Tatra Mt. Math. Publ. **40** (2008), 185–192

TATRA MOUNTAINS
Mathematical Publications

MOMENT PROBLEM FOR DOUBLE FUZZY SEQUENCES

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ABSTRACT. We present a moment problem in the context of fuzzy sets. A generalization of the Hausdorff moment theorem is formulated and proved for fuzzy double sequences

1. Introduction

The Hausdorff one-dimensional moment problem [Ha, HS, H, SC, W] is the following: given a prescribed set of real numbers $\{v_n\}_0^\infty$, find a bounded nondecreasing function $u(t)$ on the closed interval $[0,1]$ such that its moments are equal to the prescribed values; that is,

$$\int_{[0,1]} t^n du(t) = v_n, \qquad n = 0, 1, 2, \ldots$$

The integral is a Riemann-Stieltjes integral. Equivalently, find a nonnegative measure μ on Borelian subsets in $[0,1]$ with

$$\int_{[0,1]} t^n d\mu(t) = v_n, \qquad n = 0, 1, 2, \ldots$$

We shall need the operator $\nabla^k \quad (k = 0, 1, 2, \ldots)$ defined by

$$\nabla^0 v_n = v_n,$$

$$\nabla^1 v_n = v_n - v_{n+1},$$

$$\nabla^k v_n = v_n - \binom{k}{1} v_{n+1} + \binom{k}{2} v_{n+2} - \cdots + (-1)^k v_{n+k}, \qquad n = 1, 2, \ldots$$

for any sequence of real numbers $\{v_n\}_0^\infty$. If $\nabla^k v_n \geq 0, \quad n = 1, 2, \ldots$, the sequence $\{v_n\}_0^\infty$ is called completely monotone. Now Hausdorff moment theorem

2000 Mathematics Subject Classification: 28E10, 81P10.
Keywords: moment, fuzzy set, completely monotone double sequence, observable.
Supported by VEGA grant 2/4137/04.

says that for a sequence $\{v_n\}_0^\infty$ to be the moment sequence of some unique positive measure μ on $[0,1]$ it is necessary and sufficient that $\{v_n\}_0^\infty$ be completely monotone.

It was shown [DR] that the result permits a generalization to the case where $\{v_k\}$ is a completely monotone sequence with values in a fuzzy set. It is easy to see that a completely monotone sequence can be defined in the same way because the completely monotone sequence v_n is, as follows from the definition, non-increasing and so using difference $v_n - v_{n+1}$ makes sense. In this paper we consider completely monotone double sequences with values in a fuzzy set.

2. Remark on Bernstein polynomials in more dimensions

In some cases we know that $f(x,y)$ is a function of the two real variables x and y. Further, for each fixed value of x, $f(x,y)$ is a polynomial in y. For each fixed value of y, $f(x,y)$ is a polynomial in x. Is $f(x,y)$ necessarily a polynomial of the two variables x and y? It is interesting to note that it was shown (only in 1984) that $f(x,y)$ is a polynomial if it is so in each variable separately.

(This fact was published by F. V. Caroll: *A polynomial in each variable separately is a polynomial*, Amer. Math. Monthly **68** 1961, p. 42, as a solution of the problem posed in Amer. Math. Monthly **67** (1960), **68** (1961), **89** (1982) and **91** (1984).)

If we denote

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k} = C_n^k x^k (1-x)^{n-k},$$

then we have

$$\sum_{k=0}^{n} k p_{nk}(x) = nx,$$
$$\sum_{k=0}^{n} k^2 p_{nk}(x) = n^2x^2 + nx(1-x).$$

Consider a function $f : [0,1] \times [0,1] \to R$. The polynomial Bernstein form (or the Bernstein polynomial) of f is

$$B_{m,n}(f;x,y) = \sum_{j=0}^{m} \sum_{k=0}^{n} f\left(\frac{j}{m}, \frac{k}{n}\right) C_m^j C_n^k x^j (1-x)^{m-j} y^k (1-y)^{n-k}.$$

If f is a continuous function, it is bounded by a positive finite M, we consider the difference

$$\begin{aligned}R(x,y) &= f(x,y) - B_{m,n}(f;x,y)\\ &= \sum_{j,k}\left[f(x,y) - f\left(\frac{j}{m},\frac{k}{n}\right)\right] C_m^j C_n^k x^j (1-x)^{m-j} y^k (1-y)^{n-k}.\end{aligned}$$

For fixed $\epsilon > 0$, there exists $\delta > 0$ such that for $|x - x_0| < \delta$ and $|y - y_0| < \delta$ we have $|f(x,y) - f(x_0,y_0)| < \epsilon$. For fixed $(x,y) \in [0,1] \times [0,1]$,
let

$$\begin{aligned}A_1 &= \left\{(j,k) \mid \left|x - \frac{j}{m}\right| \le \delta, \left|y - \frac{k}{n}\right| \le \delta\right\},\\ A_2 &= \left\{(j,k) \mid \left|x - \frac{j}{m}\right| > \delta\right\},\\ A_3 &= \left\{(j,k) \mid \left|y - \frac{k}{n}\right| > \delta\right\}\end{aligned}$$

and

$$R_i(x,y) = \sum_{(j,k)\in A_i}\left[f(x,y) - f\left(\frac{j}{m},\frac{k}{n}\right)\right] C_m^j C_n^k x^j (1-x)^{m-j} y^k (1-y)^{n-k} \quad \text{for} \quad i = 1,2,3,$$

and we easily obtain

$$\begin{aligned}|R(x,y)| &\le |R_1(x,y)| + |R_2(x,y)| + |R_3(x,y)|\\ &\le \epsilon \sum_{j,k}^{1} C_m^j C_n^k x^j (1-x)^{m-j} y^k (1-y)^{n-k}\\ &\quad + \frac{2M}{m^2\delta^2}\sum_{j,k}(mx-j)^2 C_m^j C_n^k x^j (1-x)^{m-j} y^k (1-y)^{n-k}\\ &\quad + \frac{2M}{n^2\delta^2}\sum_{j,k}(ny-k)^2 C_m^j C_n^k x^j (1-x)^{m-j} y^k (1-y)^{n-k}\\ &\le \epsilon + \frac{M}{2m\delta^2} + \frac{M}{2n\delta^2}.\end{aligned}$$

For $n, m > \frac{M}{2\epsilon\delta^2}$ we have

$$|f(x,y) - B_{m,n}(f;x,y)| < 3\epsilon$$

which proves the following proposition.

Proposition. *Every continuous function $f : [0,1]^2 \to R$ can be uniformly approximated by its Bernstein polynomials.*

3. Double fuzzy moment problem

We consider a set $\mathcal{F} \subset [0,1]^\Omega$ of fuzzy subsets of a set Ω. Let $B[0,1]^2$ denote the family of Borel sets in $[0,1]^2 = [0,1] \times [0,1]$.

An observable is a mapping $y : B[0,1]^2 \to \mathcal{F}$ satisfying the conditions:

a) $y([0,1]^2) = 1$,

b) $A, B \in B[0,1]^2 \Rightarrow y(A \cup B) = y(A) + y(B)$ if $A \cap B = \emptyset$,

c) $A_n \in B[0,1]^2$, $n = 1, 2, \dots$, $A_n \nearrow A \Rightarrow y(A_n) \nearrow y(A)$.

Let $(a_{kl}) \subset \mathcal{F}$ be a double sequence of fuzzy elements. We say that (a_{kl}) is a solution of the double fuzzy moment problem if there exists an observable $y : B[0,1]^2 \to \mathcal{F}$ such that

$$a_{kl} = \int_{[0,1]^2} t^k s^l \, dy(t,s), \qquad k,l = 0,1,\dots,$$

i. e.,

$$a_{kl}(\omega) = \int_{[0,1]^2} t^k s^l \, dy(t,s)(\omega)\,, \quad k,l = 0,1,\dots\,, \quad \omega \in \Omega.$$

Now we shall prove double fuzzy moment problem theorem.

Theorem 1. *The double sequence $(a_{kl}) \subset \mathcal{F}$ is a solution of the double fuzzy moment problem, i.e., there exists an observable $y : B[0,1]^2 \to \mathcal{F}$ such that*

$$a_{kl}(\omega) = \int_{[0,1]^2} t^k s^l \, dy(t,s)(\omega), \qquad k,l = 0,1,\dots\,, \quad \omega \in \Omega$$

if and only if

$$0 \le \nabla_1^k \nabla_2^l a_{m,n}(\omega) \le 1, \qquad k,l,m,n = 0,1,\dots,\ a_{00}(\omega) = 1, \quad \omega \in \Omega$$

$$\nabla_1^k \nabla_2^l a_{n,m}(\omega) = \sum_{j=0}^{k} \sum_{p=0}^{l} (-1)^j (-1)^p \binom{n}{j} \binom{m}{p} a_{n+j,m+p}(\omega),$$
$$n,m = 0,1,\dots,\ k,l = 0,1,\dots$$

P r o o f. Put $x_k^{(n)}(t) z_l^{(m)}(s) = t^k (1-t)^n s^l (1-s)^m, \quad m,n,k,l = 0,1,\dots$

Necessity. If there exists an observable y such that

$$a_{kl}(\omega) = \int_{[0,1]^2} t^k s^l \, dy(t,s)(\omega), \qquad k,l = 0,1,\dots; \quad \omega \in \Omega\,,$$

then

$$\int_{[0,1]^2} x_k^{(0)} z_l^{(0)} dy(t,s)(\omega) = a_{kl}(\omega)\,,$$

$$\int_{[0,1]^2} x_k^{(n)} z_l^{(m)} dy(t,s)(\omega) = \nabla_1^n \nabla_2^m a_{kl}(\omega), \quad m,n,k,l = 0,1\ldots; \quad \omega \in \Omega\,.$$

Hence

$$0 \le \nabla_1^n \nabla_2^m a_{k,l}(\omega) \le 1, \quad k,l,m,n = 0,1,\ldots, \quad \omega \in \Omega\,.$$

So (a_{kl}) is also completely monotone.

Sufficiency. Let

$$0 \le \nabla_1^n \nabla_2^m a_{k,l}(\omega) \le 1, \quad k,l,m,n = 0,1,\ldots, \quad a_{00}(\omega) = 1, \quad \omega \in \Omega\,.$$

Define a mapping L_0 by $L_0(t^n s^m)(\omega) = a_{nm}(\omega)$, $n,m = 0,1,\ldots$, $\omega \in \Omega$.

Extend L_0 to the linear hull of $t^n s^m$, $n,m = 0,1,\ldots$, i.e., to the set of all polynomials in t and s:
if

$$x(t,s) = \sum c_{k,l} s^k t^l,$$

put

$$L(x)(\omega) = \sum c_{k,l} a_{k,l}(\omega).$$

The functions $t^n s^m$, $n,m = 0,1,\ldots$, are linear independent so the definition of L is unique, L is additive and homogeneous. We have

$$L\left(x_k^{(n)} z_l^{(m)}\right)(\omega) = \nabla_1^n \nabla_2^m a_{kl}(\omega), \quad k,m,l,n = 0,1,\ldots, \quad \omega \in \Omega\,.$$

Take any polynomial $p(t,s)$ of degree $n+m$. The sequence of Bernstein polynomials of $p(t,s)$ is:

$$p_{nm}(t,s) = B_{nm}(p,t,s) = \sum_{k=0,l=0}^{n,m} \binom{n}{k}\binom{m}{l} p\left(\frac{k}{n},\frac{l}{m}\right) t^k (1-t)^{n-k} s^l (1-s)^{m-l}.$$

The degree of polynomial

$$p_{n,m}(t,s), \quad n,m = 1,2,\ldots$$

is $\le m+n$, $p_{n,m}(t,s)$ uniformly converge to $p(t,s)$, for $n,m \to \infty$, hence

$$L(p_{n,m})(\omega) \to L(p)(\omega)\,.$$

Denote by $P_{m,n}$ the vector space of all polynomials degree not exceeding $m+n$; $P_{m,n}$ is finite-dimensional, hence for every, $\omega \in \Omega$, $L(p)(\omega)$ is a continuous linear functional on $P_{m,n}$. But, moreover,

$$L(p_{m,n})(\omega) = \sum_{k=0,l=0}^{n,m} \binom{n}{k}\binom{m}{l} p\left(\frac{k}{n}, \frac{l}{m}\right) L\left(x_n^{(n-k)} y_m^{(m-l)}\right)(\omega)$$
$$= \sum_{k=0,l=0}^{n,m} \binom{n}{k}\binom{m}{l} p\left(\frac{k}{n}, \frac{l}{m}\right) \nabla_1^{n-k}\nabla_2^{m-l} a_{k,l}(\omega)\,,$$

hence $L(p)(\omega)$ is positive if p is positive. So $L(\cdot)(\omega)$ is a positive linear functional on P (all polynomials). We may extend $L(\cdot)(\omega)$ to a continuous linear functional on $C([0,1]^2)$; it is positive. Therefore there exists a positive Borel measure ν_ω on $B[0,1]^2$, see [R, S], such that

$$L(f)(\omega) = \int_{[0,1]^2} f(t,s)\, d\nu_\omega(t,s)\,.$$

Put

$$y(A)(\omega) = \nu_\omega(A), \quad A \in B[0,1]^2\,.$$

We may write

$$L(f) = \int_{[0,1]^2} f(t,s)\, dy(t,s),$$
$$L(t^n s^m) = \int_{[0,1]^2} t^n s^m\, dy(t,s), \quad n,m = 0,1,\dots$$

Since by assumption

$$0 \le \nabla_1^k \nabla_2^l a_{n,m}(\omega) \le 1, \qquad k,l,m,n = 0,1,\dots;$$

we have

$$0 \le y(A)(\omega) \le 1, \qquad A \in B[0,1]^2, \qquad \omega \in \Omega\,.$$

So y is an observable. □

4. Moments of observables for some types of MV algebras

Consider a set $\mathcal{F} \subset [0,1]^\Omega$ of fuzzy subsets of a set Ω. We may take for example the operation $f \oplus g = \min(f+g, 1)$. This algebraic structure is an example of MV algebra. As for MV algebras and the product of observables we refer the reader to [2DR] and references given there. On the other hand, every MV algebra can be represented by a set $[0,u]^\Omega$, where $[0,u]$ is an interval in an ℓ-group G. So we are able to construct a convenient theory for a special case of MV algebras (with a (boundedly) complete vector lattice L as G).

Recall the definition of an observable in that particular context. Let G be a commutative ℓ-group,

$$u \in G, \qquad u > 0,$$
$$\mathcal{F} = [0,u]^{\Omega}, \qquad I = [0,1].$$

An observable is a mapping $x : \mathcal{B}(I) \to \mathcal{F}$ satisfying the following conditions:

(a) $x(I) = u_{\Omega}$.

(b) If $A, B \in \mathcal{B}(I)$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

(c) If $A_n \in \mathcal{B}(I)$ $(n = 1, 2, \dots)$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

If there is given a commutative binary operation $\star$ on $[0,u]$, we can define the joint observable of two observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ as a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:

(i) $h(R^2) = u_{\Omega}$.

(ii) If $A, B \in \mathcal{B}(R^2)$, $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.

(iii) If $A_n \in \mathcal{B}(R^2)$ $(n = 1, 2, \dots)$, $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.

(iv) If $A, B \in \mathcal{B}(R)$, then

$$h(A \times B) = x(A) \star y(B)\,.$$

We shall now present an application of the preceding results : an observable of two variables as the "product" of observables of one variable in a more general context.

Theorem 2. [2DR] *Let G be a commutative, weakly σ-distributive ℓ-group with a partial commutative binary operation $\star : G^+ \times G^+ \to G^+$ satisfying the distributive law. Let $a, b,\ c \in G^+$, $a \le b$ imply $a\star c \le b\star c$. Let $u \in G$, $u > 0$ be such an element that $u \cdot u = u$. Let $x, y : \mathcal{B}(I) \to [0,u]^{\Omega}$ be observables. Then there exists the joint observable of x and y.*

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Received December 10, 2007

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