in \mathcal{M}(X)$. Furthermore, $D \subset A$ and $\mu(D) > 0$. Then, by (5), we have $D \cap h^{-1}\bigl((a - \frac{1}{2n_0}, \infty)\bigr) \neq \emptyset$. Let $x \in D \cap h^{-1}\bigl((a - \frac{1}{2n_0}, \infty)\bigr)$. Then $g(x) \in f_x^{-1}\bigl((a - \frac{1}{2n_0}, \infty)\bigr)$.

Applying (Z) property of the set $f_x^{-1}\bigl((a - \frac{1}{2n_0}, \infty)\bigr)$ with respect to $\mathcal{F}(Y)$, analogously as in (6), there is a set $V(x) \in \mathcal{F}(Y)$ contained in I such that

$$g(x) \in \operatorname{Int}\bigl(V(x)\bigr) \quad \text{and} \quad \nu\left(\operatorname{Int}\left(f_x^{-1}\left(\left(a - \frac{1}{2n_0}, \infty\right)\right)\right) \cap J\right) > \frac{1}{2}\,\nu(J)$$

for each $J \in \mathcal{F}(Y)$ such that $g(x) \in \operatorname{Int}(J) \subset J \subset V(x)$. Let $C \supset B$ be an $\mathcal{M}(X)$-measurable cover of B contained in D. Similarly as above, there is $V \in \mathcal{F}(Y)$ for which the set

$$E = \bigl\{x \in C : V(x) = V\bigr\}$$

is of positive outer measure μ^*.

Let G be an $\mathcal{M}(X)$-measurable cover of E. For each $x \in A \cap G$ we can take a finite system of sets $U_i(x) \in \mathcal{F}(Y)$, $i = 1, 2 \ldots, k(x)$, with disjoint interiors contained in $V \cap \operatorname{Int}\bigl(f_x^{-1}((a - \frac{1}{2n_0}, \infty))\bigr)$ such that

$$\nu\bigl(U(x) \cap V\bigr) > \frac{1}{2}\,\nu(V),$$

where $U(x) = \bigcup_{i=1,2,\ldots,k(x)} \operatorname{Int}\bigl(U_i(x)\bigr)$.

The family of finite systems of a countable family of sets from $\mathcal{F}(Y)$ is countable. Thus there is a system of sets $\{W_i\}_{i=1,2,\ldots,k} \subset \mathcal{F}(Y)$ contained in V such that the set $H = \bigl\{x \in A \cap G : U_i(x) = W_i, i = 1, 2, \ldots, k\bigr\}$ is of positive outer measure μ^*. Let $W = \bigcup_{i=1,2,\ldots,k} W_i$. Then

$$W \subset \operatorname{Int}\left(f_x^{-1}\left(\left(a - \frac{1}{2n_0}, \infty\right)\right)\right), \qquad \text{for} \quad x \in H \quad \text{and} \quad \nu(W) > \frac{1}{2}\,\nu(V).$$

The set $A \cap G \cap g^{-1}(V)$ is of positive outer measure μ^*. Analogously as above, there is a set $K \subset A \cap G$ of positive outer measure μ^* and a finite system of sets $\{Z_i\}_{i=1,2,\ldots,l} \subset \mathcal{F}(Y)$ contained in $V \bigcup_{i=1,2,\ldots,l} Z_i = Z$ such that

$$Z \subset \operatorname{Int}\left(f_x^{-1}\left(\left(-\infty, a - \frac{1}{n_0}\right)\right)\right), \qquad \text{for} \quad x \in K \quad \text{and} \quad \nu(Z) > \frac{1}{2}\,\nu(V).$$

Note that $W \cap Z \neq \emptyset$. Let $y_0 \in W \cap Z$. Then $f_x(y_0) > a - \frac{1}{2n_0}$ for $x \in H$ and $f_x(y_0) < a - \frac{1}{n_0}$ for $x \in K$.

Let $P \subset X$ be the intersection of a measurable cover of H and a measurable cover of K. Then $\mu(P) > 0$ and the restriction $(f^{y_0})_{|P}$ is not $\mathcal{M}(X)$-measurable, which contradicts $\mathcal{M}(X)$-measurability of f^{y_0} and the proof of Theorem 1 is finished. □

Corollary 2. *If $f : Y \to \mathbb{R}$ is a function such that the sets $f^{-1}((-\infty, a))$ and $f^{-1}((a, \infty))$ have (Z) property with respect to $\mathcal{F}(Y)$ for every $a \in \mathbb{R}$, then f is $\mathcal{M}(Y)$-measurable.*

Proof. If it were not true, we would not have a $\mathcal{M}(Y)$-measurable function $\phi : Y \to \mathbb{R}$ such that $\phi^{-1}((-\infty, a))$ and $\phi^{-1}((a, \infty))$ would have (Z) property with respect to $\mathcal{F}(Y)$ for every $a \in \mathbb{R}$. Then for the function $f : Y \times Y \to \mathbb{R}$ given by formula $f(x, y) = \phi(y)$ the assumption of Theorem 1 would be fulfilled. But for the function $g(y) = y$ the Carathéodory superposition $f(y, y) = g(y)$ would not be $\mathcal{M}(Y)$-measurable, a contradiction. □

From now on we suppose (Z, d) to be a separable metric space.

Proposition 1. *If a multivalued function $\Phi : Y \to Z$ has (Z) property with respect to $\mathcal{F}(Y)$, then Φ is $\mathcal{M}(Y)$-measurable.*

Proof. Let $z \in Z$, $g_z(y) = d\big(z, \Phi(y)\big)$ for $y \in Y$ and $r \in \mathbb{R}_+$. Then

$$g_z^{-1}((-\infty, r)) = \big\{y \in Y : \Phi(y) \cap B(z, r) \neq \emptyset\big\} = \Phi^-\big(B(z, r)\big).$$

By the lower (Z) property of Φ with respect to $\mathcal{F}(Y)$ the set $g_z^{-1}((-\infty, r))$ has (Z) property with respect to $\mathcal{F}(Y)$. On the other hand

$$g_z^{-1}((r, \infty)) = \Big\{y \in Y : \Phi(y) \subset Y \setminus \mathrm{Cl}\big(B(z, r)\big)\Big\} = \Phi^+\Big(Y \setminus \mathrm{Cl}\big(B(z, r)\big)\Big)$$

and, by the upper (Z) property of Φ with respect to $\mathcal{F}(Y)$, the set $g_z^{-1}((r, \infty))$ has (Z) property with respect to $\mathcal{F}(Y)$. Then the function g_z is $\mathcal{M}(Y)$-measurable (see Corollary 2) and, by (2) (i), Φ is $\mathcal{M}(Y)$-measurable. □

Theorem 2. *Let (Y, ρ) be a Polish space. If $F : X \times Y \to Z$ is a closed valued multivalued function such that all its y-sections are $\mathcal{M}(X)$-measurable and all its x-sections have the (Z) property with respect to $\mathcal{F}(Y)$, then F is $\mathcal{M}(X)$-sup-measurable.*

Proof. Let $z \in Z$, $g_z(x, y) = d\big(z, F(x, y)\big)$ and $r \in \mathbb{R}_+$. If $x \in X$, then (analogously as in the proof of Proposition 1) the sets

$$(g_z)_x^{-1}((-\infty, r)) \quad \text{and} \quad (g_z)_x^{-1}((r, \infty))$$

have (Z) property with respect to $\mathcal{F}(Y)$. If $y \in Y$, then $(g_z)^y$ is $\mathcal{M}(X)$-measurable by (2) (i). Thus, by Theorem 1, the function g_z is $\mathcal{M}(X)$-sup-measurable. Then the function $g_z\big(x, h(x)\big)$ is $\mathcal{M}(X)$-measurable for every $\mathcal{M}(X)$-measurable function $h : X \to Y$, and then, by (2) (i),

(7) for every $\mathcal{M}(X)$-measurable function $g : X \to Y$, the multivalued function $x \to F\big(x, g(x)\big)$ is $\mathcal{M}(X)$-measurable.

Let $G : X \to Y$ be an $\mathcal{M}(X)$-measurable multivalued function with closed values. The task now is to show that the multivalued function $H(x) = F\big(x, G(x)\big)$ is $\mathcal{M}(X)$-measurable.

By (2) (ii), we can select a sequence $(g_n)_{n\in\mathbb{N}}$ of $\mathcal{M}(X)$-measurable functions $g_n : X \to Y$, $n \in \mathbb{N}$, such that $G(x) = \mathrm{Cl}\big(\{g_n(x)\}_{n\in\mathbb{N}}\big)$.

Let $U \subset Z$ be an open set and for $n \in \mathbb{N}$ let us define

$$B_n = \Big\{x \in X : F\big(x, g_n(x)\big) \cap U \neq \emptyset\Big\}.$$

Since the functions g_n, for $n \in \mathbb{N}$, are $\mathcal{M}(X)$-measurable, by (7) we have

(8) for any $n \in \mathbb{N}$ the set $B_n \in \mathcal{M}(X)$.

Furthermore, if $x \in X$, then by (Z) property of F_x with respect to $\mathcal{F}(Y)$ we have

$$\Big\{x \in X : \mathrm{Cl}\Big(\big\{g_n(x)\big\}_{n\in\mathbb{N}}\Big) \cap F_x^-(U) \neq \emptyset\Big\} = \Big\{x \in X : \big\{g_n(x)\big\}_{n\in\mathbb{N}} \cap F_x^-(U) \neq \emptyset\Big\}.$$

Observe that

$$\begin{aligned} H^-(U) &= \Big\{x \in X : F\big(x, G(x)\big) \cap U \neq \emptyset\Big\} \\ &= \Bigg\{x \in X : \bigcup_{y\in G(x)} F(x, y) \cap U \neq \emptyset\Bigg\} \\ &= \big\{x \in X : \exists\, y \in G(x) \wedge F(x, y) \cap U \neq \emptyset\big\} \\ &= \big\{x \in X : G(x) \cap F_x^-(U) \neq \emptyset\big\} \\ &= \Big\{x \in X : \mathrm{Cl}\Big(\big\{g_n(x)\big\}_{n\in\mathbb{N}}\Big) \cap F_x^-(U) \neq \emptyset\Big\} \\ &= \Big\{x \in X : \big\{g_n(x)\big\}_{n\in\mathbb{N}} \cap F_x^-(U) \neq \emptyset\Big\}. \end{aligned}$$

Therefore

$$H^-(U) = \bigcup_{n\in\mathbb{N}} \Big\{x \in X : F\big(x, g_n(x)\big) \cap U \neq \emptyset\Big\} = \bigcup_{n\in\mathbb{N}} B_n$$

and, by (8), $H^{-}(U) \in \mathcal{M}(X)$, which finishes the proof of Theorem 2. $\square$

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