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ALGEBRAIC PROPERTIES OF THE CLASSES $\mathcal{M} \cap \mathcal{B}_\alpha$

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ABSTRACT. The sums and the maximums of functions from the class $\mathcal{M} \cap \mathcal{B}_\alpha$, for each countable ordinal α are characterized. Moreover, the maximal additive class and the maximal class with respect to maximums for these classes of functions are characterized as well.

1. Preliminaries

The letters $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ stand for the set of positive integers, the set of all integers, and the real line, respectively. If $A \subset \mathbb{R}$, then $\operatorname{int} A$, $\operatorname{bd} A$, and χ_A denote the (Euclidean) interior, the boundary, and the characteristic function of the set A, respectively. The word *function* denotes a real function defined on $\mathbb{R}$. The symbol sgn denotes the sign function; i.e., $\operatorname{sgn} x = |x|/x$ if $x \neq 0$, and $\operatorname{sgn} 0 = 0$.

For each countable ordinal α, let $\mathcal{B}_\alpha$ denote the α^{th} *Baire class*; i.e., $\mathcal{B}_0$ stands for the family of all continuous functions, and for each $\alpha > 0$,

$$\mathcal{B}_\alpha \stackrel{\text{df}}{=} \Big\{ f \in \mathbb{R}^\mathbb{R} \colon \big(\exists (f_n) \subset \textstyle\bigcup_{\beta<\alpha} \mathcal{B}_\beta\big)\ f = \lim_{n\to\infty} f_n \Big\}.$$

(We require the above sequence to be pointwise convergent.)

Let $f \colon \mathbb{R} \to \mathbb{R}$. For each $a \in \mathbb{R}$ we define $[f = a] \stackrel{\text{df}}{=} \big\{x \in \mathbb{R} \colon f(x) = a\big\}$. Similarly we define the sets $[f > a]$, $[f < a]$, etc. We say that f is *Darboux*, if it has the intermediate value property; i.e.,

$$(\forall a < b)(\forall y \in \mathbb{R})\ \Big(\big(f(a) - y\big)\big(f(b) - y\big) < 0 \Rightarrow (a, b) \cap [f = y] \neq \emptyset\Big).$$

We will denote the class of all Darboux functions by $\mathcal{D}$.

Following J. G. Ceder [3], we define $\mathcal{M}$ as the family of all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$(\forall a < b)\ \big(f(a)f(b) < 0 \Rightarrow (a, b) \cap [f = 0] \neq \emptyset\big).$$

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The goal of this paper is to examine basic algebraic properties of the class $\mathcal{M} \cap \mathcal{B}_\alpha$, for each countable ordinal α.

2. Addition

THEOREM 2.1. *Let $f: \mathbb{R} \to \mathbb{R}$. For each ordinal $\alpha > 1$, the following are equivalent*:

a) *there are functions $g, h \in \mathcal{M} \cap \mathcal{B}_\alpha$ such that $f = g + h$,*

b) *there are functions $g \in \mathcal{D} \cap \mathcal{B}_\alpha$ and $h \in \mathcal{D} \cap \mathcal{B}_2$ such that $f = g + h$,*

c) *$f \in \mathcal{B}_\alpha$.*

Proof. The implication c) $\Rightarrow$ b) follows by [4, Theorem II.1.2], and the implications b) $\Rightarrow$ a) and a) $\Rightarrow$ c) are obvious. $\square$

The proof of the next theorem is analogous. (We use [2, Theorem B] instead of [4, Theorem II.1.2].)

THEOREM 2.2. *Let $f: \mathbb{R} \to \mathbb{R}$. The following are equivalent*:

a) *there are functions $g, h \in \mathcal{M} \cap \mathcal{B}_1$ such that $f = g + h$,*

b) *$f \in \mathcal{B}_1$.*

Recall that the *maximal additive class* for a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is defined as follows:

$$\mathcal{M}_{\mathrm{a}}(\mathcal{F}) \stackrel{\mathrm{df}}{=} \Big\{ f \in \mathbb{R}^{\mathbb{R}} \colon (\forall g \in \mathcal{F})\ f + g \in \mathcal{F} \Big\}.$$

It turns out that the maximal additive class for each family $\mathcal{M} \cap \mathcal{B}_\alpha$ is the smallest possible one.

THEOREM 2.3. *For each ordinal $\alpha > 0$, we have $\mathcal{M}_{\mathrm{a}}(\mathcal{M} \cap \mathcal{B}_\alpha) = \{\chi_\emptyset\}$.*

Proof. For each $g \in \mathcal{M} \cap \mathcal{B}_\alpha$, we have $g + \chi_\emptyset = g \in \mathcal{M} \cap \mathcal{B}_\alpha$. So, $\chi_\emptyset \in \mathcal{M}_{\mathrm{a}}(\mathcal{M} \cap \mathcal{B}_\alpha)$.

Now, let $f \neq \chi_\emptyset$. Then $f(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. We consider three cases.

Case 1. If x_0 is a point of continuity of f, then there is a $\delta > 0$ such that

$$|f(x) - f(x_0)| < |f(x_0)| \qquad \text{whenever} \qquad |x - x_0| \le \delta.$$

Define $g(x) \stackrel{\mathrm{df}}{=} 2|f(x_0)| \cdot \operatorname{sgn}(x - x_0)$. Then clearly $g \in \mathcal{M} \cap \mathcal{B}_1 \subset \mathcal{M} \cap \mathcal{B}_\alpha$. Moreover,

$$\begin{aligned}(f+g)(x_0 - \delta) &< \big(f(x_0) + |f(x_0)|\big) - 2|f(x_0)| \le 0 \\ &\le \big(f(x_0) - |f(x_0)|\big) + 2|f(x_0)| < (f+g)(x_0 + \delta)\end{aligned}$$

and $f + g \neq 0$ on $[x - \delta, x + \delta]$. Consequently, $f + g \notin \mathcal{M}$ and $f \notin \mathcal{M}_{\mathrm{a}}(\mathcal{M} \cap \mathcal{B}_\alpha)$.

Case 2. If $f \notin \mathcal{D}$, then there exist $a < b$ and $y \in \mathbb{R}$ such that

$$\big(f(a) - y\big)\big(f(b) - y\big) < 0 \quad \text{and} \quad (a, b) \cap [f = y] = \emptyset.$$

Define $g \stackrel{\text{df}}{=} -y\chi_{\mathbb{R}}$. Then g is continuous (so $g \in \mathcal{M}$), $(f + g)(a)(f + g)(b) < 0$ and $(a, b) \cap [f + g = 0] = \emptyset$. So, $f + g \notin \mathcal{M}$ and $f \notin \mathcal{M}_{\mathrm{a}}(\mathcal{M} \cap \mathcal{B}_\alpha)$.

Case 3. Finally, assume that f is Darboux and discontinuous at x_0. Using [6] (if $\alpha > 1$) or [1, Theorem 3.2, p. 14] (if $\alpha = 1$) we can find a function $g \in \mathcal{D} \cap \mathcal{B}_\alpha$ such that $f + g \notin \mathcal{D}$. Proceeding as in Case 2., we can find a $y \in \mathbb{R}$ such that $f + g - y\chi_{\mathbb{R}} \notin \mathcal{M}$. Clearly $g - y\chi_{\mathbb{R}} \in \mathcal{D} \cap \mathcal{B}_\alpha \subset \mathcal{M} \cap \mathcal{B}_\alpha$. It follows that $f \notin \mathcal{M}_{\mathrm{a}}(\mathcal{M} \cap \mathcal{B}_\alpha)$. □

3. Multiplication

Theorem 3.1. *For each ordinal α, the class $\mathcal{M} \cap \mathcal{B}_\alpha$ is closed with respect to multiplication.*

P r o o f. Indeed, let $f, g \in \mathcal{M} \cap \mathcal{B}_\alpha$. Then clearly $fg \in \mathcal{B}_\alpha$. J. G. C e d e r proved that $\mathcal{M}$ is the class of all finite products of Darboux functions [3]. Hence fg is also a finite product of Darboux functions, and consequently, $fg \in \mathcal{M}$. □

From the above theorem we can easily obtain the following corollaries.

Corollary 3.2. *Let $f \colon \mathbb{R} \to \mathbb{R}$. For each ordinal α, the following are equivalent*:

a) *there are functions $g, h \in \mathcal{M} \cap \mathcal{B}_\alpha$ such that $f = gh$,*
b) *$f \in \mathcal{M} \cap \mathcal{B}_\alpha$.*

Recall that the *maximal multiplicative class* for a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is defined as follows:

$$\mathcal{M}_{\mathrm{m}}(\mathcal{F}) \stackrel{\text{df}}{=} \Big\{ f \in \mathbb{R}^{\mathbb{R}} \colon (\forall g \in \mathcal{F})\ fg \in \mathcal{F} \Big\}.$$

Corollary 3.3. *For each ordinal $\alpha > 0$, we have $\mathcal{M}_{\mathrm{m}}(\mathcal{M} \cap \mathcal{B}_\alpha) = \mathcal{M} \cap \mathcal{B}_\alpha$.*

P r o o f. Indeed, if $f \in \mathcal{M}_{\mathrm{m}}(\mathcal{M} \cap \mathcal{B}_\alpha)$, then $f = f\chi_{\mathbb{R}} \in \mathcal{M} \cap \mathcal{B}_\alpha$.

On the other hand, if $f \in \mathcal{M} \cap \mathcal{B}_\alpha$, then by Theorem 3.1, $fg \in \mathcal{M} \cap \mathcal{B}_\alpha$ for each $g \in \mathcal{M} \cap \mathcal{B}_\alpha$. It follows that $f \in \mathcal{M}_{\mathrm{m}}(\mathcal{M} \cap \mathcal{B}_\alpha)$. □

4. Maximums

Theorem 4.1. *Let $f\colon \mathbb{R}\to\mathbb{R}$. For each ordinal $\alpha>0$, the following are equivalent*:

a) *there are functions $g,h\in\mathcal{M}\cap\mathcal{B}_\alpha$ such that $f=\max\{g,h\}$,*

b) *$f\in\mathcal{B}_\alpha$ and*

$$(\forall a,b\in\mathbb{R})\ \big(\emptyset\neq(a,b)\subset[f<0]\Rightarrow a,b\in[f\le 0]\big). \tag{1}$$

P r o o f. a) $\Rightarrow$ b). Let $g,h\in\mathcal{M}\cap\mathcal{B}_\alpha$ be such that $f=\max\{g,h\}$. Then clearly $f\in\mathcal{B}_\alpha$. Take $a<b$ such that $f<0$ on (a,b). Since $g\in\mathcal{M}$ and $g<0$ on (a,b), we conclude that $g\le 0$ on $\{a,b\}$. Analogously, $h\le 0$ on $\{a,b\}$. Consequently, $f=\max\{g,h\}\le 0$ on $\{a,b\}$.

b) $\Rightarrow$ a). Now assume that $f\in\mathcal{B}_\alpha$ and f fulfills condition (1). The proof is different in cases $\alpha>1$ and $\alpha=1$.

Case 1. $\alpha>1$. Let $\mathcal{I}$ be the family of all open intervals I with rational end points for which the intersection $I\cap[f>0]$ is infinite. Enumerate all the elements of $\mathcal{I}$ as $\{I_n\colon n<N\}$, where $N\in\mathbb{N}\cup\{\infty\}$. For each $n<N$ choose two distinct points

$$a_n,b_n\in I_n\cap[f>0]\setminus\big(\{a_k\colon k<n\}\cup\{b_k\colon k<n\}\big).$$

Put

$$A\stackrel{\text{df}}{=}\{a_n\colon n<N\},\qquad B\stackrel{\text{df}}{=}\{b_n\colon n<N\}.$$

Define

$$g\stackrel{\text{df}}{=}f-f\chi_A,\qquad h\stackrel{\text{df}}{=}f-f\chi_B.$$

Then clearly $f=\max\{g,h\}$ on $\mathbb{R}$. Moreover, since A and B are countable, we have $\chi_A,\chi_B\in\mathcal{B}_2$ and $g,h\in\mathcal{B}_\alpha$. To complete the proof we will show that $g,h\in\mathcal{M}$.

Let $a<b$ be such that $g(a)g(b)<0$. Let, e.g., $g(a)<0$. (The opposite case is analogous.) Then by definition, $f(a)=g(a)<0<g(b)=f(b)$. If there are $x\in(a,b]\cap[f>0]$ and $\delta>0$ such that $(x-\delta,x)\cap[f>0]=\emptyset$, then by (1), there is an $x_0\in(a,b)\cap(x-\delta,x)\cap[f=0]$. Since $[g=0]=[f=0]\cup A$, we conclude that $x_0\in(a,b)\cap[g=0]\neq\emptyset$.

So, assume the opposite case. Then, in particular, $(a,b)\cap[f>0]\neq\emptyset$, and there is an $n<N$ such that $I_n\subset(a,b)$. Consequently, $a_n\in(a,b)\cap[g=0]\neq\emptyset$.

We have proved that $g\in\mathcal{M}$. Analogously we can prove that $h\in\mathcal{M}$.

Case 2. $\alpha=1$. First we construct two disjoint isolated sets $A,B\subset\operatorname{int}[f>0]$ such that if

$$\operatorname{int}\big((a,b)\cap[f>0]\big)\neq\emptyset\quad\text{and}\quad[a,b]\not\subset[f>0],\qquad\text{for all}\quad a,b\in\mathbb{R}, \tag{2}$$

then

$$(a,b) \cap A \neq \emptyset \quad \text{and} \quad (a,b) \cap B \neq \emptyset.$$

Let $\{I_n : n < N\}$, where $N \in \mathbb{N} \cup \{\infty\}$, be the family of all connected components of $\operatorname{int}[f > 0]$. For each $n < N$ let $(a_{n,z})_{z \in \mathbb{Z}}$ be a strictly increasing sequence with limit points $\inf I_n$ and $\sup I_n$. Define

$$A \stackrel{\text{df}}{=} \{a_{n,2z} : n < N,\, z \in \mathbb{Z}\}, \qquad B \stackrel{\text{df}}{=} \{a_{n,2z+1} : n < N,\, z \in \mathbb{Z}\}.$$

One can easily verify that the sets A and B fulfill condition (2).

Define

$$g \stackrel{\text{df}}{=} f - f\chi_A, \qquad h \stackrel{\text{df}}{=} f - f\chi_B.$$

Then clearly $f = \max\{g, h\}$ on $\mathbb{R}$. Moreover, since both A and B are F_σ and G_δ sets, we have $\chi_A, \chi_B \in \mathcal{B}_1$ and $g, h \in \mathcal{B}_1$. To complete the proof we will show that $g, h \in \mathcal{M}$.

Let $a < b$ be such that $g(a)g(b) < 0$. Let, e.g., $g(a) < 0$. (The other case is analogous.) Then $f(a) = g(a) < 0 < g(b) = f(b)$. If $\operatorname{int}\big((a,b) \cap [f > 0]\big) \neq \emptyset$, then by (2),

$$(a,b) \cap [g = 0] \supset (a,b) \cap A \neq \emptyset.$$

So, assume that

$$\operatorname{int}\big((a,b) \cap [f > 0]\big) = \emptyset. \tag{3}$$

Put

$$P \stackrel{\text{df}}{=} \operatorname{bd}\big([a,b] \cap [f < 0]\big) \neq \emptyset,$$

and let $x_0 \in P$ be a continuity point of $f \restriction P$. Since $(a,b) \not\subset [f > 0]$, we can assume that $x_0 > a$.

If $f(x_0) > 0$, then there is an open interval $I \ni x_0$ such that $f > 0$ on $I \cap P$. Since $f(a) < 0$, using (3) we can find an $x_1 \in [a,b] \cap I \cap P$ which is isolated from the left in $[a,b]$. But then $x_1 \in [f > 0]$ and x_1 is an end point of some connected component of $\operatorname{int}[f < 0]$, contrary to (1).

If $f(x_0) < 0$, then there is an open interval $I \ni x$ such that $f < 0$ on $I \cap P$. By (3), we conclude that $x_0 \in (a,b) \cap I \subset \operatorname{int}[f < 0]$ and $x_0 \notin \operatorname{bd} I$, a contradiction.

It follows that $f(x_0) = 0$. Since $[g = 0] = [f = 0] \cup A$, we conclude that $x_0 \in (a,b) \cap [g = 0] \neq \emptyset$.

We have proved that $g \in \mathcal{M}$. Analogously we can prove that $h \in \mathcal{M}$. □

For any family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, we define $\mathcal{L}(\mathcal{F})$ to be the smallest lattice of functions (i.e., a family of functions which is closed both with respect to maximums and with respect to minimums) containing $\mathcal{F}$.

Theorem 4.2. *For each ordinal α, we have $\mathcal{L}(\mathcal{M} \cap \mathcal{B}_\alpha) = \mathcal{B}_\alpha$.*

P r o o f. By [5, Theorem 1], we have $\mathcal{L}(\mathcal{D}\cap\mathcal{B}_\alpha)=\mathcal{B}_\alpha$. Since we clearly have

$$\mathcal{L}(\mathcal{D}\cap\mathcal{B}_\alpha)\subset\mathcal{L}(\mathcal{M}\cap\mathcal{B}_\alpha)\subset\mathcal{L}(\mathcal{B}_\alpha)=\mathcal{B}_\alpha,$$

our theorem follows. □

Recall that the *maximal class with respect to maximums* for a family $\mathcal{F}\subset\mathbb{R}^\mathbb{R}$ is defined as follows:

$$\mathcal{M}_{\max}(\mathcal{F})\overset{\text{df}}{=}\Big\{f\in\mathbb{R}^\mathbb{R}\colon(\forall g\in\mathcal{F})\ \max\{f,g\}\in\mathcal{F}\Big\}.$$

Theorem 4.3. *For each ordinal α, $\mathcal{M}_{\max}(\mathcal{M}\cap\mathcal{B}_\alpha)$ is the family of all functions $f\in\mathcal{M}\cap\mathcal{B}_\alpha$ such that*

$$[f<0]\subset\operatorname{int}[f\le 0].\tag{4}$$

P r o o f. '$\subset$'. First, assume that $f\in\mathcal{M}_{\max}(\mathcal{M}\cap\mathcal{B}_\alpha)$. Then $\max\{f,y\chi_\mathbb{R}\}\in\mathcal{B}_\alpha$ for each $y\in\mathbb{R}$. Consequently, since for each $y\in\mathbb{R}$, we have

$$\begin{aligned}[f>y]\ &=\big[\max\{f,y\chi_\mathbb{R}\}>y\big],\\ [f<y]\ &=\Big[\max\big\{f,(y-1)\chi_\mathbb{R}\big\}<y\Big],\end{aligned}$$

we can conclude that $f\in\mathcal{B}_\alpha$.

Now, let $a<b$ be such that $f(a)f(b)<0$. Let, e.g., $f(a)<0$. Then since $\max\big\{f,f(a)\chi_\mathbb{R}\big\}\in\mathcal{M}$, there is an $x_0\in(a,b)\cap\big[\max\{f,f(a)\chi_\mathbb{R}\}=0\big]$. Hence $f(x_0)=0$ and $f\in\mathcal{M}$.

Suppose that there is a sequence $(x_n)\subset[f>0]$ such that $x_n\searrow x_0\in[f<0]$. (The case $x_n\nearrow x_0$ is analogous.) Define

$$g(x)\overset{\text{df}}{=}\begin{cases}f(x_0)&\text{if } x\le x_0,\\ 0&\text{if } x=x_n,\ n\in\mathbb{N},\\ 1&\text{otherwise.}\end{cases}$$

Then clearly $g\in\mathcal{M}\cap\mathcal{B}_1\subset\mathcal{M}\cap\mathcal{B}_\alpha$. On the other hand, $\max\{f,g\}(x_0)=f(x_0)<0$ and $\max\{f,g\}>0$ on (x_0,∞). Hence $\max\{f,g\}\notin\mathcal{M}$, contrary to the assumption $f\in\mathcal{M}_{\max}(\mathcal{M}\cap\mathcal{B}_\alpha)$.

'$\supset$'. Now, assume that $f\in\mathcal{M}\cap\mathcal{B}_\alpha$ fulfills condition (4). Let $g\in\mathcal{M}\cap\mathcal{B}_\alpha$. Clearly $h\overset{\text{df}}{=}\max\{f,g\}\in\mathcal{B}_\alpha$. We will show that $h\in\mathcal{M}$.

Let $a,b\in\mathbb{R}$ be such that $a<b$ and $h(a)h(b)<0$. Let, e.g., $h(a)<0$. Then $f(a)<0$ and $g(a)<0$. We consider the following cases.

Case 1. If $[a,b]\subset[f\le 0]$, then choose an $x_0\in[g=0]\cap(a,b)$ (we use the assumption $g\in\mathcal{M}$ and the relation $g(b)=h(b)>0$) and observe that $h(x_0)=0$.

Case 2. Put $c\overset{\text{df}}{=}\inf\big([a,b]\cap[f>0]\big)$ otherwise. By (4), we have $c>a$ and $f(c)\ge 0$.

Case 2.a) If $[a,c] \subset [g \leq 0]$, then choose an $x_0 \in [f = 0] \cap (a,c]$ (we use the assumption $f \in \mathcal{M}$ and the inequality $f(b) \geq 0$) and observe that $h(x_0) = 0$.

Case 2.b) If $g(d) > 0$ for some $d \in [a,c]$, then we repeat the argumentation of Case 1 for the interval $[a,d]$.

In any case we conclude that $h \in \mathcal{M}$. It follows that $f \in \mathcal{M}_{\max}(\mathcal{M} \cap \mathcal{B}_\alpha)$. $\square$

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